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A Regular Threefold of General Type with $p_g = 0$ and $P_2 = 6$

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Abstract. – *The range of the bigenus P_2 is one of the unsolved problems concerning smooth complex projective regular threefolds of general type with $p_g = 0$. The examples in the literature have $P_2 \leq 5$. In the present paper we present a non-singular threefold with $p_g = q_1 = q_2 = 0$, $P_2 = 6$; the bicanonical map is stably birational.*

Introduction.

It is well known that the bigenus P_2 of nonsingular complex surfaces of general type with $p_g = 0$ satisfies $2 \leq P_2 \leq 10$ and examples are known for all possible values of P_2.

In the case of nonsingular projective regular threefolds X of general type with $p_g = 0$, defined over the field $\mathbb{C}$ of complex numbers, one of the problems that remain to be solved is the behaviour of the bigenus.

Results concerning this issue are given in [6] and [9], where it is proved that the bigenus can take on the value $P_2 = 0$, as well as in [7], [10] and [1], where threefolds with $P_2 = 1, 2, 3, 4$ are presented, and in [8], which gives a threefold with the bigenus reaching $P_2 = 5$. All these varieties have the irregularities $q_1 = q_2 = 0$.

Examples of threefolds of general type with $p_g = 0$, as varieties in weighted projective spaces, are also presented in [3] and [5]; they all have $P_2 \leq 4$.

The problem regarding the bigenus therefore lies in finding an integer n_0, if one exists, such that $P_2 \leq n_0$ for any X, and in establishing by means of examples whether P_2 can assume any value less than or equal to n_0.

In order to give a contribution in this direction, we found the nonsingular threefold X of general type presented here: in addition to $p_g = q_1 = q_2 = 0$, it has $P_2 = 6$ and $P_3 = 9$. The bicanonical map $\Phi_{|2K_X|}$ is stably birational, i.e. $\Phi_{|mK_X|}$ is birational for all $m \geq 2$.

The variety X presented here is obtained as a nonsingular model of a degree six hypersurface in $\mathbb{P}^4$ endowed with suitable singularities.

1. – Construction of the example.

Let X be a nonsingular projective threefold of general type defined over the field $\mathbb{C}$ of complex numbers and let K_X be a canonical divisor on X.

As usual, we denote the m-genus $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X))$ of X by P_m and the i-th irregularity $\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X)$ of X by q_i, $i = 1, 2$. Moreover, we denote P_1 as p_g and call it the *geometric genus* of X. We also call P_2 the *bigenus* of X.

To produce a threefold with the properties listed in the title, we consider in $\mathbb{P}^4$ a hypersurface of degree 6 and we put suitable singularities on it, at the vertices A_i of the fundamental pentahedron. To be more precise, at two of them, say A_1 and A_4, we impose:

- a triple point with a triple point infinitely near and then a double surface in the subsequent neighbourhood,

and at three of them, say A_2, A_3, A_5, we impose singularities of the type:

- a triple point with a triple curve infinitely near,

and we place them in a suitable arrangement. It is worth noting that, for the purposes of obtaining a normal variety with the desired birational invariants, the position of the infinitely near singularities is just as important as the choice of their type.

In $\mathbb{P}^4$ with homogeneous coordinates $x_1, \dots, x_5$, by imposing these singularities, we obtain hypersurfaces of a (incomplete) linear system depending on 37 parameters, but to obtain the example we can confine ourselves to the hypersurfaces of the linear system:

$$\begin{aligned} &a_1 x_1{}^3 x_2{}^3 + a_2 x_1{}^2 x_2{}^2 x_3{}^2 + a_3 x_2{}^3 x_3{}^3 + a_4 x_1{}^3 x_2{}^2 x_4 + \\ &\quad a_5 x_2{}^2 x_3{}^2 x_4{}^2 + a_6 x_1 x_3{}^2 x_4{}^3 + a_7 x_3{}^3 x_4{}^3 + a_8 x_1{}^3 x_2{}^2 x_5 + \\ &\quad a_9 x_1{}^2 x_2{}^2 x_4 x_5 + a_{10} x_1 x_3{}^2 x_4{}^2 x_5 + a_{11} x_3{}^2 x_4{}^3 x_5 + a_{12} x_1{}^2 x_2{}^2 x_5{}^2 + \\ &\quad a_{13} x_1{}^2 x_2 x_3 x_5{}^2 + a_{14} x_1{}^2 x_4{}^2 x_5{}^2 + a_{15} x_2 x_3 x_4{}^2 x_5{}^2 + \\ &\quad a_{16} x_3{}^2 x_4{}^2 x_5{}^2 + a_{17} x_1{}^2 x_4 x_5{}^3 + a_{18} x_1 x_4{}^2 x_5{}^3 = 0. \end{aligned}$$

We denote a generic element of the above linear system (corresponding to a generic choice of the parameters) as V; the threefold X in this paper is a desingularization of V.

Other singularities appear on V in addition to those imposed so, to be sure that the only ones that affect the plurigenera are the five imposed at the vertices of the fundamental pentahedron, we have to find all the actual singularities and, by means of the desingularization, all the infinitely near ones.

We can check the actual singularities by using Bertini's theorem; they belong to the base point locus of the linear system whose V is a generic element and they can be found by means of the derivatives of the polynomial F defining V: in addition to the imposed triple points, the actual singularities of V belong to five double lines: $\mathcal{L}_1 : x_1 = x_2 = x_4 = 0$; $\mathcal{L}_2 : x_1 = x_3 = x_4 = 0$; $\mathcal{L}_3 : x_1 = x_3 = x_5 = 0$; $\mathcal{L}_4 : x_2 = x_3 = x_5 = 0$; $\mathcal{L}_5 : x_2 = x_4 = x_5 = 0$. V is therefore a normal variety.

To solve all the actual or infinitely near singularities that appear on V, we construct, in a classical way, a variety $\widetilde{\mathbb{P}^4}$ by means of appropriate blow-ups, which are suggested, step by step, by the singularities of V or its strict transforms.

First, we cover $\mathbb{P}^4$ with the affine open sets $U_i = \{x_i \neq 0\}$ and V with $V_i = U_i \cap V$. Then, by means of the blow-ups used to solve the singularities of V, we determine affine open sets $\widetilde{U}_{i,j}$, $1 \leq i \leq 5$, $j \in J_i$, and birational maps $\varphi_{k,l;i,j} : \widetilde{U}_{i,j} \to \widetilde{U}_{k,l}$ (J_i being the set of symbols denoting the sequence of the open sets that we choose in the local representation of the blow-ups adopted). The open sets $\widetilde{U}_{i,j}$ that we consider are those in which the strict transform $X_{i,j}$ of V_i is nonsingular.

Finally, we represent $\widetilde{\mathbb{P}^4}$ by glueing the schemes $\widetilde{U}_{i,j}$ by means of $\varphi_{k,l;i,j}$.

The birational morphism from $\widetilde{\mathbb{P}^4}$ to $\mathbb{P}^4$, obtained by composing the blow-ups, will be denoted by σ: $\widetilde{\mathbb{P}^4} \to \mathbb{P}^4$.

The desingularization X of V, the strict transform of V by σ, is thus obtained by pasting the hypersurfaces $X_{i,j} \subset \widetilde{U}_{i,j}$. For the sake of simplicity, we shall denote the restriction of $\varphi_{k,l;i,j}$ to $X_{i,j}$ and the restriction of σ to X again as $\varphi_{k,l;i,j} : X_{i,j} \to X_{k,l}$ and $\sigma : X \to V$.

2. – Some details of the desingularization of V.

In our case, to bring the desingularization of V to the end, we only need blow-ups along varieties that are locally linear.

As an example, we describe some blow-ups that have a leading role in computing the plurigenera of X.

For the sake of simplicity, we denote as $(x_1, \ldots, \widehat{x_i}, \ldots, x_5)$ the affine coordinates on $U_i = \{x_i \neq 0\}$.

• First of all, we consider the blow-up $\pi_1 : \mathbb{P}_2 \to \mathbb{P}_1 = \mathbb{P}^4$ at the point A_1, the origin in U_1. By construction, A_1 is a triple point of V with a triple point infinitely near and then a double surface in the subsequent neighbourhood.

Since A_1 belongs only to U_1, $\mathbb{P}_2$ is covered by U_2, U_3, U_4, U_5 and by $\pi_1^{-1}(U_1)$ that, in turn, can be covered by four affine open sets $\widetilde{U}_{1,2}$, $\widetilde{U}_{1,3}$, $\widetilde{U}_{1,4}$ and $\widetilde{U}_{1,5}$.

If $(y'_2, \ldots, y'_5)$, $(y''_2, \ldots, y''_5)$, $\ldots$ denote affine coordinates on $\widetilde{U}_{1,2}$, $\widetilde{U}_{1,3}, \ldots$ respectively, the restriction of π_1 to $\widetilde{U}_{1,2}$, $\widetilde{U}_{1,3}$, $\ldots$ is given by:

$$\begin{array}{llll} x_2 = y'_2, & x_3 = y'_3 y'_2, & x_4 = y'_4 y'_2, & x_5 = y'_5 y'_2; \\ x_2 = y''_2 y''_3, & x_3 = y''_3, & x_4 = y''_4 y''_3, & x_5 = y''_5 y''_3; \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

The exceptional divisor E_1 of π_1 is given in $\widetilde{U}_{1,2}$, $\widetilde{U}_{1,3}$, $\dots$ by $y'_2 = 0$, $y''_3 = 0$, $\dots$ respectively.

We write the polynomials that define the strict transform W_2 of V_1 in $\widetilde{U}_{1,j}$ using the previous formulas and we focus our attention on the origin $A_{1,3}(0,0,0,0) \in E_1$ in the affine open set $\widetilde{U}_{1,3}$. This is a triple point of W_2 and it is one of the singularities imposed on V.

We now consider a second blow-up $\pi_2 : \mathbb{P}_3 \to \mathbb{P}_2$ at $A_{1,3}$. We denote the strict transform of W_2, with respect to π_2, as W_3.

$A_{1,3}$ belongs only to $\widetilde{U}_{1,3}$; so we cover $\pi_2^{-1}(\widetilde{U}_{1,3})$ with four affine open sets $\widetilde{U}_{1,32}$, $\widetilde{U}_{1,33}$, $\widetilde{U}_{1,34}$ and $\widetilde{U}_{1,35}$.

For instance, if $(z_2, \dots, z_5)$ denote affine coordinates on $\widetilde{U}_{1,33}$, the restriction of π_2 to it is given by:

$$y''_2 = z_2 z_3, \quad y''_3 = z_3, \quad y''_4 = z_4 z_3, \quad y''_5 = z_5 z_3.$$

If we write the polynomial that defines W_3 in $\widetilde{U}_{1,33}$, we can see that it has a surface $\mathcal{S}_3$ as a locus of double points on the exceptional divisor E_2, given locally by $z_3 = 0$: this is the surface given by $z_3 = z_2 = 0$.

So we need a third blow-up $\pi_3 : \mathbb{P}_4 \to \mathbb{P}_3$ along this surface.

We cover $\pi_3^{-1}(\widetilde{U}_{1,33})$ with two open sets: $\widetilde{U}_{1,332}$ and $\widetilde{U}_{1,333}$. If we analyse $\widetilde{U}_{1,332}$, for instance, where affine coordinates are denoted by $(u_2, \dots, u_5)$, we can see that the restriction of π_3 to this open set is given by

$$z_2 = u_2, \quad z_3 = u_2 u_3, \quad z_4 = u_4, \quad z_5 = u_5$$

and the exceptional divisor E_3 is given by $u_2 = 0$.

Using Bertini's theorem and the derivatives of the polynomial $F_{1,332}$ defining the strict transform W_4 of W_3 in $\widetilde{U}_{1,332}$, we see that W_4 has no singularities in $\widetilde{U}_{1,332}$. In fact, the local representation of W_4 in $\widetilde{U}_{1,332}$ is:

$$\begin{aligned} X_{1,332} : {} & a_2 + a_1 u_2 + a_3 {u_2}^4 {u_3}^3 + a_4 u_4 + a_5 {u_2}^4 {u_3}^4 {u_4}^2 + a_6 {u_3}^2 {u_4}^3 + \\ & a_7 u_2 {u_3}^3 {u_4}^3 + a_8 u_5 + a_9 {u_2}^2 {u_3}^2 u_4 u_5 + a_{10} {u_3}^2 {u_4}^2 u_5 + a_{11} {u_2}^2 {u_3}^4 {u_4}^3 u_5 + \\ & a_{13} u_3 {u_5}^2 + a_{12} {u_2}^2 {u_3}^2 {u_5}^2 + a_{14} {u_3}^2 {u_4}^2 {u_5}^2 + a_{16} {u_2}^2 {u_3}^4 {u_4}^2 {u_5}^2 + \\ & a_{15} {u_2}^4 {u_3}^5 {u_4}^2 {u_5}^2 + a_{17} {u_3}^2 u_4 {u_5}^3 + a_{18} {u_2}^2 {u_3}^4 {u_4}^2 {u_5}^3 = 0. \end{aligned}$$

$\widetilde{U}_{1,332}$ is consequently one of the $\widetilde{U}_{1,j}$ that give rise to $\widetilde{\mathbb{P}^4}$.

• As a second example, we check the singularity at the point $A_2(0,1,0,0,0)$, which is the origin in U_2. By construction, A_2 is a triple point of V with a triple curve infinitely near.

To analyse this singularity, we consider the blow-up $\pi_4 : \mathbb{P}_5 \to \mathbb{P}_4$ at this point and the strict transform W_5 of W_4 by π_4. Among the four open sets $\widetilde{U}_{2,1}$, $\widetilde{U}_{2,3}$, $\widetilde{U}_{2,4}$ and $\widetilde{U}_{2,5}$ that cover $\pi_4^{-1}(U_2)$, we concentrate on $\widetilde{U}_{2,4}$ and we report the corresponding local study.

After denoting affine coordinates on $\widetilde{U}_{2,4}$ as (y_1, y_3, y_4, y_5), the local representation of π_4 is

$$x_1 = y_1 y_4, \quad x_3 = y_3 y_4, \quad x_4 = y_4, \quad x_5 = y_5 y_4,$$

while the exceptional divisor E_4 is locally represented by $y_4 = 0$.

W_5 has a triple curve $\mathcal{C}_5$ that is given in $\widetilde{U}_{2,4}$ by: $y_1 = y_3 = y_4 = 0$. We therefore consider the blow-up $\pi_5 : \mathbb{P}_6 \to \mathbb{P}_5$ of $\mathbb{P}_5$ along $\mathcal{C}_5$.

After covering $\pi_5^{-1}(\widetilde{U}_{2,4})$ with $\widetilde{U}_{2,41}$, $\widetilde{U}_{2,43}$, $\widetilde{U}_{2,44}$ and denoting affine coordinates on $\widetilde{U}_{2,41}$ as (u_1, u_3, u_4, u_5), we can represent π_5 locally as:

$$y_1 = u_1, \quad y_3 = u_1 u_3, \quad y_4 = u_1 u_4, \quad y_5 = u_5.$$

The exceptional divisor on $\widetilde{U}_{2,41}$ is then $E_5 : u_1 = 0$ and the strict transform W_6 of W_5 is locally represented by:

$$\begin{aligned} X_{2,41} : a_1 + a_3 {u_3}^3 + u_4(&a_4 u_1 + a_5 {u_3}^2 + a_2 {u_1}^2 {u_3}^2 + a_6 {u_1}^3 {u_3}^2 {u_4}^2 + \\ &a_7 {u_1}^3 {u_3}^3 {u_4}^2 + a_9 u_5 + a_8 u_1 u_5 + a_{11} {u_1}^2 {u_3}^2 {u_4}^2 u_5 + a_{10} {u_1}^3 {u_3}^2 {u_4}^2 u_5 + \\ &a_{12} {u_5}^2 + a_{15} u_3 u_4 {u_5}^2 + a_{13} {u_1}^2 u_3 u_4 {u_5}^2 + a_{14} {u_1}^2 {u_4}^2 {u_5}^2 + \\ &a_{16} {u_1}^2 {u_3}^2 {u_4}^2 {u_5}^2 + a_{18} u_1 {u_4}^2 {u_5}^3 + a_{17} {u_1}^2 {u_4}^2 {u_5}^3) = 0. \end{aligned}$$

So, for a generic choice of the parameters, W_6 has no singularities in this open set. $\widetilde{U}_{2,41}$ is consequently among the $\widetilde{U}_{2,j}$ that give rise to $\widetilde{\mathbb{P}^4}$.

For the remaining imposed singularities, we need blow-ups similar to those just described, i.e. resembling those used at A_2 for $A_3(0,0,1,0,0)$ and $A_5(0,0,0,0,1)$, and like those used at A_1 for $A_4(0,0,0,1,0)$.

Finally, to complete the desingularization, we have to consider the other actual or infinitely near singularities that appear on V. They can all be solved, however, with two kinds of blow-up: along varieties that are locally double lines of V (actual or infinitely near), or along varieties that are locally simple planes on V (or on its strict transforms) and that contain double curves. Since these singularities have no influence, as we shall see, on the computation of the plurigenera of X, we can disregard their analysis here.

3. – The canonical class of X.

Let

$$\widetilde{\mathbb{P}^4} = \mathbb{P}_{r+1} \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} \mathbb{P}_3 \xrightarrow{\pi_2} \mathbb{P}_2 \xrightarrow{\pi_1} \mathbb{P}_1 = \mathbb{P}^4$$

be the sequence of blow-ups we use to solve the singularities of V.

In our case, each π_i is the blow-up of $\mathbb{P}_i$ along a (locally linear) subvariety Z_i of dimension j_i.

Assuming that $W_1 = V$, let W_{i+1} be the strict transform of $W_i \subset \mathbb{P}_i$ by π_i. W_{r+1} is therefore X. After denoting the multiplicity of W_i at the generic point of $Z_i \subset W_i$ as m_i, $1 \leq i \leq r$, we have (cf. [2], p. 602, for instance)

$$W_{i+1} = \pi_i^*(W_i) - m_i E_i \quad \text{and} \quad K_{\mathbb{P}_{i+1}} = \pi_i^*(K_{\mathbb{P}_i}) + (3 - j_i)E_i, \tag{1}$$

where $E_i = \pi_i^{-1}(Z_i)$ is the exceptional divisor of π_i and π_i^* is the homomorphism between the Cartier divisor groups $\pi_i^* : Div(\mathbb{P}_i) \to Div(\mathbb{P}_{i+1})$.

We remind that $K_{\mathbb{P}^4} = -5H$ and $V = W_1 \equiv 6H$, where H is a hyperplane in $\mathbb{P}^4$. If we put $n_i = j_i + m_i - 3$, then by iteration of (1) we can deduce

$$K_{\widetilde{\mathbb{P}^4}} + X \equiv \pi_r^*\{\cdots \pi_3^*\{\pi_2^*\{\pi_1^*(H) - n_1 E_1\} - n_2 E_2\} - n_3 E_3 \cdots\} - n_r E_r \ .$$

The blow-ups we need to bring the desingularization to the end are of the following types:

- a blow-up at a triple point; in this case $n_i = 0 + 3 - 3 = 0$;
- a blow-up along a triple curve; in this case $n_i = 1 + 3 - 3 = 1$;
- a blow-up along a double surface; in this case $n_i = 2 + 2 - 3 = 1$;
- a blow-up along a double curve; in this case $n_i = 1 + 2 - 3 = 0$;
- a blow-up along a simple surface; in this case $n_i = 2 + 1 - 3 = 0$.

So, in our case, $n_i = j_i + m_i - 3 \geq 0$ for any i.

Since $X = W_{r+1}$ is nonsingular, we can apply the adjunction formula and obtain:

$$K_X \equiv [\pi_r^*\{\cdots \pi_3^*\{\pi_2^*\{\pi_1^*(H) - n_1 E_1\} - n_2 E_2\} - n_3 E_3 \cdots\} - n_r E_r] \,|_X$$

or equivalently

$$K_X \equiv [\sigma^*(H) - \cdots - \pi_r^*(\pi_{r-1}^*(n_{r-2} E_{r-2})) - \pi_r^*(n_{r-1} E_{r-1}) - n_r E_r] \,|_X \ . \tag{2}$$

Bearing in mind that we present X as a pasting of the affine schemes $X_{i,j}$, $1 \leq i \leq 5, j \in J_i$, we can rewrite K_X as a collection $\{d_{i,j}, X_{i,j}\}$.

So, if U_i, V_i, $\widetilde{U}_{i,j}$ are as in section 1, and if $\sigma_{i,j} : X_{i,j} \to V_i$ denotes the restriction of σ to $X_{i,j}$ and $\sigma_{i,j}^* : \mathbb{C}[V_i] \to \mathbb{C}[X_{i,j}]$ the homomorphism between the coordinate rings of V_i and $X_{i,j}$, from (2) we have:

$$K_X = \left\{\sigma_{i,j}^*(\overline{\theta_i})/\overline{s_{i,j}} \ , \ X_{i,j}\right\}, \tag{3}$$

where: $\theta(x_1, \ldots, x_5)$ is an arbitrary linear form, θ_i is the canonical projection of θ in U_i and $\overline{\theta_i}$ is its image in $\mathbb{C}[V_i]$. In our case $s_{i,j}$ belongs to $\mathbb{C}[\widetilde{U}_{i,j}]$; so $\overline{s_{i,j}}$ denotes its image in $\mathbb{C}[X_{i,j}]$.

If we multiply (3) by the integer $m \geq 1$, we obtain:

$$mK_X = \left\{ \sigma^*_{i,j}(\overline{\vartheta_i})/\overline{s_{i,j}}^m \ , \ X_{i,j} \right\}, \tag{4}$$

where: $\vartheta(x_1, \ldots, x_5)$ is an arbitrary form of degree m.

REMARK 1. – Since the $s_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ appearing in (3) are due to the addendum

$$- \cdots - \pi^*_r(\pi^*_{r-1}(n_{r-2}E_{r-2})) - \pi^*_r(n_{r-1}E_{r-1}) - n_r E_r$$

of (2), we have to pay careful attention to the $n_i > 0$, i.e. to the blow-ups of the second or third type in the above-mentioned list. As seen when developing the desingularization, they are only due to the imposed singularities, whereas the actual or infinitely near unimposed singularities need blow-ups of the fourth or fifth type, so the corresponding n_i are equal to zero. That is why we usually say that the imposed singularities of V affect the birational invariants of X, while the other singularities of V do not. □

REMARK 2. – Since, for any π_i, we know the local equation of its exceptional divisor E_i and the value of the corresponding n_i, it is easy to write every $\pi^*_r(\pi^*_{r-1}(\ldots(\pi^*_{i+1}(n_i E_i))\ldots))$ and consequently every $s_{i,j}$. All the $s_{i,j}$ here have a very straightforward expression. For example, referring to the open sets $\widetilde{U}_{1,332}$ and $\widetilde{U}_{2,41}$ of the previous section, we have

$$s_{1,332} = u_2 \qquad s_{2,41} = u_1,$$

bearing in mind that E_3 is given by $u_2 = 0$ in $\widetilde{U}_{1,332}$, and E_5 is given by $u_1 = 0$ in $\widetilde{U}_{2,41}$.

More in general, if $(u_1, \ldots, \widehat{u_i}, \ldots, u_5)$ denote affine coordinates on $\widetilde{U}_{i,j}$, we can see that all the $s_{i,j}$, if they are not constant, are given by $s_{i,j} = u_k$, with $k \in \{1, \ldots, \widehat{i}, \ldots, 5\}$ depending on j. □

4. – How to compute the plurigenera of X.

We denote the form of degree 6 defining $V \subset \mathbb{P}^4$ by F, the polynomials defining V_i in U_i by F_i and those defining $X_{i,j}$ in $\widetilde{U}_{i,j}$ by $F_{i,j}$. As before, let ϑ_i be the canonical projection in U_i of a form $\vartheta(x_1, \ldots, x_5)$ and let $\sigma_{i,j}$ be the restriction of σ to $X_{i,j}$.

The following proposition is already contained in [6], page 141, but we repeat the proof here for the reader's convenience and for the sake of completeness.

PROPOSITION 1. – *Let* $s_{i,j}$ *be as in* (3). *Then, for any effective divisor* $L = \{\ell_{i,j}\,,\, X_{i,j}\}$ *linearly equivalent to* mK_X, *there exists a form* $\varepsilon(x_1, \ldots, x_5)$ *of degree* m *such that* $\ell_{i,j} = \sigma^*_{i,j}(\overline{\varepsilon_i})/\overline{s_{i,j}}^m$, *for any* i *and* j.

PROOF. – The $\varphi_{k,l;i,j} : X_{i,j} \to X_{k,l}$ are birational maps; so for any (i, j) and (k, l) (non-empty) open subsets $X_{i,j;k,l}$ and $X_{k,l;i,j}$ of $X_{i,j}$ and $X_{k,l}$, respectively, exist such that the $\eta_{k,l;i,j} = \varphi_{k,l;i,j} |_{X_{i,j;k,l}}: X_{i,j;k,l} \to X_{k,l;i,j}$ are isomorphisms.

X, indeed, is obtained by glueing the $X_{i,j}$ along $X_{i,j;k,l}$ via the isomorphisms $\eta_{k,l;i,j}$.

As usual, we denote as $\varphi^*_{i,j;k,l}$ and $\eta^*_{i,j;k,l}$ the isomorphisms $\varphi^*_{i,j;k,l} : \mathbb{C}(X_{i,j}) \to \mathbb{C}(X_{k,l})$ and $\eta^*_{i,j;k,l} : \mathbb{C}[X_{i,j;k,l}] \to \mathbb{C}[X_{k,l;i,j}]$.

Now, let $L = \{\ell_{i,j}, X_{i,j}\}$ be an effective divisor on X linearly equivalent to mK_X. Then $L - mK_X = \{\ell_{i,j}\, \overline{s_{i,j}}^m / \sigma^*_{i,j}(\overline{\vartheta_i}),\, X_{i,j}\}$ is the divisor of a rational function $\tilde{p}$ of $\mathbb{C}(X)$; in other words,

$$(\varphi_{i,j;k,l})^* \left(\ell_{i,j}\, \overline{s_{i,j}}^m / \sigma^*_{i,j}(\overline{\vartheta_i})\right) = \ell_{k,l}\, \overline{s_{k,l}}^m / \sigma^*_{k,l}(\overline{\vartheta_k}), \tag{5}$$

for any (i, j) and (k, l).

Since, for any (i, j) and (i, l), we have

$$\varphi_{i,j;i,l} = \sigma^{-1}_{i,j} \circ \sigma_{i,l},$$

then $\varphi^*_{i,j;i,l}(\sigma^*_{i,j}(\overline{\vartheta_i})) = \sigma^*_{i,l}(\overline{\vartheta_i})$.

We deduce from (5), in the case of $k = i$, that $\varphi^*_{i,j;i,l}(\ell_{i,j}\, \overline{s_{i,j}}^m) = \ell_{i,l}\, \overline{s_{i,l}}^m$. In particular, $(\eta_{i,j;i,l})^*(\ell_{i,j}\, \overline{s_{i,j}}^m) = \ell_{i,l}\, \overline{s_{i,l}}^m$. Therefore, by definition of glueing of schemes, for any arbitrarily fixed i, the collection $\{\ell_{i,j}\, \overline{s_{i,j}}^m\}, j \in J_i$, gives rise to an element of $\Gamma(X_i, \mathcal{O}_X)$, where $X_i = \bigcup_{j \in J_i} X_{i,j}$.

The domain of regularity of the (dominant) rational map $\sigma^{-1} : V \to X$ is $V \setminus I$, where I is a closed subset of V with $\mathrm{codim}(I) = 2$. Thus, for any arbitrarily fixed i, the collection $\{(\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m)\}$, $j \in J_i$, leads to an element of $\Gamma(V_i \setminus I_i, \mathcal{O}_{V_i})$, where $I_i = I \cap V_i$.

Since V is a normal hypersurface in $\mathbb{P}^4$, $\mathbb{C}[V_i]$ is an integrally closed domain, so the collection $\{(\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m)\}$ gives rise to an element of $\Gamma(V_i, \mathcal{O}_{V_i})$ (cf. [4], Theorem 2.15, for instance).

Let us denote as p the image of $\tilde{p}$ by $(\sigma^{-1})^* : \mathbb{C}(X) \to \mathbb{C}(V)$. We have:

$$p = (\sigma^{-1}_{i,j})^*(\tilde{p}) = (\sigma^{-1}_{i,j})^* \left(\ell_{i,j}\, \overline{s_{i,j}}^m / \sigma^*_{i,j}(\overline{\vartheta_i})\right) = (\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m) \Big/ \overline{\vartheta_i},$$

for any $1 \leq i \leq 5, j \in J_i$. The collection $\{(\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m), V_i\}$ therefore gives an effective divisor on V linearly equivalent to $\{\overline{\vartheta_i}, V_i\}$, defined by the form ϑ.

V is a normal hypersurface in $\mathbb{P}^4$, so its homogeneous coordinate ring $\mathbb{C}[V]$ is an integrally closed domain. The effective divisor $\{(\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m), V_i\}$ is thus defined by a form ε of degree m (the same degree of ϑ):

$$(\sigma^{-1}_{i,j})^*(\ell_{i,j}\, \overline{s_{i,j}}^m) = \overline{\varepsilon_i}.$$

It follows that $\ell_{i,j} \overline{s_{i,j}}^m = \sigma^*_{i,j}(\overline{\varepsilon_i})$ and this concludes the proof. □

REMARK 3. – The divisors $\varepsilon = 0$ on $\mathbb{P}^4$, where ε are forms of degree m such that $\sigma^*_{i,j}(\overline{\varepsilon_i})/\overline{s_{i,j}}^m \in \mathbb{C}[X_{i,j}]$, i.e. that define effective m-canonical divisors on X, are classically called m-canonical adjoints to V. Proposition 1 tells us that an isomorphism exists between $|mK_X|$ and the linear system of the m-canonical adjoints to V, restricted to V. □

REMARK 4. – Let $\widetilde{U}_{i,j}$ be an open set in which the $s_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ is not a constant. If, as before, we use $(u_1, \dots, \widehat{u_i}, \dots, u_5)$ to denote affine coordinates on this open set, for the example presented here we shall have $s_{i,j} = u_k$, for some k (cf. Remark 2). We can therefore order $F_{i,j}$ by the powers of u_k

$$F_{i,j} = q_i^{(0)} + u_k\, q_i^{(1)} + u_k^2\, q_i^{(2)} + \cdots$$

and consider the addendum $p_i^{(0)}$ of F_i that corresponds, in the strict transform $F_{i,j}$, to $q_i^{(0)}$.

For any arbitrarily fixed i, there are several of such open sets (depending on j), but the polynomial $p_i^{(0)}$ does not depend on the particular $\widetilde{U}_{i,j}$ we use to compute it. Specifically:

$$\begin{aligned}
p_1^{(0)} =& a_2\, x_2{}^2\, x_3{}^2 + a_4\, x_2{}^2\, x_4 + a_6\, x_3{}^2\, x_4{}^3 + a_8\, x_2{}^2\, x_5 + a_{10}\, x_3{}^2\, x_4{}^2\, x_5 + \\
& a_{13}\, x_2\, x_3\, x_5{}^2 + a_{14}\, x_4{}^2\, x_5{}^2 + a_{17}\, x_4\, x_5{}^3 \\
p_2^{(0)} =& a_1\, x_1{}^3 + a_3\, x_3{}^3 + a_5\, x_3{}^2\, x_4{}^2 + a_9\, x_1{}^2\, x_4\, x_5 + a_{12}\, x_1{}^2\, x_5{}^2 + \\
& a_{15}\, x_3\, x_4{}^2\, x_5{}^2 \\
p_3^{(0)} =& a_2\, x_1{}^2\, x_2{}^2 + a_3\, x_2{}^3 + a_7\, x_4{}^3 + a_{10}\, x_1\, x_4{}^2\, x_5 + a_{13}\, x_1{}^2\, x_2\, x_5{}^2 + \\
& a_{16}\, x_4{}^2\, x_5{}^2 \\
p_4^{(0)} =& a_4\, x_1{}^3\, x_2{}^2 + a_5\, x_2{}^2\, x_3{}^2 + a_6\, x_1\, x_3{}^2 + a_9\, x_1{}^2\, x_2{}^2\, x_5 + a_{11}\, x_3{}^2\, x_5 + \\
& a_{14}\, x_1{}^2\, x_5{}^2 + a_{15}\, x_2\, x_3\, x_5{}^2 + a_{18}\, x_1\, x_5{}^3 \\
p_5^{(0)} =& a_3\, x_2{}^3\, x_3{}^3 + a_{12}\, x_1{}^2\, x_2{}^2 + a_{13}\, x_1{}^2\, x_2\, x_3 + a_{15}\, x_2\, x_3\, x_4{}^2 + a_{16}\, x_3{}^2\, x_4{}^2 + \\
& a_{17}\, x_1{}^2\, x_4 + a_{18}\, x_1\, x_4{}^2
\end{aligned}$$

We can see that all the polynomials $p_i^{(0)}$ are different from F_i and that none of them is divisible by any of the x_k ($k \in \{1, \dots, \widehat{i}, \dots, 5\}$). In addition, they have degree 5 for $1 \le i \le 4$, whereas $p_5^{(0)}$ has degree 6.

Finally, we should mention that the total transform $\sigma^*_{i,j}(p_i^{(0)})$ of $p_i^{(0)}$ is divisible by u_k exactly 8 times in the case of $i = 1$ or $i = 4$, and by u_k exactly 6 times in the case of $i = 2, 3, 5$, according to the total transform $\sigma^*_{i,j}(F_i)$ of F_i. □

Computing the m-canonical adjoints to V is generally not so easy, but the following proposition and Remark 5 make it simpler when $m \le 6$.

PROPOSITION 2. – *Let ε be a form of degree m that defines an effective m-canonical divisor (i.e. $\sigma_{i,j}^*(\overline{\varepsilon_i})/\overline{s_{i,j}}^m \in \mathbb{C}[X_{i,j}]$, for any i and j).*

If, for i_0 and j_0, $\sigma_{i_0,j_0}^*(\varepsilon_{i_0})/s_{i_0,j_0}^m$ does not belong to $\mathbb{C}[\widetilde{U}_{i_0,j_0}]$, then there is an addendum of ε_{i_0} (essential and different from 0) that is divisible by the polynomial $p_{i_0}^{(0)}$ defined in Remark 4.

PROOF. – Let ε_i be the canonical projection of ε in U_i, as above. Based on the hypothesis, for any i and j, there are $h_{i,j}$ and $A_{i,j}$ in $\mathbb{C}[\widetilde{U}_{i,j}]$ such that

$$\sigma_{i,j}^*(\varepsilon_i) = h_{i,j}s_{i,j}^m + A_{i,j}F_{i,j}. \tag{6}$$

We also know that there is an open set $\widetilde{U}_{i_0,j_0}$ where $\sigma_{i_0,j_0}^*(\varepsilon_{i_0})/s_{i_0,j_0}^m$ does not belong to $\mathbb{C}[\widetilde{U}_{i_0,j_0}]$, i.e. s_{i_0,j_0} is not a constant and $\sigma_{i_0,j_0}^*(\varepsilon_{i_0})$ is divisible by s_{i_0,j_0} only β times, $\beta < m$. Since, from Remark 2, $s_{i_0,j_0} = u_k$, we can rewrite (6) in the form

$$\sigma_{i_0,j_0}^*(\varepsilon_{i_0}) = u_k^m h_{i_0,j_0} + A_{i_0,j_0}F_{i_0,j_0}. \tag{7}$$

If we now order $\sigma_{i_0,j_0}^*(\varepsilon_{i_0})$ and F_{i_0,j_0} by the powers of u_k:

$$\sigma_{i_0,j_0}^*(\varepsilon_{i_0}) = u_k^\beta \omega_\beta + u_k^{\beta+1}\omega_{\beta+1} + \cdots \tag{8}$$

$$F_{i_0,j_0} = q_{i_0}^{(0)} + u_k\, q_{i_0}^{(1)} + u_k^2\, q_{i_0}^{(2)} + \cdots \tag{9}$$

on comparing the two members of (7), we can deduce that ω_β have to be divisible by $q_{i_0}^{(0)}$, i.e. a polynomial $B_{i_0} \in \mathbb{C}[u_1, \ldots, \widehat{u_{i_0}}, \ldots, u_5]$ exists such that

$$\omega_\beta = B_{i_0} q_{i_0}^{(0)}. \tag{10}$$

Multiplying (10) by u_k^β we have

$$u_k^\beta \omega_\beta = u_k^\beta B_{i_0} q_{i_0}^{(0)}. \tag{11}$$

In order to deduce the expression of the addendum $\varepsilon_{i_0}^{(\beta)}$ (different from 0) of ε_{i_0} such that $\sigma_{i_0,j_0}^*(\varepsilon_{i_0}^{(\beta)}) = u_k^\beta \omega_\beta$, we can apply $(\sigma_{i_0,j_0}^{-1})^*$ to the members of (11). So, if we bear in mind the expression of σ_{i_0,j_0}^{-1}, and that $p_{i_0}^{(0)}$ is the addendum of F_{i_0} corresponding to $q_{i_0}^{(0)}$ in the strict transform F_{i_0,j_0} of F_{i_0}, then from (11) we deduce

$$\varepsilon_{i_0}^{(\beta)} = (\sigma_{i_0,j_0}^{-1})^*(u_k^\beta \omega_\beta) = (\sigma_{i_0,j_0}^{-1})^*(u_k^\beta B_{i_0} q_{i_0}^{(0)}) = \frac{C_{i_0} p_{i_0}^{(0)}}{M_{i_0}}, \tag{12}$$

where C_{i_0} and M_{i_0} are a polynomial and a monomial in $\mathbb{C}[x_1, \ldots, \widehat{x_{i_0}}, \ldots, x_5]$, respectively.

As we can see in Remark 4, the factors of M_{i_0} cannot divide $p_{i_0}^{(0)}$; so they have to divide C_{i_0} and then the addendum $\varepsilon_{i_0}^{(\beta)}$ of ε_{i_0} is divisible by $p_{i_0}^{(0)}$:

$$\varepsilon_{i_0}^{(\beta)} = p_{i_0}^{(0)} D_{i_0}, \tag{13}$$

$D_{i_0} \in \mathbb{C}[x_1, \ldots, \widehat{x_{i_0}}, \ldots, x_5]$. $\square$

REMARK 5. – Let ε be a form of degree m defining an effective m-canonical divisor and let $\sigma^*_{i_0,j_0}(\varepsilon_{i_0})$ be divisible by s_{i_0,j_0} only β times, $\beta < m$. From Proposition 2, the addendum $\varepsilon^{(\beta)}_{i_0}$ of ε_{i_0} is divisible by the polynomial $p^{(0)}_{i_0}$.

Recalling that $\sigma^*_{i_0,j_0}(p^{(0)}_{i_0})$ is divisible by u_k 6 times at least (see final part of Remark 4), we deduce that $6 \leq \beta$. So, if ε defines an effective m-canonical divisor, but $\sigma^*_{i_0,j_0}(\varepsilon_{i_0})/s^m_{i_0,j_0} \notin \mathbb{C}[\widetilde{U}_{i_0,j_0}]$ for some i_0 and j_0, then $6 < m$. □

Whenever $m \leq 6$, the computation of the forms ε defining effective m-canonical divisors is then simplified and brought back to the computation of forms ε such that $\sigma^*_{i,j}(\varepsilon_i)/s^m_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ for any i and j (instead of $\sigma^*_{i,j}(\overline{\varepsilon_i})/\overline{s_{i,j}}^m \in \mathbb{C}[X_{i,j}]$). The following Proposition tells us how to do that.

PROPOSITION 3. – *Let ε be a form of degree m defining an effective m-canonical divisor. Then $\sigma^*_{i,j}(\varepsilon_i)/s^m_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$, for any i and j, if and only if the monomials $x_1^a x_2^b x_3^c x_4^d x_5^e$ of ε satisfy the conditions:*

$$a \leq 2b + d + e, \quad b \leq a + c, \quad c \leq b + d, \quad d \leq a + 2c + e, \quad e \leq a + d. \tag{14}$$

PROOF. – From Remark 2, we know that $s_{i,j} = u_k$, where u_k is one of the affine coordinates on $\widetilde{U}_{i,j}$. So $\sigma^*_{i,j}(\varepsilon_i)/s^m_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ if and only if all the monomials of $\sigma^*_{i,j}(\varepsilon_i)$ are divisible by u_k^m.

Let $\mathcal{M} = x_1^a x_2^b x_3^c x_4^d x_5^e$ be a monomial of ε and $\mathcal{M}_i$ its canonical projection in U_i.

First of all, we analyze $\sigma^*_{1,332}(\mathcal{M}_1) = \sigma^*_{1,332}(x_2^b x_3^c x_4^d x_5^e)$. By composing the local expressions of the blow-ups used to solve the singularities of V (compare section 2 for more details), we can write $\sigma_{1,332}$:

$$x_2 = u_2^3 u_3^2, \quad x_3 = u_2 u_3, \quad x_4 = u_2^2 u_3^2 u_4, \quad x_5 = u_2^2 u_3^2 u_5,$$

calculate $\sigma^*_{1,332}(\mathcal{M}_1)$ and therefore see that u_2 appears in it to the power $3b + c + 2d + 2e$. Bearing in mind that $s_{1,332} = u_2$ and that $m = a + b + c + d + e$, we deduce that $\sigma^*_{1,332}(\mathcal{M}_1)$ is divisible by $s^m_{1,332}$ if and only if

$$a \leq 2b + d + e.$$

This situation holds, however, as we can easily check, not only on $\widetilde{U}_{1,332}$, but also on any $\widetilde{U}_{1,j}$, $j \in J_1$, where $s_{1,j}$ is not a constant.

Similarly, analysing $\sigma^*_{i,j}(\mathcal{M}_i)$ when $i = 2, 3, 4, 5$, we deduce that:
$b \leq a + c$ referring to $\sigma^*_{2,j}(\mathcal{M}_2)$, $j \in J_2$;
$c \leq b + d$ referring to $\sigma^*_{3,j}(\mathcal{M}_3)$, $j \in J_3$;
$d \leq a + 2c + e$ referring to $\sigma^*_{4,j}(\mathcal{M}_4)$, $j \in J_4$;
$e \leq a + d$ referring to $\sigma^*_{5,j}(\mathcal{M}_5)$, $j \in J_5$. □

The conditions of Proposition 3 make it easy for us to calculate the vector space of the forms defining effective m-canonical divisors on X for any $m \leq 6$, according to Remark 5.

Computation of p_g.

The above conditions oblige the polynomials ε of degree 1 defining effective canonical divisors on X to be identically zero, so $p_g = 0$. If we want to express this result in geometrical terms, we can recall that K_X is given by (3) and that, in order for it to be effective, the hyperplane $\theta = 0$ has to pass through the five vertices A_i of the fundamental pentahedron. Since there are no such hyperplanes, $p_g = 0$ here again.

Computation of P_2 and P_3.

Given the conditions (14), we see that the vector space W_2' of the forms (of degree 2) defining effective 2-canonical divisors on X is spanned by

$$x_1x_2, \quad x_1x_4, \quad x_1x_5, \quad x_2x_3, \quad x_3x_4, \quad x_4x_5,$$

and the vector space W_3' of the forms defining effective 3-canonical divisors on X is spanned by

$$x_1^2x_2, \quad x_1x_2x_3, \quad x_1x_2x_4, \quad x_1x_2x_5, \quad x_1x_3x_4,$$
$$x_1x_4x_5, \quad x_2x_3x_4, \quad x_3x_4^2, \quad x_3x_4x_5,$$

so $P_2 = 6$ and $P_3 = 9$.

If we want to emphasize the geometrical point of view of the conditions (14), as before, we must first recall that 2-canonical adjoints belong to the linear system of hyperquadrics:

$$b_1x_1x_2 + b_2x_1x_3 + b_3x_1x_4 + b_4x_1x_5 + b_5x_2x_3 + b_6x_2x_4 + b_7x_2x_5 + b_8x_3x_4 +$$
$$b_9x_3x_5 + b_{10}x_4x_5 + b_{11}x_1^2 + b_{12}x_2^2 + b_{13}x_3^2 + b_{14}x_4^2 + b_{15}x_5^2 = 0.$$

Then, we see that the tangent cone of V at A_1 is $x_2^2(a_1x_2 + a_4x_4 + a_8x_5) = 0$ and that the first condition in (14), due to the singularity at A_1, obliges the hyperquadrics to pass through A_1 ($b_{11} = 0$) and to have their tangent plane at this point of the type $b_1x_2 + b_3x_4 + b_4x_5 = 0$ ($b_2 = 0$).

Finally, referring to a singularity of the second type, we see that the tangent cone of V at A_2 is $a_1x_1^3 + a_3x_3^3 = 0$ and that the second condition in (14) obliges the hyperquadrics to pass through A_2 ($b_{12} = 0$) and to have their tangent plane at this point of the type $b_1x_1 + b_5x_3 = 0$ ($b_6 = b_7 = 0$).

It is worth noting that the values of P_2 and P_3 enable us to deduce, by means of the Rieman-Roch formula, that a canonical divisor on X, or on any nonsingular variety birationally equivalent to X, is not numerically effective. To do so, we can refer to [8], section 17, so we omit the proof here. X therefore cannot have a nonsingular minimal model.

Lastly, even if the value of other plurigenera of X is not essential for the purposes of the present paper, it is worth adding that, in accordance with Remark 5, the conditions of Proposition 3 help us to compute P_4, P_5 and P_6 just as easily as before, which are 22, 36 and 66, respectively. To tell the truth, in the case of the present example, we could use Proposition 2 to prove that the computation of any P_m (even if $m > 6$) can be brought down to computing forms ε such that $\sigma^*_{i,j}(\varepsilon_i)/s^m_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ for any i and j. We do not provide the proof here, however, because these considerations go beyond the scope of this paper.

5. – The bicanonical map $\Phi_{|2K_X|}$ is stably birational.

In this section we aim to prove that X is of general type and that the rational maps associated with the linear systems $|mK_X|$ are birational for all $m \geq 2$.

Bearing in mind the isomorphism between $|mK_X|$ and the linear system on V of the m-canonical adjoints, let us consider the (incomplete) linear system L_m defined on V by the hypersurfaces $\varepsilon = 0$ that give effective m-canonical divisors, and let Φ_{L_m} be the rational map defined by L_m.

Let us consider the commutative triangle

$$\begin{array}{ccc} X & \overset{\Phi_{|mK_X|}}{\dashrightarrow} & \mathbb{P}^{P_m-1} \\ \sigma\big\downarrow & \nearrow_{\Phi_{L_m}} & \\ V & & \end{array}$$

where we represent the two rational maps $\Phi_{|mK_X|}$ and Φ_{L_m} by dotted arrows to emphasize that they are not regular.

Since $\sigma : X \to V$ is an isomorphism on an open set of X, and since the above triangle is commutative, we deduce that

$$\Phi_{|mK_X|} \text{ is birational} \iff \Phi_{L_m} \text{ is birational.}$$

On the other hand, the study previously performed on the forms ε to compute the plurigenera can also be used to analyse Φ_{L_m}.

Let us prove the birationality of Φ_{L_2} or, equivalently, of $\Phi_{|2K_X|}$.

Among the polynomials defining effective 2-canonical divisors on X, we choose

$$x_1x_2, \quad x_1x_5, \quad x_2x_3, \quad x_3x_4, \quad x_4x_5\,.$$

The five polynomials define the rational map $\tau : \mathbb{P}^4 \to \mathbb{P}^4$ given by:

$$Y_1 = x_1x_2, \quad Y_2 = x_1x_5, \quad Y_3 = x_2x_3, \quad Y_4 = x_3x_4, \quad Y_5 = x_4x_5.$$

It is straightforward to prove that τ is a birational map. V is not contained in the indeterminacy locus of τ, so the restriction of τ to V is also a birational map. Φ_{L_2} is therefore birational.

It is equally straightforward to prove that $\Phi_{|3K_X|}$ is birational. Since $P_2 > 0$, we deduce that Φ_{L_m}, and therefore $\Phi_{|mK_X|}$, is also a birational map for any $m \geq 4$.

6. – The irregularities of X.

To verify that the irregularities of X vanish, we can apply the Castelnuovo-Enriques criterion generalized in [8], section 4.

We consider a generic hyperplane $H \subset \mathbb{P}^4$, the surface $\mathcal{H} = H \cap V$ on V and the strict transform $S \subset X$ of $\mathcal{H}$ by σ. Since the linear system $|\mathcal{H}|$ on V is base-point free, S is nonsingular and the restriction of σ to S yields a desingularization of $\mathcal{H}$.

$\mathcal{H}$ has singular points at the intersections of H with the double lines of V, i.e. $\mathcal{H}$ has five double points at the intersections of H with $\mathcal{L}_i$, $1 \leq i \leq 5$.

To analyse these singularities, we can look at the restriction of σ to S. If we examine the singularities of V, we find that all the double points of $\mathcal{H}$ only have a finite number of infinitely near double points. So the irregularity $q(S) = \dim H^1(S, \mathcal{O}_S)$ and the geometric genus of S are the same as those of a smooth surface of degree 6 in $\mathbb{P}^3$ (cf. [2], p. 636 or [8], sections 3 and 4). It follows that $q(S) = 0$ and $p_g(S) = 10$. Thus, from [8], Remark 8, we deduce that

$$q_1 = \dim H^1(X, \mathcal{O}_X) = q(S) = 0.$$

To compute the second irregularity of X, we refer to [8], formula (36):

$$q_2 = p_g(X) + p_g(S) - \dim W_2, \tag{15}$$

where W_2 is the vector space of the forms ϕ of degree 2, such that $\sigma_{i,j}^*(\phi_i)/s_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ for any i and j.

To compute $\dim W_2$, we first consider a monomial $\mathcal{M}' = x_1^{a'} x_2^{b'} x_3^{c'} x_4^{d'} x_5^{e'}$ of ϕ ($a' + b' + c' + d' + e' = 2$); then, bearing the previously-mentioned local expressions of the $\sigma_{i,j}$ in mind and following the method used to prove Proposition 3, we can see that $\sigma_{i,j}^*(\phi_i)/s_{i,j} \in \mathbb{C}[\widetilde{U}_{i,j}]$ for any i and j, if and only if

$$a' \leq 2b' + d' + e' + 1,\ \ b' \leq a' + c' + 1,\ \ c' \leq b' + d' + 1,\ \ d' \leq a' + 2c' + e' + 1$$
$$e' \leq a' + d' + 1$$

for any $\mathcal{M}'$ of ϕ.

W_2 is then spanned by

$$x_1x_2, \quad x_1x_3, \quad x_1x_4, \quad x_1x_5, \quad x_2x_3$$
$$x_2x_4, \quad x_2x_5, \quad x_3x_4, \quad x_3x_5, \quad x_4x_5$$

so $\dim W_2 = 10$. Since $p_g(X) = 0$ and $p_g(S) = 10$, from (15) we obtain $q_2 = \dim H^2(X, \mathcal{O}_X) = 0$.

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