Wei Yang, Jie Lou


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# The Dynamics of an Interactional Model of Rabies Transmitted between Human and Dogs

WEI YANG - JIE LOU

**Abstract.** – *Assuming that the population of dogs is constant and the population of human satisfies the Logistical model, an interactional model of rabies transmitted between human and dogs is formulated. Two thresholds $R_0$ and $R_1$ which determine the outcome of the disease are identified. Utilizing the method of Lyapunov function and the property of the cooperative systems, we get the global asymptotic stability for both the disease-free equilibrium and the endemic equilibrium. A critical vaccination rate is obtained, which determines whether the dog rabies dies out or becomes endemic. Some suggestions are provided to the prevention and control of rabies according to the results of analysis and simulations.*

## 1. – Introduction.

Rabies (Hydrophobia) is a viral disease that affects the central nervous system of all warm-blooded animals. It can be transmitted through a bite, scratch, lick of the infectious animals, or even the seemingly innocuous act of petting the family dogs. The virus stays in the reservoir's body fluids, including saliva. Nowadays in China, the rabies-caused mortality is the highest among the infectious diseases, nearly 100% once getting infectious [1]. According to the annual report data from the Ministry of Health of China, the human rabies situation in China is very severe recent years. We examined the achieved data of human rabies cases in China from 1950 to 2007 [2] and plotted the figure to show the situation more clearly (see Figure 1).

The reservoir hosts of rabies in nature could be dogs, cats, rats, raccoons, bats, and so on. In China, about 80% $\sim$ 90% of rabies transmitted to human is from the infectious dogs. The latent period of rabies can be less than a week or more than ten years, and the average is 66.9 days according to the reports in China. The infected (or exposed) but not infectious dogs, in their latent period, can also transmit the disease, which makes the prevention and the control of rabies hard to handle. The symptoms of rabies, once getting affected, can be divided into two types. One is the furious type, about 80% of the cases, which appear to be furious, excited and sensible to water; the other is the dumb or the paralytic type, about 20% of the cases, which don't have the period of showing


Fig. 1. – Annual rabies cases reported in China from 1950 to 2007.

excited or sensible to water, however, appear to be dumb or paralytic. Generally, the cases of the bites of the infectious vampire bats belong to the latter type. The vaccines against rabies for animals have two types. One is valid for one year and the other lasts for three years [3]. But when it comes to human, there is no valid vaccine that can be vaccinated beforehand. When you were likely to get infected, such as bit or scratched by a dog, you have to get vaccinated immediately, but this vaccine cannot last any longer immune period. It's lucky to know that since the virus does't change the genes, rabies wouldn't be transmitted vertically, namely, the infectious mothers wouldn't pass the disease to their children.

Mathematical models presenting the transmission dynamics of rabies have been considered a lot. Anderson et al. [4] formulate a model of rabies transmitted between foxes, quantitatively study the population dynamics, and conclude that it is possible to control the disease through vaccination and culling. Rhodes et al. [5] study the transmission of rabies in Zimbabwe using a compartment model, and obtain the reproduction number. Kallen et al. [6] study the spatial spread model. Allen et al. [7] discuss the discrete-time deterministic and stochastic models for the spread of rabies. Wang and Lou [8] formulate two *SI* models to describe the interaction of rabies between human and dogs. But they don't consider latent state in their models, which actually plays an important role in the transmission of rabies, because the infected dogs in latent period can cause infection.

In this paper, considering the features of the transmission of rabies, an interactional model of rabies transmitted between human and dogs is formulated. In the model, we assume that the population of the dogs is constant, and that the population of human satisfies the Logistical model. Supposed furthermore that the susceptible dogs and the infected (or exposed) human would be vaccinated continuously, an interactional SEIV model is formulated and studied. Discussing the corresponding differential systems, we identify the thresholds $R_0$ and $R_1$, which determine the existence of the disease-free equilibrium and the endemic equilibrium. Then utilizing the method of Lyapunov function and the property of cooperative systems, we obtain the global asymptotic stability of the equilibria.

The present paper is arranged as follows: in the next section, we establish the interactional model; in section 3, we do the model analysis and obtain the equilibria and their stabilities; in section 4, we get a critical vaccination rate $p^*$ and do the simulations; in the last section, we give some discussion.

## 2. – The Model.

The population of human is partitioned susceptible, exposed (in the latent period), infectious, vaccinated, with sizes denoted by $S_p(t)$, $E_p(t)$, $I_p(t)$, and $V_p(t)$ respectively. Similar symbols can be got for dog population. Then according to the features of rabies transmitted between human and dogs, an interactional SEIV model is formulated. The transfer diagram is depicted in Figure 2.

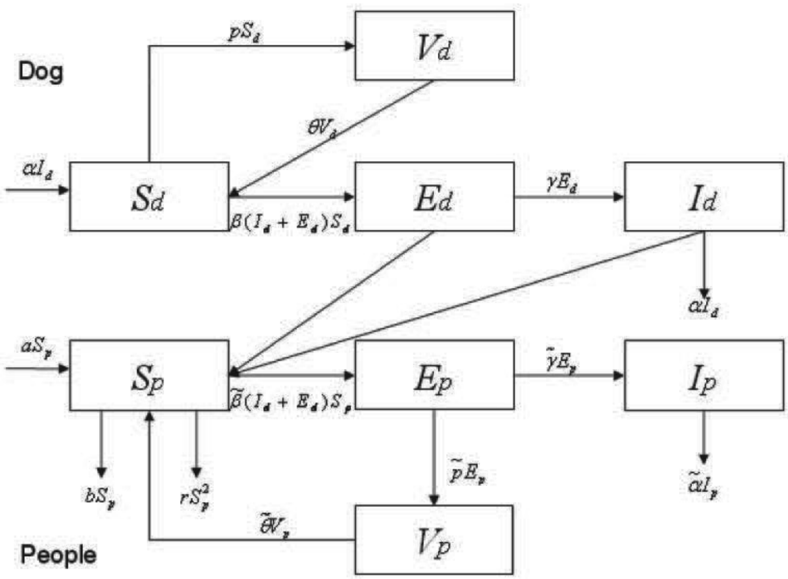


Fig. 2. -- Rabies transmitted between human and dogs.

The transfer diagram leads to the following differential equations:

$$(2.1)\qquad \begin{cases} \dot{S}_d &= aI_d - \beta(I_d + E_d)S_d - pS_d + \theta V_d, \\ \dot{E}_d &= \beta(I_d + E_d)S_d - \gamma E_d, \\ \dot{I}_d &= \gamma E_d - aI_d, \\ \dot{V}_d &= pS_d - \theta V_d, \\ \dot{N}_d &= 0 \end{cases}$$

and

$$(2.2)\qquad \begin{cases} \dot{S}_p &= \eta S_p(1 - S_p/K) - \tilde{\beta}(I_d + E_d)S_p + \tilde{\theta}V_p, \\ \dot{E}_p &= \tilde{\beta}(I_d + E_d)S_p - \tilde{p}E_p - \tilde{\gamma}E_p, \\ \dot{I}_p &= \tilde{\gamma}E_p - \tilde{a}I_p, \\ \dot{V}_p &= \tilde{p}E_p - \tilde{\theta}V_p, \\ \dot{N}_p &= \eta S_p(1 - S_p/K) - \tilde{a}I_p\,. \end{cases}$$

The total populations of human and dog are denoted by

$$N_p(t) = S_p(t) + E_p(t) + I_p(t) + V_p(t)$$

and

$$N_d(t) = S_d(t) + E_d(t) + I_d(t) + V_d(t)$$

respectively.

In this model we assume that the total population of dogs keeps constant in a local area (such as some city), namely, we assume people will adopt another dog when the old one dies for some reason.

In the equations of dog population, $p$ denotes the vaccinated rate of the susceptible dogs, $\theta$ denotes the removal rate of dogs from the vaccinated class to the susceptible class, $\beta$ denotes the bilinear incidence rate of dogs, $a$ denotes the rabies-caused mortality rate of dogs and $\gamma$ denotes the rate at which the exposed dogs become infectious.

In the equations of human population, $\tilde{p}$ denotes the vaccinated rate of the exposed human, $\tilde{\theta}$ denotes the removal rate of human from the vaccinated class to the susceptible class, $\tilde{\beta}$ denotes the bilinear incidence rate of rabies transmitted from dogs to human, $\tilde{a}$ denotes the rabies-caused mortality rate of human, $\tilde{\gamma}$ denotes the rate at which the exposed human become infectious, $\eta$ denotes the intrinsic growth rate of human and $K$ denotes the environmental capacity.

Thanks to the improvement of the medical services, human getting affected (such as bit or scratched by a dog) will be isolated immediately. So we assume that they couldn't transmit the disease to others any more.

## 3. – Model analysis.

From the model we can see that system (2.1) can be discussed separately.

3.1 – *Analysis of system* (2.1).

Denote $N_d = N$ and $S_d = N - E_d - I_d - V_d$. System (2.1) is reduced as the following:

$$\tag{3.1} \begin{cases} \dot{E}_d &= \beta(I_d + E_d)(N - E_d - I_d - V_d) - \gamma E_d, \\ \dot{I}_d &= \gamma E_d - aI_d, \\ \dot{V}_d &= p(N - E_d - I_d - V_d) - \theta V_d. \end{cases}$$

Its feasible set is $\Omega_1 = \{(E_d, I_d, V_d) | E_d \geq 0, I_d \geq 0, V_d \geq 0, E_d + I_d + V_d \leq N\}$.

About the existence and stability of equilibria for system (3.1) we have the following result.

THEOREM 3.1. – *Define*

$$\tag{3.2} R_0 = \frac{\theta N}{p+\theta} \cdot \frac{\beta(a+\gamma)}{a\gamma} \ ,$$

*then*

1. *one disease-free equilibrium* $E_1\left(0, 0, \dfrac{pN}{p+\theta}\right)$ *always exists. It is globally asymptotically stable when* $R_0 < 1$ *and unstable when* $R_0 > 1$.

2. *when* $R_0 > 1$, *there exists a unique endemic equilibrium*

$$E_2(E_d^*, I_d^*, V_d^*) = \left(\frac{aN}{a+\gamma}\left(1 - \frac{1}{R_0}\right), \frac{\gamma N}{a+\gamma}\left(1 - \frac{1}{R_0}\right), \frac{pN}{(p+\theta)R_0}\right).$$

PROOF. – The existences of equilibria are easy to check. In the following we prove the global stability of $E_1$.

Let $x = E_d,\ y = I_d,\ z = V_d - \dfrac{pN}{p+\theta}$. Then system (3.1) is equivalent to the following system:

$$\tag{3.3} \begin{cases} \dot{x} &= \left(\dfrac{\beta\theta N}{p+\theta} - \gamma\right)x + \dfrac{\beta\theta N}{p+\theta}\, y - \beta(x+y)(x+y+z), \\ \dot{y} &= \gamma x - ay, \\ \dot{z} &= -px - py - (p+\theta)z, \end{cases}$$

and the corresponding positively invariant set changes to

$$\Omega_2 = \left\{(x, y, z) | x \geq 0, y \geq 0, x + y + z \leq N, -\frac{pN}{p+\theta} \leq z \leq \frac{\theta N}{p+\theta}\right\}.$$

If $R_0 < 1$, then $\gamma - \dfrac{\beta\theta N}{p+\theta} > \dfrac{\gamma^2}{a+\gamma} > 0$. Let

$$L = \gamma x + \left(\gamma - \frac{\beta\theta N}{p+\theta}\right) y + \frac{\gamma\beta}{2p} z^2 \geq 0, \quad \text{in } \Omega_2.$$

The derivative of $L$ with respect to $t$ along system (3.3) is

$$\begin{aligned}
\frac{dL}{dt} &= \frac{dL(x(t), y(t), z(t))}{dt} \\
&= \gamma\dot{x} + \left(\gamma - \frac{\beta\theta N}{p+\theta}\right)\dot{y} + \frac{\gamma\beta}{p} z\dot{z} \\
&= \gamma\frac{\beta\theta N}{p+\theta} y - \beta\gamma(x+y)(x+y+z) - a\left(\gamma - \frac{\beta\theta N}{p+\theta}\right) y \\
&\quad - \frac{\gamma\beta}{p} z(px + py + (p+\theta)z) \\
&= a\gamma(R_0 - 1)y - \beta\gamma(x+y+z)^2 - \frac{\gamma\beta\theta}{p} z^2 \\
&\leq 0\ .
\end{aligned}$$

Since $\dot{L} = 0$ if and only if $x = y = z = 0$, therefore, the largest invariant set in $E = \{(x, y, z) \in \Omega_1 : \dot{L} = 0\}$ is $M = \{(0, 0, 0)\}$. So if $R_0 < 1$, then $E_1$ is globally asymptotically stable.

If $R_0 > 1$, the Jacobian matrix of system (3.3) at point $(0, 0, 0)$ is

$$J_1 = \begin{pmatrix} \dfrac{\beta\theta N}{p+\theta} - \gamma & \dfrac{\beta\theta N}{p+\theta} & 0 \\ \gamma & -a & 0 \\ -p & -p & -(p+\theta) \end{pmatrix}.$$

It's obvious that the matrix $J_1$ exists an eigenvalue $\lambda_1 = -(p+\theta) < 0$. Let $\lambda_2$, $\lambda_3$ be the other two eigenvalues.

$$\lambda_2 \cdot \lambda_3 = \left(\frac{\beta\theta N}{p+\theta} - \gamma\right)(-a) - \frac{\beta\theta N}{p+\theta}\gamma = a\gamma(1 - R_0) < 0.$$

Namely, the matrix $J_1$ exists a positive eigenvalue, then $E_1$ is unstable. □

Since the rabies-caused mortality is much larger than the recovery rate of the vaccinated dogs becoming susceptible again, we suppose that $a > \theta$ in the following discussions.

First we state two useful lemmas as the following.

LEMMA 3.2 [9, 10]. – *Let $\Omega \subset R_+^n$ be bounded and consider the cooperative system $\dot{x} = F(x)$, $x \in \Omega$. Suppose that every forward (or positive) semi-orbit has* *compact closure in $\Omega$, and that there is a unique positive equilibrium $P$ in $\Omega$, which is locally asymptotically stable, then $P$ is globally asymptotically stable in $\Omega$.*

LEMMA 3.3 [11]. – *Assume that*

A1) *$\Phi_t$ has a global attractor;*

A2) *there exists an $M = \{M_1, \cdots, M_k\}$ of pair-wise disjoint, compact, and isolated invariant sets on $\partial X_0$, such that*

a) $\bigcup_{x \in M_\partial} \omega(x) \subset \bigcup_{i=1}^{k} M_i,\ M_\partial = \{x \in \partial X_0 | \Phi_t x \in \partial X_0\}$;

b) *no subsets of $M$ forms a cycle on $\partial X_0$;*

c) *each $M_i$ is isolated in $X$;*

d) *$W^s(M_i) \cap X_0 = \phi$ for each $1 \le i \le k$, $W^s(M_i)$ is the stable manifold of $M_i$.*

*Then $\Phi_t$ is uniformly persistent with respect to $X_0$.*

About the stability of $E_2$, we have the following theorem.

THEOREM 3.4. – *The endemic equilibrium $E_2$ is locally asymptotically stable in the interior of $\Omega_1$, if $R_0 > 1$; and, furthermore suppose that $p > \gamma$, it is globally asymptotically stable.*

PROOF. – First, if $R_0 > 1$, we prove the local asymptotic stability of $E_2$.

Let $x = E_d - E_d^*,\ y = I_d - I_d^*,\ z = V_d - V_d^*$, the linearized system of system (3.1) at point $E_2$ is

$$\begin{cases} \dot{x} &= (\beta S_d^* - \beta(E_d^* + I_d^*) - \gamma)x + (\beta S_d^* - \beta(E_d^* + I_d^*))y - \beta(E_d^* + I_d^*)z, \\ \dot{y} &= \gamma x - ay, \\ \dot{z} &= -px - py - (p + \theta)z, \end{cases}$$

and the corresponding Jacobian matrix is

$$J_2 = \begin{pmatrix} \beta S_d^* - \beta(E_d^* + I_d^*) - \gamma & \beta S_d^* - \beta(E_d^* + I_d^*) & -\beta(E_d^* + I_d^*) \\ \gamma & -a & 0 \\ -p & -p & -(p + \theta) \end{pmatrix}.$$

Let $det(\lambda I - J_2) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$. Here

$$\begin{aligned} c_0 &= (a + \gamma)\theta\beta(E^* + I^*), \\ c_1 &= \beta(E^* + I^*)(a + \theta) + \left(\gamma + \frac{a^2}{a + \gamma}\right)(p + \theta), \\ c_2 &= \beta(E^* + I^*) + p + \theta + \frac{\gamma^2}{a + \gamma}. \end{aligned}$$

If $R_0 > 1$, then $c_2, c_1, c_0 > 0$ and $c_2c_1 > c_0$. Utilizing Hurwitz's theorem, we get the local asymptotic stability of $E_2$.

In the following we prove the global asymptotic stability of $E_2$ in interior $\Omega_1$.

Let $L_d = S_d + E_d$, system (3.1) is equivalent to the following system

$$\begin{cases} \dot{L_d} &= (a-\theta)I_d - (p+\theta)L_d + \theta N + (p-\gamma)E_d, \\ \dot{E_d} &= \beta(E_d + I_d)(L_d - E_d) - \gamma E, \\ \dot{I_d} &= \gamma E_d - aI_d, \end{cases} \tag{3.4}$$

The invariant set of system (3.4) changes to $\Omega_3 = \{(L_d, E_d, I_d) | 0 \le L_d, E_d, I_d \le N, L_d + I_d \le N, L_d \ge E_d\}$ and the equilibria of system (3.4) are $\tilde{E_1}\left(\dfrac{\theta N}{p+\theta}, 0, 0\right)$ and $\tilde{E_2}(L_d^*, E_d^*, I_d^*) = (S_d^* + E_d^*, E_d^*, I_d^*)$.

The Jacobian matrix of system (3.4) is

$$\tilde{J_2} = \begin{pmatrix} -(p+\theta) & p-\gamma & a-\theta \\ \beta(I_d + E_d) & \beta(L_d - I_d - 2E_d) - \gamma & \beta(L_d - E_d) \\ 0 & \gamma & -a \end{pmatrix}. \tag{3.5}$$

Then the Jacobian matrix of the linearized system of system (3.4) at the point $\tilde{E_1}$ is

$$J_3 = \begin{pmatrix} -(p+\theta) & p-\gamma & a-\theta \\ 0 & \dfrac{\beta\theta N}{p+\theta} - \gamma & \dfrac{\beta\theta N}{p+\theta} \\ 0 & \gamma & -a \end{pmatrix}.$$

Obviously, $J_3$ exists an eigenvalue $\lambda_1 = -(p+\theta)$, and $\lambda_2 \cdot \lambda_3 = a\gamma(1 - R_0)$. So when $R_0 > 1$, $J_3$ has a positive eigenvalue, then $\tilde{E_1}$ is unstable.

Now we prove that every forward semi-orbit has compact closure in $\Omega_3$. Namely, if $R_0 > 1$, system (3.4) is uniformly persistent. Following the notation in Lemma 3.3, we choose $X = \Omega_3$, $X_0 = \{(L_d, E_d, I_d) \in X, E_d > 0\}$, $\partial X_0 = X \setminus X_0$. We have proved that $\tilde{E_1}$ is unstable. And $J_3$ has two negative eigenvalues and one positive eigenvalue. It's obvious that $\tilde{E_1}$is stable on $\partial X_0$, then $\tilde{E_1}$ is isolated in $X_0$. So $M_\partial = \partial X_0$, $M = \{\tilde{E_1}\}$. Moreover, all the solution is ultimately bounded in $X_0$, then there admits a global attractor. All the hypotheses in Lemma 3.3 hold, then system (3.4) is uniformly persistent with respect to $X_0$. We have the conclusion that every forward semi-orbit has compact closure in $\Omega_3$.

From matrix (3.5), we can see that all the off-diagonal elements of the Jacobian matrix of system (3.4) are non-negative, namely, system (3.4) is a co-operative system. Noticing that $\tilde{E_2}$ is the unique equilibrium in the interior of $\Omega_3$ and utilizing Lemma 3.2, we get the global asymptotic stability of $\tilde{E_2}$. Since system (3.1) is equivalent to system (3.4), the global asymptotic stability of the endemic equilibrium $E_2$ is obtained. □

3.2 – *Analysis of system* (2.2).

First we state a definition and a lemma, which will be used to analyze system (2.2).

DEFINITION 3.1 [12]. – *Consider the non-autonomous system*

$$\frac{dx}{dt} = f(t,x), \quad f : R \times \Omega(\subseteq R^n) \to R^n \tag{3.6}$$

*and the autonomous system*

$$\frac{dy}{dt} = g(y), \quad y : \Omega(\subseteq R^n) \to R^n. \tag{3.7}$$

*Suppose that both of the solutions of the systems exist uniquely for all* $t \geq 0$. *If* $f(t,x) \to g(x)$ *uniformly as* $t \to +\infty$, *then system* (3.7) *is called the* **limit system** *of system* (3.6)*; and system* (3.6) *is called the* **asymptotically autonomous system** *with the limit system* (3.7).

LEMMA 3.5 [13]. – *Let* $f \in C(R \times R^n)$ *in system* (3.6) *and* $g \in C(R^n)$ *in system* (3.7) *satisfy the local Lipschitz condition with respect to* $x$ *and* $y$. *If every solution of system* (3.6) *is bounded for all* $t \geq 0$, *and the limit system* (3.7) *exists a equilibrium which is globally asymptotically stable, then*

$$\lim_{t\to+\infty} x(t) = P.$$

Let $g = \tilde{\beta}(I_d^* + E_d^*)$, where $I_d^*, E_d^*$ is the value of the globally asymptotically stable equilibrium of system (2.1). Then the asymptotically autonomous system (2.2) has the following limit system:

$$\begin{cases} \dot{S}_p &= \eta S_p(1 - S_p/K) - gS_p + \tilde{\theta}V_p, \\ \dot{E}_p &= gS_p - \tilde{p}E_p - \tilde{\gamma}E_p, \\ \dot{I}_p &= \tilde{\gamma}E_p - \tilde{a}I_p, \\ \dot{V}_p &= \tilde{p}E_p - \tilde{\theta}V_p. \end{cases} \tag{3.8}$$

Its positively invariant set is $\Omega_4 = \{(S_p, E_p, I_p, V_p) | S_p, E_p, I_p, V_p \geq 0, S_p + E_p + I_p + V_p \leq K\}$.

About the existences and stabilities of equilibria for system (3.8), we can obtain the following theorem.

THEOREM 3.6. – *Define*

$$R_1 = \frac{\eta\omega}{\tilde{\gamma}},$$

*where* $\omega = (\tilde{p} + \tilde{\gamma})/g$. *Then*

1. *when* $R_0 < 1$, *we have* $g = 0$. *System* (3.8) *has two equilibria* $E_3(0,0,0,0)$ *and* $E_4(K,0,0,0)$. $E_3$ *is unstable, and* $E_4$ *is globally asymptotically stable.*

2. *when* $R_0 > 1$ *and* $R_1 > 1$, *we have* $g = \tilde{\beta}(I_d^* + E_d^*)$. *Two equilibria exist: the disease-free equilibrium* $E_3$ *and the unique endemic equilibrium* $E_5(S_p^*, E_p^*, I_p^*, V_p^*)$. $E_3$ *is unstable, and* $E_5$ *is globally asymptotically stable in the interior of* $\Omega_4$.

PROOF. – The existence of equilibria are easy to check. And we also omit the proof of instability of $E_3$ here. It is easy to arrive using the Hurwitz's theorem.

Now we begin to prove the stability of the disease-free equilibrium $E_4$.

Let $x = S_p - K$. Let $g = 0$. System (3.8) is equivalent to the following system:

$$\begin{cases} \dot{x} &= -\eta x + \tilde{\theta} V_p - \dfrac{\eta}{K} x^2, \\ \dot{E}_p &= -\tilde{p} E_p - \tilde{\gamma} E_p, \\ \dot{I}_p &= \tilde{\gamma} E_p - \tilde{a} I_p, \\ \dot{V}_p &= \tilde{p} E_p - \tilde{\theta} V_p. \end{cases} \tag{3.9}$$

The corresponding Jacobian matrix of the system at point (0,0,0) is

$$J_5 = \begin{pmatrix} -\eta & 0 & 0 & \tilde{\theta} \\ 0 & -(\tilde{p} + \tilde{\gamma}) & 0 & 0 \\ 0 & \tilde{\gamma} & -\tilde{a} & 0 \\ 0 & \tilde{p} & 0 & -\tilde{\theta} \end{pmatrix}.$$

Obviously, the eigenvalues of the above matrix are $-\eta, -\tilde{\theta}, -\tilde{a}, -(\tilde{p} + \tilde{\gamma})$, which are all negative, then $E_4$ is locally asymptotically stable.

Solve system(3.9), we have

$$E_p = E_0 e^{-(\tilde{p}+\tilde{\gamma})t},$$

$$I_p = -\frac{E_0 \tilde{\gamma}}{(\tilde{p} + \tilde{\gamma}) - \tilde{a}} e^{-(\tilde{p}+\tilde{\gamma})t} + I_0 e^{-\tilde{a}t},$$

$$V_p = -\frac{E_0 \tilde{\gamma}}{(\tilde{p} + \tilde{\gamma}) - \tilde{\theta}} e^{-(\tilde{p}+\tilde{\gamma})t} + V_0 e^{-\tilde{\theta}t},$$

therefore

$$\lim_{t\to+\infty} E_p = 0, \qquad \lim_{t\to+\infty} I_p = 0, \qquad \lim_{t\to+\infty} V_p = 0.$$

So the limit equation of the first equation of system (3.10) is

$$\dot{x} = -\eta x - \frac{\eta}{K}x^2\ ,\ x \in [-K, 0]. \tag{3.10}$$

Then if $x \neq -K$, namely, $S_p \neq 0$, which means that the solution does't start from the disease-free equilibrium $E_3$, then $\lim_{t\to+\infty} x = 0$. Therefore we have the global attraction of the equilibrium $E_4$ in $\Omega_4 \backslash \{E_3\}$.

In a word, the global asymptotic stability of the disease-free equilibrium $E_4$ in $\Omega_4 \backslash \{E_3\}$ is obtained.

Moreover, the Jacobian matrix of system (3.8) is

$$J_6 = \begin{pmatrix} \eta\left(1 - \frac{2S_p}{K}\right) - g & 0 & 0 & \tilde{\theta} \\ g & -(\tilde{p} + \tilde{\gamma}) & 0 & 0 \\ 0 & \tilde{\gamma} & -\tilde{a} & 0 \\ 0 & \tilde{p} & 0 & -\tilde{\theta} \end{pmatrix}.$$

Using the similar technique in Theorem 3.4, we can prove that $E_5$ is globally asymptotically stable in the interior of $\Omega_4$. We omit the proof here. □

From Lemma 3.5, we can see that when $R_0 < 1$, every orbit of system (2.2) goes to $E_4$, namely, human rabies dies out; when $R_0 > 1$ and $R_1 > 1$, every orbit of system (2.2) goes to $E_5$, namely, human rabies becomes endemic.

## 4. – Simulations.

In this section, we present some numerical simulations with different values of parameters. We will see that the key to control the human rabies is to control the dog rabies.

Let $R_0 = 1$, there is

$$p^* = \frac{\theta\beta N(a + \gamma)}{a\gamma} - \theta.$$

If $p > p^*$, then $R_0 < 1$. Namely, when the vaccination rate of dogs is greater then some critical value $p^*$, the rabies will die out. When $R_0 < 1$, there is no chance that human rabies would become endemic.

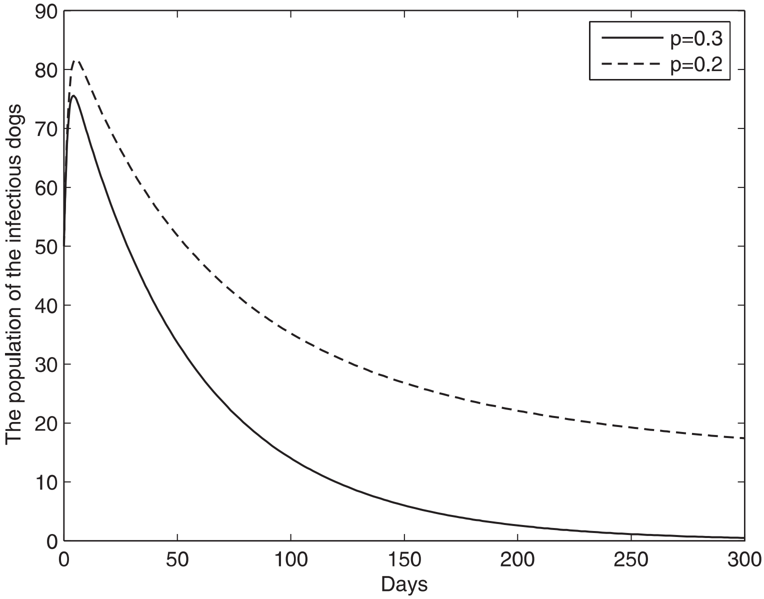


Fig. 3. – The critical value $p^* = 0.224$. When $p = 0.3 > p^*, R_0 = 0.7634 < 1$; when $p = 0.2 < p^*, R_0 = 1.1104 > 1$.

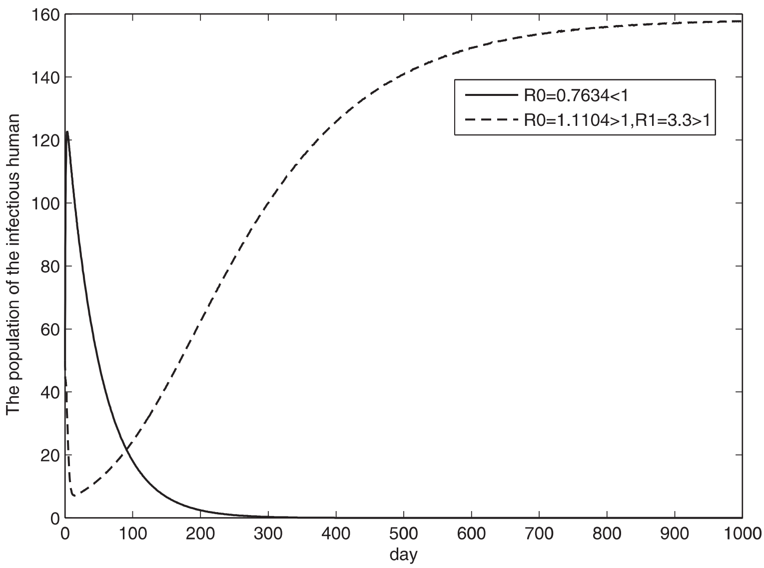


Fig. 4. – $\mathrm{K} = 1200$, $\tilde{p} = 0.3$, $\tilde{\theta} = 0.02$, $\tilde{a} = 0.8$, $\tilde{\gamma} = 0.02$, $\eta = 0.3$.

In Figure 3, we choose two pair of parameters, with different vaccination rates. Fix $N = 150, \gamma = 0.02, a = 0.5, \beta = 0.005, \theta = 0.02$. From calculation, we get $p^* = 0.224$. Then, we choose different vaccination rates. The simulation shows that once the vaccination rate is larger than $p^*$, the dog rabies will extinct (for example $p = 0.3$).

In Figure 4, we fix the parameters for human, and change the parameter $R_0$. Namely, we simulate two situations: one is that the dog rabies dies out; the other is that the dog rabies becomes endemic. When $R_0 < 1$, namely, the dog rabies dies out, then the human rabies dies out; when $R_0 > 1$ and $R_1 < 1$, the whole population of human would die out, which is biologically meaningless; when $R_0 > 1$ and $R_1 > 1$, namely, the dog rabies becomes endemic, then the human rabies will become endemic.

## 5. – Discussion.

The main results in the paper can be summarized in the following table (DEF stands for disease-free equilibrium and EE stands for endemic equilibrium).

| System | Threshold | Equilibrium | Stability |
|---|---|---|---|
| Dog | $R_0 < 1$ | DFE $E_1$ | $E_1$ is globally asymptotically stable |
| | $R_0 > 1$ | DFE $E_1$ & EE $E_2$ | $E_1$ is unstable; $E_2$ is globally asymptotically stable, provided that $a > \theta,\ p \geq \gamma$ |
| Human | $R_0 < 1$ | DFE $E_3$ & DFE $E_4$ | $E_3$ is unstable; $E_4$ is globally asymptotically stable |
| | $R_0 > 1$ $R_1 > 1$ | DFE $E_3$ & EE $E_5$ | $E_3$ is unstable; $E_5$ is globally asymptotically stable |

In this paper, the interactional model of rabies transmitted between human and dogs is formulated and the stability of the equilibria is studied, under some assumptions. Firstly, the total population of the dogs is supposed to be constant, namely, the input of the dogs is equivalent to the rabies-caused death. Actually, although the rabies-caused mortality is extremely high, the quantity of the disease-caused dead dogs isn't so much, at the meantime considering a local area, such as a city, the birth of the dogs and the migration of the dogs are also not so many, therefore we suppose both of them are equivalent. Secondly, we suppose the rabies-caused mortality is larger then the recovery rate of the vaccinated dogs becoming susceptible again. Here, since the vaccine of dogs is valid for one year or three years, namely, $\theta$ is rather small, the assumption is natural. Finally, we suppose $p \geq \gamma$ to obtain the global stability of the endemic equilibrium of dogs. It means that once the vaccination reaches certain standard, the population dynamics of dogs affected can be control at a low standard. In fact, we can see from the endemic equilibrium of dogs

$$E_d^* = \frac{aN}{a+\gamma}\left(1 - \frac{1}{R_0}\right), \quad I_d^* = \frac{\gamma N}{a+\gamma}\left(1 - \frac{1}{R_0}\right),$$

where $R_0 = \dfrac{\theta N}{p+\theta} \cdot \dfrac{\beta(a+\gamma)}{a\gamma}$. $R_0$ will decrease with the growth $p$, and at the meantime, $E_d^*$, $I_d^*$ will also decrease. Since the equilibrium is globally asymptotically stable, we can see that the population of dogs affected with rabies will be controlled at a low standard. Maybe this can give some suggestions to the prevention and the control of rabies at a local area.

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Wei Yang, School of Mathematical Sciences, Fudan University
Shanghai, 200433, P. R. China.
E-mail: weiyang@fudan.edu.cn

Jie Lou, Department of Mathematics, Shanghai University
99 Shangda Road, Shanghai 200444, P. R. China.
E-mail: jie.lou@staff.shu.edu.cn

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