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General Gagliardo Inequality and Applications to Weighted Sobolev Spaces

ANTONIO AVANTAGGIATI - PAOLA LORETI

Abstract. – *In this paper we obtain a more general inequality with respect to a well known inequality due to Gagliardo (see* [4], [5]*). The inequality contained in* [4], [5] *has been extended to weighted spaces, obtained as cartesian product of measurable spaces. As application, we obtain a first order weighted Sobolev inequality. This generalize a previous result obtained in* [2].

1. – Introduction and Main Result.

In this paper we study an inequality due to Gagliardo, extending the inequality to weighted spaces, obtained as cartesian product of measurable spaces. As application, we obtain a first order Sobolev inequality in weighted spaces. More precisely, we are concerned with N (>1) $\sigma-$finite measurable spaces $(\mathcal{X}_i, \mathcal{V}_i, \mu_i)$, $i = 1, 2, \dots, N$, and their cartesian product $(\mathcal{X}, \mathcal{S}, \mu)$. Therefore

- $\mathcal{X}$ is the set of all ordered N-tuples $(x_1, x_2, \dots, x_N)$ with $x_i \in \mathcal{X}_i$ $i = 1, 2, \dots, N$, and denoted by $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_N$
- $\mathcal{V}$ is the $\sigma-$algebra generated by the class of measurable rectangles $\mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_N$, with $\mathcal{A}_i \in \mathcal{V}_i, i = 1, 2, \dots, N$.
- μ is the unique $\sigma-$ finite measure on $\mathcal{V}$, verifying the condition

$$\mu(\mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_N) = \prod_{i=1}^{N} \mu_i(A_i),$$

for all measurable rectangles, and denoted by

$$\mu = \mu_1 \times \mu_2 \times \dots \times \mu_N.$$

For any fixed k with $1 \le k \le N$, we will denote by $\sigma = (i_1, i_2, \dots, i_k)$ a $k-$tupla of the indexes $1, 2, \dots, N$ (eventually also denoted by σ_k),with $i_1 < i_2 < \dots < i_k \le N$, also we denote by σ' the $(N-k)$-pla $(i'_1, i'_2, \dots, i'_{N-k})$, obtained from $\{1, 2, \dots, N\} \setminus \{i_1, i_2, \dots, i_k\}$.

Moreover we denote by $\mathcal{S}_k^N$ the set of all k-tupla of the set $\{1, 2, \dots, N\}$, (eventually also denoted by $\mathcal{S}$). The number of element of $\mathcal{S}$ is $\binom{N}{k}$. Given

$x = (x_1, x_2, \dots, x_N)$, $\sigma = (i_1, i_2, \dots, i_k)$ and $\sigma' = (i'_1, i'_2, \dots, i'_{N-k})$ we put

$$x_\sigma = (x_{i_1}, x_{i_2}, \dots, x_{i_k}) \qquad x_{\sigma'} = (x_{i'_1}, x_{i'_2}, \dots, x_{i'_{N-k}}); \qquad x = (x_\sigma, x'_\sigma).$$

We extend such a convention to all $N-$tuples we are concerned with; therefore

$$\mathcal{X}_\sigma = \mathcal{X}_{i_1} \times \mathcal{X}_{i_2} \times \dots \mathcal{X}_{i_k}; \qquad \mathcal{X}_{\sigma'} = \mathcal{X}_{i'_1} \times \mathcal{X}_{i'_2} \times \dots \mathcal{X}_{i'_{N-k}}; \qquad \mathcal{X} = \mathcal{X}_\sigma \times \mathcal{X}'_\sigma,$$

and

$$\mu_\sigma = \mu_{i_1} \times \mu_{i_2} \times \dots \mu_{i_k}; \qquad \mu_{\sigma'} = \mu_{i'_1} \times \mu_{i'_2} \times \dots \mathcal{X}_{i'_{N-k}}; \qquad \mu = \mu_\sigma \times \mu'_\sigma.$$

In this paper we are interested to real valued functions. For a function f defined on $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \mathcal{X}_N$ we will use $f(x)$ or $f(x_1, x_2, \dots, x_N)$ or $f(x_\sigma, x'_\sigma)$. The parameter

$$\lambda = \binom{N-1}{k-1} \qquad \Big[=: \lambda(N, K) \Big] \tag{1.1}$$

will have a particular rule in the paper.

The main result of this paper is the following inequality

THEOREM 1.1. – *Given* $\binom{N}{k}$ *functions* $F_\sigma(x_\sigma)$ *with* $\sigma \in \mathcal{S}_k^N$, *satisfying the hypothesis* $F_\sigma \in L^\lambda(\mathcal{X}_\sigma, \mu_\sigma)$ *we define*

$$F(x) = \prod_{\sigma \in \mathcal{S}} F_\sigma(x_\sigma), \tag{1.2}$$

then $F \in L^1(\mathcal{X}, \mu)$ *and*

$$\Big(\int_{\mathcal{X}} |F(x)| d\mu(x) \Big)^\lambda \leq \prod_{\sigma \in \mathcal{S}} \Big(\int_{\mathcal{X}^\sigma} |F_\sigma(x_\sigma)|^\lambda d\mu_\sigma(x_\sigma) \Big). \tag{1.3}$$

REMARK 1.2. – The theorem (1.1) generalizes Gagliardo's lemma in $\mathbb{R}^N$, since the last one follows by the theorem (1.1), taking $\mathcal{X}_i = \mathbb{R}$, $i = 1, 2, \dots N$, $\mathcal{V}_i$ the $\sigma-$algebra of Lebesgue measurable sets, and $\mu(x) = dx_1 dx_2 \dots, dx_N$ the Lebesgue measure. In a previous paper we establish a Gagliardo type inequality in Gaussian spaces ([3]).

2. – Proof of the Theorem.

If $k = 1$ the result is true for any N, since each function F_i depends only by one variable x_i, $\lambda = 1$, and the result follows with equality, by the Fubini's theorem. If $k = N$, the result is also true, it follows that the result is true for $N = 2$.

We prove the result arguing by induction on N. For $N = 2$ the result is true by the previous remark. Assume $N \geq 3$, and $k \geq 2$. The inductive hypothesis is to assume the result true for all $N \leq M - 1$, and to prove the result for $N = M$. We split the set $\mathcal{S}_k^M$ of all k-tuples in two sets $\mathcal{A}$ and $\mathcal{B}$. In the set $\mathcal{A}$ we put all k-tupla which do not contain the integer M, and $\mathcal{B}$ is $S_k^M \setminus \mathcal{A}$. The number of k-tupla which are in $\mathcal{A}$ are exactly $\binom{M-1}{k}$ and the value of the corresponding parameter is

$$\mu = \binom{M-2}{k-1} \qquad \Big[=: \lambda(M-1,k) \Big]. \tag{2.1}$$

If we take $\sigma \in \mathcal{A}$, since by hypothesis $F_\sigma(x_\sigma) \in \mathcal{L}^\lambda(\mathcal{X}_\sigma, d\mu_\sigma)$ we have $|F_\sigma(x_\sigma)|^{\frac{\lambda}{\mu}} \in \mathcal{L}^\mu(\mathcal{X}_\sigma, d\mu_\sigma)$. Applying the inductive hypothesis

$$\int\limits_{\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \mathcal{X}_{M-1}} \prod_{\sigma \in \mathcal{A}} |F_\sigma(x_\sigma)|^{\frac{\lambda}{\mu}} d\mu_1 \times d\mu_2 \ldots \times d\mu_{M-1} \leq \prod_{\sigma \in \mathcal{A}} \left(\int\limits_{X_\sigma} |F_\sigma(x_\sigma)|^\lambda d\mu_\sigma \right)^{\frac{1}{\mu}} \tag{2.2}$$

The number of the elements in the set $\mathcal{B}$ is

$$\binom{M}{k} - \binom{M-1}{k} = \binom{M-1}{k-1} = \lambda.$$

On the other hand, since any k-tupla of the set $\mathcal{B}$ has the index M, we can write it as $\sigma = (\sigma_{k-1}, M)$. We can have all the k-tupla of $\mathcal{B}$, just making σ_{k-1} describe all the $(k-1)$-ple of index $1, 2, \ldots, M-1$, that is, in our notation, all the elements of $\mathcal{S}_{k-1}^{M-1}$, denoted from now by $\mathcal{B}^-$. The value of the corresponding parameter is

$$\nu = \binom{M-2}{k-2} \qquad \Big[=: \lambda(M-1,k-1) \Big]. \tag{2.3}$$

Applying the inductive hypothesis, we get

$$\begin{aligned} &\int\limits_{\mathcal{X}_1 \times \mathcal{X}_2 \ldots \times \mathcal{X}_{M-1}} \prod_{\sigma \in \mathcal{B}} |F_\sigma(x_\sigma)|^{\frac{\lambda}{\nu}} d(\mu_1 \times d\mu_2 \times \ldots \times d\mu_{M-1}) \\ &= \int\limits_{\mathcal{X}_1 \times \mathcal{X}_2 \ldots \times \mathcal{X}_{M-1}} \prod_{\sigma_{k-1} \in \mathcal{B}^-} |F_\sigma(x_{\sigma_{k-1}}, x_M)|^{\frac{\lambda}{\nu}} d(\mu_1 \times d\mu_2 \times \ldots \times d\mu_{M-1}) \\ &\leq \prod_{\sigma_{k-1} \in \mathcal{B}^-} \left(\int\limits_{\mathcal{X}^{\sigma_{k-1}}} |F_\sigma(x_{\sigma_{k-1}}, x_M)|^\lambda d\mu_{\sigma_{k-1}} \right)^{\frac{1}{\nu}}. \end{aligned} \tag{2.4}$$

Next, we observe that $\lambda = \nu + \mu$, then $\dfrac{\lambda}{\mu}$ and $\dfrac{\lambda}{\nu}$ are complementary exponent of summability. Applying Holder inequality we have

$$\int\limits_{\mathcal{X}} |F(x)| d\mu = \int\limits_{\mathcal{X}_M} d\mu_M \int\limits_{\mathcal{X}_1 \times \mathcal{X}_2 \ldots \times \mathcal{X}_{M-1}} \prod_{\sigma \in \mathcal{S}} |F_\sigma(x_\sigma)| d(\mu_1 \times d\mu_2 \times \ldots \times d\mu_{M-1}) \tag{2.5}$$

$$=\int_{\mathcal{X}_M} d\mu_M \int_{\mathcal{X}_1\times\mathcal{X}_2\ldots\times\mathcal{X}_{M-1}} \prod_{\sigma\in\mathcal{A}} |F_\sigma(x_\sigma)| \prod_{\sigma\in\mathcal{B}} |F_\sigma(x_\sigma)| d(\mu_1\times d\mu_2\times\ldots\times d\mu_{M-1})$$

$$\leq \int_{\mathcal{X}_M} d\mu_M \Bigg(\int_{\mathcal{X}_1\times\mathcal{X}_2\ldots\times\mathcal{X}_{M-1}} \prod_{\sigma\in\mathcal{A}} |F_\sigma(x_\sigma)|^{\frac{\lambda}{\mu}} d(\mu_1\times d\mu_2\times\ldots\times d\mu_{M-1})\Bigg)^{\frac{\mu}{\lambda}} \cdot$$

$$\Bigg(\int_{\mathcal{X}_1\times\mathcal{X}_2\ldots\times\mathcal{X}_{M-1}} \prod_{\sigma\in\mathcal{B}} |F_\sigma(x_\sigma)|^{\frac{\lambda}{\nu}} d(\mu_1\times d\mu_2\times\ldots\times d\mu_{M-1})\Bigg)^{\frac{\nu}{\lambda}}$$

Applying (2.2) and (2.4) we have

$$\int_{\mathcal{X}} \prod_{\sigma\in\mathcal{S}} |F_\sigma(x_\sigma)| d\mu \leq \prod_{\sigma\in\mathcal{A}} \Bigg(\int_{\mathcal{X}_\sigma} |F_\sigma(x_\sigma)|^\lambda d\mu_\sigma \Bigg)^{\frac{1}{\lambda}} \cdot \tag{2.6}$$

$$\int_{\mathcal{X}_M} d\mu_M \prod_{\sigma_{k-1}\in\mathcal{B}^-} \Bigg(\int_{\mathcal{X}_{\sigma_{k-1}}} |F_\sigma(x_{\sigma_{k-1}}, x_M)|^\lambda d\mu_{\sigma_{k-1}} \Bigg)^{\frac{1}{\lambda}}.$$

Applying Holder inequality to the last product of λ factors, each of which belongs to $L^\lambda(\mathcal{X}_M, d\mu_M)$ we conclude the proof, since

$$\int_{\mathcal{X}_M} d\mu_M \prod_{\sigma_{k-1}\in\mathcal{B}^-} \Bigg(\int_{\mathcal{X}_{\sigma_{k-1}}} |F_\sigma(x_{\sigma_{k-1}}, x_N)|^\lambda d\mu_{\sigma_{k-1}} \Bigg)^{\frac{1}{\lambda}}$$

$$\leq \prod_{\sigma_{k-1}\in\mathcal{B}^-} \Bigg(\int_{\mathcal{X}_M} d\mu_M \int_{\mathcal{X}_{\sigma_{k-1}}} |F_\sigma(x_{\sigma_{k-1}}, x_M)|^\lambda d\mu_{\sigma_{k-1}} \Bigg)^{\frac{1}{\lambda}}$$

$$= \prod_{\sigma\in\mathcal{B}} \Bigg(\int_{\mathcal{X}_\sigma} |F_\sigma(x_\sigma)|^\lambda d\mu_\sigma \Bigg)^{\frac{1}{\lambda}}.$$

□

REMARK 2.1. – The theorem (1.1) was established in [2] in the particular case

$$d\mu_j(x_j) = (2\pi)^{-\frac{1}{2}} \exp\Bigg(-\frac{1}{2}|x_j|^2\Bigg) dx_j, \quad j = 1, \ldots N$$

$$d\mu = (2\pi)^{-\frac{N}{2}} \exp\Bigg(-\frac{1}{2}|x|^2\Bigg) dx, \qquad \mathcal{X} = \mathbb{R}^N.$$

3. – Applications.

3.1 – *Non isotropic Gaussian spaces.*

Take $\mathcal{X}_i = \mathbb{R}[\,=: \mathbb{R}_i]$, $i = 1, 2, \dots N$, $\mathcal{V}_i$ the $\sigma-$ algebra of Lebesgue measurable sets, and assume μ_i absolutely continuous with respect to Lebesgue measure

$$d\mu_i = \frac{1}{c_i}\exp(-v_i(x_i))dx_i \qquad i = 1, 2, \dots, N$$

with $v_i : \mathbb{R}_i \to \mathbb{R}_+$, even and convex functions, and

$$c_i = \int_{\mathbb{R}} \exp(-v_i(t))dt \qquad i = 1, 2, \dots, N.$$

We set

$$v(x) = \sum_{j=1}^{N} v_j(x_j), \quad c = \prod_{j=1}^{N} c_j,$$

and

$$d\mu(x) = \prod_{j=1}^{N} d\mu_i = \prod_{j=1}^{N} \frac{1}{c_i}\exp(-v_i(x_i))dx_i = \frac{1}{c}\exp(-v(x))dx$$

The measure spaces we are dealing with $(\mathbb{R}_i, \mathcal{V}_i, \frac{1}{c_i}\exp(-v_i(x_i))$, $\quad i = 1, 2, \dots, N$, and their cartesian product $(\mathbb{R}^N, \mathcal{V}, \frac{1}{c}\exp(-v(x))$

By theorem (1.1) we have

THEOREM 3.1. – *Given a subset E of $\mathcal{S}_k^N$ $(\neq \emptyset)$ and a family $(F_\sigma)_{\sigma \in E}$ satisfying the hypothesis $F_\sigma \in L^\lambda(\mathbb{R}^\sigma, d\mu_\sigma)$ $\forall \sigma \in E$, we define*

$$F(x) = \prod_{\sigma \in E} F_\sigma(x_\sigma), \tag{3.1}$$

then $F \in L^1(\mathbb{R}^N, d\mu)$ and

$$\left(\int_{\mathbb{R}^N} |F(x)| d\mu(x)\right)^\lambda \leq \prod_{\sigma \in E} \int_{\mathbb{R}^\sigma} |F_\sigma(x_\sigma)|^\lambda d\mu_\sigma(x_\sigma) \tag{3.2}$$

PROOF. – We observe that

$$\int_{\mathbb{R}^\sigma} d\mu_\sigma(x_\sigma) = 1, \qquad \forall \sigma \in \mathcal{S}_k^N. \tag{3.3}$$

This means that the characteristic function $\chi_{\mathbb{R}_\sigma}(x_\sigma)$ of the space $\mathbb{R}_\sigma$ verifies

$$\int_{\mathbb{R}_\sigma} \chi_{\mathbb{R}^\sigma}(x_\sigma) d\mu_\sigma(x_\sigma) = 1, \tag{3.4}$$

and it is $\lambda-$ summable. We introduce for every $\sigma \in \mathcal{S}_k^N \setminus E$ the functions

$$F_\sigma = \chi_{\mathbb{R}_\sigma}, \qquad \forall \sigma \in \mathcal{S}_k^N \setminus E.$$

We have a family of $\binom{N}{k}$ functions (F_σ), $\sigma \in \mathcal{S}_k^N$ which verifies the hypothesis of (1.1), therefore the assert follows by thesis of the theorem (1.1). □

We set

$$d\gamma = \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} e^{-\frac{1}{2}|x|^2} dx.$$

REMARK 3.2. – Let $1 \leq p < N$ and $q = \dfrac{Np}{N-p}$, dx the Lebesgue measure. It is well-known that

- $f \in L^p(\mathbb{R}^N, dx), \qquad$ and $|Df| \in L^p(\mathbb{R}^N, dx) \Longrightarrow f \in L^q(\mathbb{R}^N, dx)$

If we are looking for a similar result in the Gaussian space $(\mathbb{R}^N, \mathcal{V}, d\gamma)$, then the answer is negative, as the following example (3.3) shows.

EXAMPLE 3.3. – Let

$$u_\star(x) = \exp\left(\frac{1}{2}\frac{\log(1+|x|^2)}{1+\log(1+|x|^2)}|x|^2\right), \quad x \in \mathbb{R}^N$$

Then

$$u_\star(x)\exp\left(-\frac{1}{2}|x|^2\right) = \exp\left(-\frac{1}{2}\frac{1}{1+\log(1+|x|^2)}|x|^2\right), \quad x \in \mathbb{R}^N.$$

This shows that $u_\star(x) \in L^1(\mathbb{R}^N, d\gamma)$.

Similarly $|Du_\star(x)| \in L^1(\mathbb{R}^N, d\gamma)$.

However $u_\star(x) \notin L^p(\mathbb{R}^N, d\gamma) \qquad \forall p > 1$.

Indeed we have

$$(u_\star(x))^p \exp\left(-\frac{1}{2}|x|^2\right) = \exp\left(\frac{1}{2}\frac{(p-1)\log(1+|x|^2)-1}{\log(1+|x|^2)+1}|x|^2\right), \quad x \in \mathbb{R}^N. \qquad □$$

3.2 – *Sobolev's Inequalities.*

For any $p \in [1, N[$, $q = \dfrac{Np}{N-p}$ the well-known Sobolev inequality related to the space $W^{1,p}(\mathbb{R}^N)$ is

$$\left(\int_{\mathbb{R}^N} |u(x)|^q dx \right)^{\frac{1}{q}} \leq c \left(\int_{\mathbb{R}^N} |Du(x)|^p dx \right)^{\frac{1}{p}} \qquad \forall u \in W^{1,p}(\mathbb{R}^N), \tag{3.5}$$

for a real, positive constant c.

We are going to find, using (1.1), how to extend (3.5) to a class of weighted Sobolev spaces. To this end let $\mathcal{X}_i$ and $\mathcal{V}_i$ as before and assume μ_i absolutely continuous with respect to Lebesgue measure, $\mathbb{R}$ will be also denotes by $\mathbb{R}_i$. To cut short, we assume that there exist measurable functions $\omega_i(x_i)$ locally summable on $\mathbb{R}_i$ such that

$$d\mu_i(x_i) = \omega_i(x_i) dx_i, \qquad i = 1, 2, \ldots, N. \tag{3.6}$$

We are now ready to state the result

THEOREM 3.4. – *Let $\omega_i(x_i)$ be the function introduced in* (3.6) *verifying the hypothesis*

$$\omega_i(x_i) \geq 1 \qquad \text{a.e. in } \mathbb{R}_i, \qquad i = 1, 2, \ldots, N. \tag{3.7}$$

Define μ as follows

$$\omega(x) = \omega_1(x_1)\omega_2(x_2)\ldots\omega_N(x_N), \qquad d\mu = \omega(x)dx. \tag{3.8}$$

Let

$$1 \leq p < N, \qquad q = \frac{Np}{N-p}, \qquad \gamma = \frac{(N-1)p}{N-p}, \tag{3.9}$$

then for evey $u \in C_0^1(\mathbb{R}^n)$ we have

$$\left(\int_{\mathbb{R}^N} |u(x)|^q \omega(x) dx \right)^{\frac{1}{q}} \leq \gamma \left(\int_{\mathbb{R}^N} |Du(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \tag{3.10}$$

PROOF. – To simplify the notations, we introduce the vectors $e_i = (\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_N})$, where δ_{hk} is the Kronecher symbol, and, since $u \in C_0^1(\mathbb{R}^n)$, we have

$$|u(x)|^\gamma = -\int_0^{+\infty} \frac{d}{dt} |u(x + te_i)|^\gamma dt.$$

From which

$$|u(x)|^\gamma \le \gamma \int_0^{+\infty} |u(x+te_i)|^{\gamma-1} |\partial_{x_i} u(x+te_i)| dt$$

$$\le \gamma \int_{\mathbb{R}_i} |u(x)|^{\gamma-1} |\partial_{x_i} u(x)| dx_i \qquad i = 1, 2 \quad , N.$$

Let $i^\star$ the $(N-1)-$ tupla $\{1, 2, \dots, N\} \setminus \{i\}$, we get

$$\sup_{x_i \in \mathbb{R}_i} |u(x_{i^\star}, x_i)|^\gamma \le \gamma \int_{\mathbb{R}_i} |u(x)|^{\gamma-1} |\partial_{x_i} u(x)| dx_i.$$

We apply the theorem (1.1), and we obtain

$$\int_{\mathbb{R}^N} |u(x)|^q \omega(x) dx = \int_{R^N} \prod_{i=1}^N |u(x_{i^\star}, x_i)|^{\frac{p}{N-p}} \omega(x) dx$$

$$\le \int_{R^N} \prod_{i=1}^N \big(\sup_{x_i \in \mathbb{R}} |u(x_{i^\star}, x_i)|^{\frac{p}{N-p}} \big) \omega(x) dx$$

$$\prod_{i=1}^N \left(\int_{R^{i^\star}} \left(\sup_{x_i \in \mathbb{R}} |u(x_{i^\star}, x_i)|^{\frac{p}{N-p}} \right)^{N-1} \omega_{i^\star}(x_{i^\star}) dx_{i^\star} \right)^{\frac{1}{N-1}}$$

$$= \prod_{i=1}^N \left(\int_{R^{i^\star}} \sup_{x_i \in \mathbb{R}} |u(x_{i^\star}, x_i)|^\gamma \omega_{i^\star}(x_{i^\star}) dx_{i^\star} \right)^{\frac{1}{N-1}}.$$

Then

$$\int_{\mathbb{R}^N} |u(x)|^q \omega(x) dx \le \prod_{i=1}^N \left(\gamma \int_{R^{i^\star}} \omega_{i^\star}(x_{i^\star}) dx_{i^\star} \int_{\mathbb{R}_i} |u(x)|^{\gamma-1} |\partial_{x_i} u(x)| dx_i \right)^{\frac{1}{N-1}}$$

$$\le \left(\gamma \int_{R^N} |u(x)|^{\gamma-1} |Du(x)| \omega(x) dx \right)^{\frac{N}{N-1}}$$

$$\le \gamma^{\frac{N}{N-1}} \left(\int_{R^N} |u(x)|^{(\gamma-1)p'} d\omega(x) dx \right)^{\frac{N}{p'(N-1)}} \left(\int_{\mathbb{R}^N} |Du(x)|^p \omega(x) dx \right)^{\frac{N}{p(N-1)}}.$$

Since $(\gamma - 1)p' = q$ and

$$\left(1 - \frac{N}{N-1} \frac{p-1}{p}\right) \frac{N-1}{N} = \frac{N-p}{(N-1)p} \frac{N-1}{N} = \frac{1}{q},$$

from the last estimate we conclude the proof. □

Concerning with Sobolev inequality in Gaussian spaces we can state the following result, which proof follows by theorem (3.4).

THEOREM 3.5. – *Let* $\omega_i(x_i)$ *be the function introduced in* (3.6) *verifying* (3.7) *and* $\mu(x)$ *defined as* (3.8), *and given* N *positive, convex functions,* $v_1(x_1)$, $v_2(x_2), \dots, v_N(x_N)$, *with* $v = \sum_{i=1}^{N} v_i(x_i)$. *Let*

$$1 \leq p < N, \qquad q = \frac{Np}{N-p}, \tag{3.11}$$

then for every function u *such that* $ue^{-v(x)} \in W^{1,p}(\mathbb{R}^N, d\mu)$, *we have*

$$\left(\int_{\mathbb{R}^N} |u(x)|^q e^{-qv(x)} \omega(x) dx \right)^{\frac{1}{q}} \leq c \left(\int_{\mathbb{R}^N} |Du(x) - u(x)Dv(x)|^p e^{-pv(x)} \omega(x) dx \right)^{\frac{1}{p}}, \tag{3.12}$$

for some positive constant c.

4. – Further generalizations.

We remark that the original Gagliardo inequality, as our previous results, are related to product of factors, each contains the same number k of variables and the same summability exponent.

In what follows, we are now dealing with a set of non negative functions.

$$f_{j,\sigma_k} : \mathbb{R}^{\sigma_k} \to \overline{\mathbb{R}}_+, \; j = 1, \dots, J_k; \sigma_k \in \mathcal{S}_k^N, \; k = 1, \dots, N.$$

The further result is the following

THEOREM 4.1. – *We assume that* $f_{j,\sigma_k} \in L^1(\mathbb{R}^{\sigma_k}, d\mu_{\sigma_k})$ *and that there exist positive, real, numbers* $c_{j,k}$

$$\sum_{k=1}^{N} \sum_{j=1}^{J_k} c_{j,k} k \binom{N}{k} = N. \tag{4.1}$$

Then

$$\int_{\mathbb{R}^N} \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} \left(f_{j,\sigma_k}(x_\sigma)\right)^{c_{j,k}} d\mu \leq \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} \left(\int_{\mathbb{R}^{\sigma_k}} f_{j,\sigma_k}(x_{\sigma_k}) d\mu_{\sigma_k} \right)^{c_{j,k}} \tag{4.2}$$

PROOF. – We set

$$\sum_{j=1}^{J_k} c_{j,k} =: Q_k, \qquad q_k = \frac{N}{kQ_k} \binom{N}{k}^{-1}, \; k = 1, \dots, N. \tag{4.3}$$

By (4.1) it follows

$$\sum_{k=1}^{N} kQ_k \binom{N}{k} = N, \text{ and } \sum_{k=1}^{N} \frac{1}{q_k} = 1. \tag{4.4}$$

We set

$$p_{j,k} = \frac{Q_k}{c_{j,k}} \text{ for } k = 1, \dots N \; j = 1, \dots J_k. \tag{4.5}$$

It follows

$$\sum_{j=1}^{J_k} \frac{1}{p_{j,k}} = 1.$$

We apply Hölder inequality and Theorem (1.1) with $\lambda =: \lambda_k$; then we get:

$$\int_{\mathbb{R}^N} \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} (f_{j,\sigma_k}(x_\sigma))^{c_{j,k}} d\mu \le \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \left(\int_{\mathbb{R}^{\sigma_k}} \prod_{j=1}^{J_k} (f_{j,\sigma_k}(x_{\sigma_k}))^{c_{j,k} q_k \lambda_k} d\mu_{\sigma_k} \right)^{\frac{1}{\lambda_k q_k}} \tag{4.6}$$

Since

$$\lambda_k q_k = \binom{N-1}{k-1} \frac{N}{kQ_k} \binom{N}{k}^{-1} = \frac{1}{Q_k},$$

We apply Hölder inequality

$$\prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \left(\int_{\mathbb{R}^{\sigma_k}} \prod_{j=1}^{J_k} (f_{j,\sigma_k}(x_{\sigma_k}))^{c_{j,k} q_k \lambda_k} d\mu_{\sigma_k} \right)^{\frac{1}{\lambda_k q_k}}$$

$$\le \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} \left(\int_{\mathbb{R}^{\sigma_k}} (f_{j,\sigma_k}(x_{\sigma_k}))^{p_{j,k} c_{j,k} q_k \lambda_k} d\mu_{\sigma_k} \right)^{\frac{Q_k}{p_{j,k}}}.$$

Since

$$\prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} \left(\int_{\mathbb{R}^{\sigma_k}} (f_{j,\sigma_k}(x_{\sigma_k}))^{p_{j,k} c_{j,k} q_k \lambda_k} d\mu_{\sigma_k} \right)^{\frac{Q_k}{p_{j,k}}} = \prod_{k=1}^{N} \prod_{\sigma_k \in \mathcal{S}_k^N} \prod_{j=1}^{J_k} \left(\int_{\mathbb{R}^{\sigma_k}} f_{j,\sigma_k}(x_{\sigma_k}) d\mu_{\sigma_k} \right)^{c_{j,k}},$$

we conclude the proof. □

REMARK 4.2. – The inequality (4.2) is optimal. Indeed we fix $a \in \mathbb{R}_+^N$ and we define (now $j_k = 1$)

$$f_{\sigma_k} = f_{1,\sigma_k} = \exp(a_{\sigma_k} x_{\sigma_k}), \qquad \forall \sigma_k \in \mathcal{S}_k^N \text{ and } k = 1, 2, \dots, N. \tag{4.7}$$

and

$$c_k = c_{1,k} = \frac{1}{N\lambda(N,k)}. \tag{4.8}$$

A direct computation gives equality in (4.2) for (4.7), (4.8), with respect to the gaussian measure $\gamma(x)$.

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