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Giovanna Ilardi, Paola Supino, Jean Vallès

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Geometry of Syzygies via Poncelet Varieties

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Abstract. – *We consider the Grassmannian $\mathbb{G}r(k, n)$ of $(k+1)$-dimensional linear subspaces of $V_n = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. We define $\mathfrak{X}_{k,r,d}$ as the classifying space of the k-dimensional linear systems of degree n on $\mathbb{P}^1$, whose bases realize a fixed number r of polynomial relations of fixed degree d, say r syzygies of degree d. Firstly, we compute the dimension of $\mathfrak{X}_{k,r,d}$. In the second part we make a link between $\mathfrak{X}_{k,r,d}$ and the Poncelet varieties. In particular, we prove that the existence of linear syzygies implies the existence of singularities on the Poncelet varieties.*

1. – Introduction and set up.

In this paper we are interested in linear systems Λ on $\mathbb{P}^1$ of degree n and projective dimension k (so, from now we assume that $n > k$), more particularly in those having an algebraic limitation, namely the syzygies. A *syzygy* of degree d for Λ is a $(k+1)$-uple of homogeneous forms of degree d, $(g_0, \dots, g_k)$, such that $\sum_{i=0}^{k} g_i f_i = 0$, where $(f_0, \dots, f_k)$ is a basis of Λ. We say that Λ has r syzygies of degree d if there exist r linearly independent $(k+1)$-uples $(g_{0,j}, \dots, g_{k,j})$, where $1 \leq j \leq r$.

We define $\mathfrak{X}_{k,r,d}$ as the classifying space of the k-dimensional linear systems of degree n on $\mathbb{P}^1$ whose basis realize a fixed number r of polynomial relations of fixed degree d, say r syzygies of degree d. It lives inside $\mathbb{G}r(k, n)$ in a natural way:

$$\mathfrak{X}_{k,r,d} := \{\Lambda \in \mathbb{G}r(k, n) \text{ having at least } r \text{ syzygies of degree } d\}.$$

The first result of this paper is the computation of the dimension of $\mathfrak{X}_{k,r,d}$. The subvarieties $\mathfrak{X}_{k,r,d}$ turn out to be determinantal varieties for a suitable map of vector bundles on the Grassmannian. This extends the main result (corollary 4.4) in [3], where the computation was only proved in the $r = 1$ case. In the second part we give a geometric interpretation of the varieties $\mathfrak{X}_{k,r,d}$ in terms of Poncelet varieties. These varieties were introduced by Trautmann in [6], but, except for the case of curves, they have not been actually studied. We prove that the existence of linear syzygies implies the existence of singularities on the Poncelet varieties.

2. – The dimension of the varieties $\mathfrak{X}_{k,r,d}$.

Let Λ be a linear system on $\mathbb{P}^1$ of degree n and dimension k. We choose u, v a system of coordinates on $\mathbb{P}^1$, and denote by V_n the $n+1$-dimensional vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ of binary homogeneous forms of degree n, otherwise said binary *quantics* of degree n. A basis for V_n is $x_0, \dots, x_n$ where $x_i = u^i v^{n-i}$.

Choose a linear subspace Λ of V_n, and let $\{f_0, \dots, f_k\}$ be a basis for Λ. It defines a morphism of vector bundles on $\mathbb{P}^1$

$$\phi_\Lambda : \Lambda \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n) \tag{1}$$

which is surjective when Λ has no base points. The system Λ gives a map from $\mathbb{P}^1$ to $\mathbb{P}^k$

$$f_\Lambda : \mathbb{P}^1\Lambda \longrightarrow \mathbb{P}^k. \tag{2}$$

Its image is a rational curve of degree n when ϕ_Λ is surjective, and less than n when Λ has base points. For general Λ, ϕ_Λ is a surjective morphism of vector bundles on $\mathbb{P}^1$, thus there is an exact sequence

$$0 \longrightarrow E_\Lambda \longrightarrow \Lambda \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0, \tag{3}$$

where the kernel is a vector bundle E_Λ on $\mathbb{P}^1$ of rank k and degree $-n$.

The short exact sequence (3) twisted by $\mathcal{O}_{\mathbb{P}^1}(d)$

$$0 \longrightarrow E_\Lambda(d) \longrightarrow \Lambda \otimes \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n+d) \longrightarrow 0$$

suggests that Λ has exactly r independent syzygies of degree d if and only if $h^0(E_\Lambda(d)) = r$.

Since $H^0(\phi_\Lambda)$ is injective one has $E_\Lambda = \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_k)$ for suitable positive $a_1 \leq \cdots \leq a_k$ such that $a_1 + \cdots + a_k = n$. Then, one can stratify the varieties $\mathfrak{X}_{k,r,d}$ by all possible splitting of the integer n in k pieces. This point of view is developed by Ramella in [4] to study the stratification of the Hilbert scheme of rational curves C embedded in projective space, by the splitting of the restriction of the tangent bundle to C, and by the splitting of the normal bundle. We will use this point of view (in theorem 2.2) in order to prove that the dimension is the expected one.

EXAMPLE 2.1. – When $n = 5$ and $k = 3$ the only possible cases for E_Λ are:

$$E_\Lambda = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2),$$

which is the general case, and

$$E_\Lambda = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3).$$

In the general case $h^0(E_\Lambda(1)) = 1$, while $h^0(E_\Lambda(1)) = 2$ in the second case. Thus the general stratum is $\mathfrak{X}_{3,1,1} = \mathbb{G}r(3,5)$, and the stratum $\mathfrak{X}_{3,2,1}$ is strictly contained in $\mathbb{G}r(3,5)$.

Note that if $n < 2k$ the general splitting has the first term $a_0 = 1$, hence $h^0(E_\Lambda(1)) \geq 1$, which implies that $\mathfrak{X}_{k,1,1} = \mathbb{G}r(k,n)$, that is, there always exists a linear syzygy.

THEOREM 2.2. – $\operatorname{codim}(\mathfrak{X}_{k,r,d}, \mathbb{G}r(k,n)) = (dk + k - n + r)r$. *Moreover the varieties* $\mathfrak{X}_{k,r,d}$ *are Cohen-Macaulay with singular locus* $\mathfrak{X}_{k+1,r,d}$.

PROOF. – Consider the universal vector bundle $\mathcal{U} = \mathcal{U}_{\mathbb{G}r(k,n)}$ on the Grassmannian

$$\mathcal{U} = \{(f, \Lambda) \in V_n \times \mathbb{G}r(k,n) \mid f \in \Lambda\},$$

and the canonical map of vector bundles

$$\mathcal{U} \hookrightarrow V_n \otimes \mathcal{O}_{\mathbb{G}r(k,n)}.$$

On the product variety $\mathbb{G}r(k,n) \times \mathbb{P}^1$ let p and q be the two projections

$$\mathbb{G}r(k,n) \xleftarrow{p} \mathbb{G}r(k,n) \times \mathbb{P}^1 \xrightarrow{q} \mathbb{P}^1.$$

The map p composed with the evaluation map gives on $\mathbb{G}r(k,n) \times \mathbb{P}^1$

$$p^*\mathcal{U} \longrightarrow V_n \otimes \mathcal{O}_{\mathbb{G}r(k,n)\times\mathbb{P}^1} \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(n).$$

For all d we also have a morphism

$$p^*\mathcal{U} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(n+d).$$

If we take now the direct image p_* on the Grassmannian, we obtain

$$\mathcal{U} \otimes V_d \longrightarrow V_{n+d}$$

which is just the relative version of our map ϕ_Λ in (1) twisted by $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$

$$\Lambda \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(n+d)).$$

Therefore

$$\mathfrak{X}_{k,r,d} = \{\Lambda \in \mathbb{G}r(k,n) \mid rk(\Phi_\Lambda) \leq (k+1)(d+1) - r\}.$$

Applying Thom-Porteous formula we compute the expected codimension for $\mathfrak{X}_{k,r,d}$ as $r(n + r - (d+1)k)$. It is a classical fact that, when the codimension is exactly the expected one, the Chow class is $det(c_{n-k(d+1)+r+j-i}(\mathbb{C}^{(d+1)(n+d+1)} \otimes \mathcal{U}^*))$.

We compute now the codimension of the tangent space in a generic point Λ. Consider its associated bundle E_Λ, by genericity we have

$$E_\Lambda = \mathcal{O}^r_{\mathbb{P}^1}(-d) \oplus \mathcal{O}^{k-r-B}_{\mathbb{P}^1}(-A) \oplus \mathcal{O}^B_{\mathbb{P}^1}(-A-1)$$

where A and B are uniquely defined by

$$(n-dr) = A(k-r) + B,\ 0 \leq B < k-r$$

and, by hypothesis on the syzygy, $A > d$. The codimension of the tangent space in the point Λ is then $h^1(E_\Lambda \otimes E^\vee_\Lambda)$. Since

$$E_\Lambda \otimes E^\vee_\Lambda = \mathcal{O}^{r(k-r-B)}_{\mathbb{P}^1}(d-A) \oplus \mathcal{O}^{rB}_{\mathbb{P}^1}(d-A-1) \oplus R,$$

where R is a suitable bundle with $h^1(R) = 0$, we have $h^1(E_\Lambda \otimes E^\vee_\Lambda) = r(n + r - (d+1)k)$ which is the expected codimension. $\square$

3. – Geometric description as Poncelet varieties.

At the end of his paper [6], Trautmann has introduced a generalization of Poncelet curves, namely the Poncelet varieties, in higher dimension. Those are in bijective correspondence with the points of the Grassmannian. The aim of this part is to describe the points of $\mathfrak{X}_{k,r,d}$ as Poncelet varieties. In particular we will show that Poncelet varieties corresponding to $\mathfrak{X}_{k,1,1}$ are singular (see theorem 3.9). Following [7], we define the Poncelet varieties as determinant of sections of Schwarzenberger bundles, therefore we start by recalling the definition of Schwarzenberger bundle and we describe the zero locus of their section (see proposition 3.2).

3.1 – *Schwarzenberger bundles.*

We denote by $(x_i = u^i v^{k+1-i})$, $(y_j = u^j v^{n-k-1-j})$, and $(z_l = u^l v^{n-l})$ the chosen basis of V_{k+1}, V_{n-k-1} and V_n respectively. The multiplication of homogeneous polynomials in two variables

$$V_{k+1} \otimes V_{n-k-1} \xrightarrow{\phi_\times} V_n,$$

$$(u^i v^{k+1-i}, u^j v^{n-k-1-j}) \longmapsto u^{i+j} v^{n-i-j}$$

can be relativized on the bundles on $\mathbb{P}V_{k+1}$. The bundles E_n are then defined by the following exact sequences

$$0 \longrightarrow V_{n-k-1} \otimes \mathcal{O}_{\mathbb{P}V_{k+1}}(-1) \xrightarrow{A} V_n \otimes \mathcal{O}_{\mathbb{P}V_{k+1}} \longrightarrow E_n \longrightarrow 0 \tag{4}$$

where A is the $(n+1) \times (n-k)$ matrix such that

$$A^T = \begin{pmatrix} x_0 & x_1 & \cdots & x_{k+1} & & & & \\ & x_0 & x_1 & \cdots & x_{k+1} & & & \\ & & x_0 & x_1 & \cdots & x_{k+1} & & \\ & & & \ddots & \ddots & & \ddots & \\ & & & & x_0 & x_1 & \cdots & x_{k+1} \end{pmatrix}$$

The bundles E_n have rank $k+1$ and are called *Schwarzenberger bundles*, because for $n=2$ they were first introduced by Schwarzenberger in [5].

Into the product $\mathbb{P}V_{k+1} \times \mathbb{P}V_n$ the projective bundle $\mathbb{P}E_n$ is defined by the equations

$$A^T Z = \left(\sum_{i=0}^{i=k+1} x_i z_{i+j} = 0 \right)_{j=0,\cdots,n-k-1}$$

where $Z = (z_0, \ldots, z_n)^T$ are the global sections of E_n. The multiplication of homogeneous polynomials in two variables induces also the following exact sequence on $\mathbb{P}(V_n^\vee)$, where $V^\vee$ means the dual vector space

$$V_{n-k-1} \otimes \mathcal{O}_{\mathbb{P}(V_n^\vee)}(-1) \xrightarrow{M} V_{k+1}^\vee \otimes \mathcal{O}_{\mathbb{P}(V_n^\vee)} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{5}$$

Here M is the $(k+2) \times (n-k)$ matrix

$$M = \begin{pmatrix} z_0 & z_1 & z_2 & \cdots & z_{n-k-1} \\ z_1 & z_2 & \cdots & \cdots & z_{n-k} \\ z_2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & z_{n-1} \\ z_{k+1} & \cdots & \cdots & z_{n-1} & z_n \end{pmatrix}.$$

The support of coherent sheaf $\mathcal{F}$ is the variety X_{k+1} of k-planes $(k+1)$-secant to the rational normal curve defined by the 2×2 minors of the matrix M ([2], proposition 9.7 p. 103). The blowing up of X_{k+1} along X_k (defined by the subminors of M) is embedded in $\mathbb{P}V_{k+1} \times \mathbb{P}V_n$ by the same equation as before,

$$\left(\sum_{i=0}^{i=k+1} x_i z_{i+j} = 0 \right)_{j=0,\cdots,n-k-1}$$

(see for instance, [3], corollary 4.4 and theorem 4.9). For this reason the bundle E_n is sometimes called *secant bundle*. In fact its fibers are identified by this way to the k-planes $(k+1)$-secant to the underlying rational normal curve. This point will be crucial to prove the next proposition.

3.2 – *Zero locus of a section.*

First, let us fix some notations.

We call $C_N \in \mathbb{P}(V_N)$ the rational normal curve of degree $N \geq 1$, image of $\mathbb{P}(V_1)$ by the Veronese embedding. We will say that a hyperplane H osculates the rational normal curve C_N when $H \cap C_N$ is supported by one single point. In the dual projective space $\mathbb{P}(V_N^\vee) = \mathbb{P}_N^\vee$ we denote by C_N^{osc} the rational normal curve of the osculating planes to C_N. When x is a point of $\mathbb{P}(V_N)$ (resp. of $\mathbb{P}(V_N^\vee)$) we will denote by H_x the corresponding hyperplane in $\mathbb{P}(V_N^\vee)$ (resp. in $\mathbb{P}(V_N)$). We recall that a general point in $\mathbb{P}(V_N)$ is uniquely defined by the intersection of N osculating hyperplanes to C_N (it is the classical polarity related to a rational normal curve).

DEFINITION 3.1. – *Let $s \in H^0(E_n) = V_n$ be a non zero section. It corresponds to a hyperplane $H_s \subset \mathbb{P}(V_n)$, so it gives an effective divisor $H_s \cap C_n$ of degree n on the rational curve C_n. By the canonical Veronese isomorphism $\mathbb{P}(V_1) \cong C_n \cong C_{k+1}$ we obtain an unique divisor of degree n on C_{k+1}. The n osculating planes in these points give a divisor on C_{k+1}^{osc}. We denote this last divisor by $D_n(s)$ and call it the divisor corresponding to s.*

Let us describe now the zero locus $Z(s) \subset \mathbb{P}V_{k+1}$ geometrically.

PROPOSITION 3.2. – *Let $s \in H^0(E_n)$ be a non zero section and $D_n(s)$ be the corresponding effective divisor of degree n on C_{k+1}^{osc}. Then,*

$$a \in Z(s) \Leftrightarrow H_a \cap C_{k+1}^{osc} \subset D_n(s).$$

REMARK 3.3. – In particular when $D_n(s)$ is smooth (that is $D_n(s)$ consists in n distinct osculating hyperplanes to C_{k+1}) the section s vanishes along $\binom{n}{k+1}$ points in $\mathbb{P}^{k+1}$ which are the intersection points of $(k+1)$ osculating hyperplanes of C_{k+1} chosen among the previous n points of $D_n(s)$.

PROOF. – The section s

$$\underline{Hom}(E_n, \mathcal{O}_{\mathbb{P}V_{k+1}}) = E_n^\vee \xrightarrow{s} \mathcal{O}_{\mathbb{P}V_{k+1}} \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0$$

induces a rational map $\mathbb{P}(V_{k+1}) \longrightarrow \mathbb{P}(E_n^\vee)$ which is not defined over the zero-scheme $Z(s)$. To describe $Z(s)$ we will give explicitely the locus of indetermination of this map.

Let us consider the canonical projection $\pi : \mathbb{P}(E_n) \longrightarrow \mathbb{P}(V_{k+1})$.
Over a point $a \in \mathbb{P}(V_{k+1})$ the fiber is

$$\pi^{-1}(a) = \left\{ \left(\sum_{i=0}^{i=k+1} a_i z_{i+j} = 0 \right)_{j=0,\dots,n-k-1} \right\} = \mathbb{P}(E_n(a)) = \mathbb{P}^k.$$

which can be thought as a $\mathbb{P}^k$ $(k+1)$-secant to C_n.

Let H_s be the hyperplane in $\mathbb{P}(V_n)$ corresponding to s. The rational map induced by the section $s : \mathbb{P}(V_{k+1}) \longrightarrow \mathbb{P}(E_n^\vee)$ sends a point $a \in \mathbb{P}(V_{k+1})$ on $H_s \cap \pi^{-1}(a)$ which is, in general, a space $\mathbb{P}^{k-1}$ in $\mathbb{P}(E_n(a))$, that is a point in the dual space $\mathbb{P}(E_n^\vee(a))$. This map is not defined when $\pi^{-1}(a) \subset H_s$. The hyperplane H_s cuts the rational curve C_n along an effective divisor D of degree n. When this divisor is smooth, it contains $\binom{n}{k+1}$ subschemes of length $(k+1)$; they generate the k-planes $(k+1)$ secant to C_n which are contained in H_s. Since $\mathbb{P}V_1 \simeq C_n \simeq C_{k+1}^{osc}$ it is clear that D corresponds to a degree n divisor on C_{k+1}^{osc}. We will denote it by $D_n(s)$.

Since the fiber $\pi^{-1}(a)$ is a k-plane $(k+1)$-secant to C_n the zero-scheme $Z(s)$ is the set of points $a \in \mathbb{P}(V_{k+1})$ such that the divisor $H_a \cap C_{k+1}^{osc}$ of degree $(k+1)$ belongs to $D_n(s)$. When $D_n(s)$ is smooth, we get n osculating hyperplanes of C_{k+1} in $\mathbb{P}(V_{k+1})$. Every subset of $(k+1)$ osculating hyperplanes gives a point in $\mathbb{P}(V_{k+1})$. These points are the zero-scheme of the section s. □

3.3 – *Poncelet varieties.*

The group $SL(2,\mathbb{C})$ acts on $\mathbb{G}r(k,n)$ and we have an equivariant morphism $\mathbb{G}r(k,n) \hookrightarrow \mathbb{P}(\bigwedge^{k+1} V_n)$. The $SL(2)$–modules $\bigwedge^{k+1} V_n$ and $S^{k+1}V_{n-k}$ are isomorphic (see [1], p. 160). Moreover by Hermite reciprocity formula (see [1], p. 82 and p. 160), we have $S^{k+1}V_{n-k} \cong S^{n-k}V_{k+1}$. So the Plucker embedding becomes

$$T\colon \mathbb{G}r(k,n) \hookrightarrow \mathbb{P}(S^{n-k}V_{k+1}). \tag{6}$$

It associates to $\Lambda = \langle f_0, \dots, f_k \rangle$ the hypersurface of degree $n-k$ in $\mathbb{P}V_{k+1}$ with equation $f_0 \wedge \dots \wedge f_k = 0$. Since from (4)

$$H^0(E_n) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(n)),$$

with abuse of notation, we consider f_i as section of E_n. We summarize these facts in the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & \Lambda \otimes \mathcal{O}_{\mathbb{P}^{k+1}} & = & \Lambda \otimes \mathcal{O}_{\mathbb{P}^{k+1}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{n-k-1} \otimes \mathcal{O}_{\mathbb{P}^{k+1}}(-1) & \longrightarrow & V_n \otimes \mathcal{O}_{\mathbb{P}^{k+1}} & \longrightarrow & E_n & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{n-k-1} \otimes \mathcal{O}_{\mathbb{P}^{k+1}}(-1) & \longrightarrow & V_n/\Lambda \otimes \mathcal{O}_{\mathbb{P}^{k+1}} & \longrightarrow & \mathcal{L} & \longrightarrow & 0.
\end{array}
\tag{7}
$$

where the support of the sheaf $\mathcal{L}$ is given by $f_0 \wedge \ldots \wedge f_k = 0$.

Definition 3.4. – *The varieties defined as determinant of* $(k+1)$ *sections of Schwarzenberger bundles on* $\mathbb{P}^{k+1}$ *are called* Poncelet varieties *of dimension k. These varieties have degree* $c_1(E_n) = n - k$.

We point out that a section of E_n corresponds to n points on the rational normal curve C_{k+1}, and it vanishes along $\binom{n}{k+1}$ points in $\mathbb{P}^{k+1}$ which are the intersection points of $(k+1)$ osculating hyperplanes of C_{k+1} in $k+1$ points chosen among the previous n points. Hence these varieties are characterized by the nice following geometric fact: they contain the vertices of polytopes with $(k+1)$-dimensional faces osculating C_{k+1}. The case of curves is well-known since Darboux, but in higher dimension quite nothing exists in literature.

Proposition 3.5. – *Let* Λ *be a linear system in* $\mathbb{G}r(k,n)$ *with* d *base points. Then* $T(\Lambda)$ *is the union of* d *osculating hyperplanes to* $C_{k+1} \subset \mathbb{P}V_{k+1}$ *and is a degree* $(n-k-d)$ *Poncelet variety of dimension k.*

Remark 3.6. – In particular when Λ is a pencil, a base point corresponds to a linear syzygy. The morphism T sends $\mathfrak{X}_{1,1,d}$ on the locus of Poncelet curves which are union of a Poncelet curve of degree $d-1$ with $n-d$ tangent lines to C_2.

Proof. – In the case of curves it is proved by Trautmann (see proposition 1.11 of [6]). In general assume that Λ has $d < n$ base points and let $f = 0$ be one equation for this base locus. Then we have a factorization

$$\mathcal{O}_{\mathbb{P}^1}^{k+1} \xrightarrow{\frac{\Lambda}{f}} \mathcal{O}_{\mathbb{P}^1}(n-d) \overset{f}{\hookrightarrow} \mathcal{O}_{\mathbb{P}^1}(n).$$

The first arrow gives the following vector bundle map

$$0 \longrightarrow \mathcal{O}^{k+1}_{\mathbb{P}(V_{k+1})} \xrightarrow{\frac{\Lambda}{f}} E_{n-d} \longrightarrow K \longrightarrow 0,$$

where K is supported on a Poncelet variety of degree $n-k-d$. The second arrow gives

$$0 \longrightarrow E_{n-d} \longrightarrow E_n \longrightarrow \oplus_{i=1}^{d} \mathcal{O}_{H_i} \longrightarrow 0$$

where H_i is the osculating hyperplane to C_{k+1} which corresponds to a base point on $C^{\vee}_{k+1}$. Now the result follows from the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^{k+1}} & = & \mathcal{O}_{\mathbb{P}^{k+1}} & & & & \\
 & & \frac{\Lambda}{f}\downarrow & & \Lambda\downarrow & & & & \\
0 & \longrightarrow & E_{n-d} & \longrightarrow & E_n & \longrightarrow & \oplus_{i=1}^{d}\mathcal{O}_{H_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & =\downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{L} & \longrightarrow & \oplus_{i=1}^{d}\mathcal{O}_{H_i} & \longrightarrow & 0,
\end{array}$$

where $\mathcal{L}$ is a sheaf supported by the Poncelet variety corresponding to Λ. □

The following examples are done as suggested by the commutative diagram in ((7)).

EXAMPLE 3.7. – In $\mathbb{G}r(2,4)$, $\mathfrak{X}_{2,1,2} = \mathbb{G}r(2,4)$ and $\operatorname{codim} \mathfrak{X}_{2,1,1} = 1$.

a) The net $\Lambda = \langle u^4, u^2v^2, v^4\rangle \in \mathfrak{X}_{2,1,2} \setminus \mathfrak{X}_{2,1,1}$ corresponds to the smooth Poncelet quadric

$$\det\begin{pmatrix} x_1 & x_3 \\ x_0 & x_2 \end{pmatrix} = x_1x_2 - x_0x_3 = 0.$$

b) The net $\Lambda = \langle u^4, u^3v, v^4\rangle \in \mathfrak{X}_{2,1,1}$ has associated Poncelet cone

$$\det\begin{pmatrix} x_2 & x_3 \\ x_1 & x_2 \end{pmatrix} = x_2^2 - x_1x_3 = 0.$$

EXAMPLE 3.8. – The net $\Lambda = \langle u^3v, uv^3, v^4\rangle$ has a base point, its associated Poncelet quadric is

$$\det\begin{pmatrix} x_0 & x_2 \\ 0 & x_1 \end{pmatrix} = x_1x_0 = 0.$$

This quadric consists in two planes, one of which osculating the rational normal curve C_3.

THEOREM 3.9. – *The Poncelet variety associated to any element of* $\mathfrak{X}_{k,1,1}$ *is singular. Moreover the Poncelet variety associated to a general element of* $\mathfrak{X}_{k,1,1}$ *contains* $\binom{n-1}{k}$ *lines and is singular in the* $\binom{n-1}{k+1}$ *vertices of this configuration.*

PROOF. – Let Λ have a syzygy of degree one: we can say that $\Lambda = \langle uf, vf, f_2, \ldots, f_k \rangle$, where $f \in V_{n-1}$. The curve Γ in $\mathbb{P}^{k+1}$ defined by the determinant of the two sections uf, vf of E_n is obtained as follows.

The pencil $\langle uf, vf \rangle$ defines $n-1$ fixed points and a moving point p on the rational normal curve C^{osc}_{k+1}. Therefore, the curve Γ in $\mathbb{P}^{k+1}$ consists in $\binom{n-1}{k}$ lines. They are the lines of k-planes in $\mathbb{P}^{(k+1)\vee}$ passing through p and each subset of k points chosen among the fixed ones.

Each $(k+1)$-uple of points on C^{osc}_{k+1}, chosen among the $(n-1)$ fixed one, gives a point on $\Gamma \subset \mathbb{P}^{(k+1)}$. It is the intersection point of $(k+1)$ lines corresponding to each choice of k points among the $(k+1)$. Since the $(k+1)$ points are distinct on the rational normal curve C^{osc}_{k+1}, the $\mathbb{P}^{k-1}$ generated by each choice of k points do not have a common point, dually, the configuration of lines is not contained in an hyperplane. Then, since the hypersurface defined by Λ contains the curve Γ, it has singularities in the $\binom{n-1}{k+1}$ vertices of the configuration of lines Γ. □

EXAMPLE 3.10. – The net $\Lambda = \langle u^5 + v^5, u^5 - u^4v + u^3v^2 - u^2v^3 + uv^4, u^5 - v^5 \rangle$ has a syzygy of degree 1. It corresponds to the Poncelet cubic surface

$$\det \begin{pmatrix} x_1 + x_2 & x_2 + x_3 & x_3 \\ x_0 + x_1 & x_1 + x_2 & x_3 + x_2 \\ x_0 & x_1 + x_0 & x_1 + x_2 \end{pmatrix} = 0.$$

This surface has four singular points, which is the maximum number of ordinary double points. Thus it is a Cayley cubic.

It could be interesting to explore the link between the sygyzies of higher degrees and the singularities of the associated Poncelet varieties.

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Giovanna Ilardi: Dip. Matematica e Applicazioni
Univ. di Napoli, via Cinthia, 80126 Napoli (Italy)
e-mail: giovanna.ilardi@dma.unina.it

Paola Supino: Dip. Matematica
Univ. Roma Tre, Largo L. Murialdo, 00146 Roma (Italy)
e-mail: psupino@uniroma3.it

Jean Vallès: Laboratoire de Mathematiques Appliques de Pau et de Pays de l'Adour
Avenue de l'Université 64000 Pau (France)
e-mail: jean.valles@univ-pau.fr

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