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Singular Dirichlet Problems with Quadratic Gradient

PEDRO J. MARTÍNEZ-APARICIO (*)

Abstract. – *We study the existence of solution for nonlinear elliptic problems with singular lower order terms that have natural growth with respect to the gradient.*

1. – Introduction.

In the framework of quasilinear elliptic equations with quadratic growth, we are concerned about the existence of solutions for the boundary value problem

$$\begin{cases} -\mathrm{div}\,(M(x,u)\nabla u) + \dfrac{Q(x,u)\nabla u \nabla u}{u} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, $0 \leq f \in L^m(\Omega)$ with $m \geq \dfrac{2N}{N+2}$, $f \not\equiv 0$ in Ω, $M(x,s)$ and $Q(x,s)$ are matrices which coefficients are Carathéodory i.e. are measurable with respect to x and continuous with respect to s. We suppose also that $M(x,s)$ is elliptic and bounded, i.e. that there exist positive constants α, β such that

$$\alpha|\xi|^2 \leq M(x,s)\xi \cdot \xi, \tag{1.2}$$

$$|M(x,s)| \leq \beta, \quad \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \quad \text{a.e } x \in \Omega,$$

and $Q(x,s)$ is symmetric, such that, for some $a, b > 0$ we have

$$a|\xi|^2 \leq Q(x,s)\xi\xi \leq b|\xi|^2. \tag{1.3}$$

There is a huge literature (see [6, 8] and the references given there) about the problems with quadratic term in the gradient which is called natural growth. The classical works do not consider a singularity in the lower order term.

We are interested in finding solutions of boundary value problems with lower

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order term having quadratic dependence on the gradient and singular dependence on u. As far as we know, it is studied for the first time in [2] the existence of positive solution for the model problem

$$(1.4) \qquad \begin{cases} -a\Delta u + \dfrac{|\nabla u|^2}{u} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for a datum $f \in L^\infty(\Omega)$ which is strictly positive on every compact subset of Ω. We have to mention that uniqueness of solutions for (1.4) is proved in [3].

Recently, the existence of positive solutions of the more general problem (1.1) is proved in [4] for data $0 \not\equiv f \in L^m(\Omega)$ for some $m \geq 2N/(N+2)$ with $f \geq 0$ and $a > 2b$. A different but related equation with a singularity in the lower order term is also studied in [10].

In this work, the result in [4] is improved by extending the existence to the case $a > b$. Specifically, we prove the following result.

THEOREM 1.1. – *Let* $0 \leq f \in L^m(\Omega)$ *for some* $m \geq \dfrac{2N}{N+2}$ *with* $f \not\equiv 0$ *in* Ω *and assume that* (1.2), (1.3) *and* $a > b$ *hold. Then there exists* $u \in H_0^1(\Omega)$, $u > 0$ *in* Ω, *with* $\dfrac{Q(x,u)\nabla u \nabla u}{u} \in L^1(\Omega)$, *weak solution of the singular-quadratic Dirichlet problem* (1.1).

The proof of Theorem 1.1 is given in Section 2. Its idea consists in approximating the problem (1.1) by a sequence of nonsingular problems (P_n). We emphasize that the lower order term blows up as $u_n(x)$ is converging to zero and $u = 0$ in $\partial\Omega$. This is the reason why it is not possible to apply the ideas of [6, 8] to show the strong convergence of ∇u_n in $L^2(\Omega)$ (and thus the strong convergence of the approximated solutions u_n in $H_0^1(\Omega)$ to a solution of (1.1)). The main point is to establish that u_n are uniformly away from zero in every compact set in Ω (see Proposition 2.1). To prove this fact it is required that $a > b$. This improves the argument in [4] where the author only proves that the limit of u_n is strictly positive in Ω. In contrast with the proof of [4] which requieres that $a > 2b$ to pass to the limit, this improvement allows us to prove the convergence of the approximated solutions to a solution of (1.1).

Section 3 is devoted to study a more general lower order term. Specifically, we consider the more general quasilinear Dirichlet problem

$$(1.5) \qquad \begin{cases} -\operatorname{div}(M(x,u)\nabla u) + g(x,u)Q(x,u)\nabla u \nabla u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g : \Omega \times (0, +\infty) \longrightarrow \mathbb{R}$ is a Carathéodory function. It is usual to require that g to satisfy the so-called "sign condition"

$$g(x,s)s \geq 0, \quad \forall s > 0. \tag{1.6}$$

Observe that Theorem 1.1 covers the case $g(x,s) = \dfrac{1}{s}$, which verifies this condition. Indeed, using the same arguments of Theorem 1.1 it is easy to extend it to the case of a general nonlinear term g satisfying the sign condition. Even more, combining these ideas with those in [1], we prove the existence of solution provided that, roughly speaking, g is between a positive hyperbola and a negative hyperbola near to 0 (see hypothesis (3.1) in Section 3).

2. – Proof of the existence result.

Let us denote by $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ and for $k > 0$, we will use the symbols T_k and G_k to denote the real functions given by

$$T_k(s) := \begin{cases} k, & s \geq k, \\ s, & -k \leq s \leq k, \\ -k, & s \leq -k, \end{cases} \qquad \text{and} \qquad G_k(s) := s - T_k(s), \quad s \in \mathbb{R}.$$

PROOF OF THEOREM 1.1. – Consider the boundary value problems

$$\begin{cases} -\mathrm{div}\,(M(x,u_n)\nabla u_n) + \dfrac{u_n}{(u_n + \frac{1}{n})^2} Q(x,u_n)\nabla u_n \nabla u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f_n = T_n(f)$. Since $f \in L^m(\Omega)$ with $m \geq \dfrac{2N}{N+2}$, then the sequence f_n converges to f in $L^m(\Omega)$. In addition, note that $0 \leq f_n \leq f$. By applying [14] there exists a solution u_n of (2.1) that belongs to $H_0^1(\Omega)$ and to $L^\infty(\Omega)$ (see [15]).

Taking u_n as test function in (2.1) and using Hölder and Sobolev inequalities leads to

$$\int_\Omega M(x,u_n)\nabla u_n \nabla u_n + \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{(u_n + \frac{1}{n})^2} u_n \leq \mathcal{S}\|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \|\nabla u_n\|_{L^2(\Omega)}.$$

By the ellipticity condition (1.2) and the positivity of the lower order term, it may be concluded that the sequence u_n is bounded in $H_0^1(\Omega)$. In fact, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in H_0^1(\Omega)$.

Taking $u_n^- \equiv \min\{u_n, 0\}$ as test function in (2.1) we obtain

$$\int_\Omega M(x,u_n)|\nabla u_n^-|^2 + \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{(u_n + \frac{1}{n})^2} u_n^- = \int_\Omega f_n u_n^-.$$

From (1.2), the positivity of the lower order term and of f_n, it follows that

$$a\int_\Omega |\nabla u_n^-|^2 \le \int_\Omega f_n u_n^- \le 0,$$

which establishes that $u_n \ge 0$.

Now, we are in a position to show that u_n are uniformly away from zero in every compact set in Ω.

PROPOSITION 2.1. – *Let* $0 \le f \in L^m(\Omega)$ *for some* $m \ge \dfrac{2N}{N+2}$ *with* $f \not\equiv 0$ *and assume that* (1.2) *and* (1.3) *hold. If* u_n *is a solution of* (2.1), *then for every* $\Omega_0 \subset\subset \Omega$ *there exists a constant* $c_{\Omega_0} > 0$ *such that*

$$u_n(x) \ge c_{\Omega_0}, \quad \text{a.e.} \quad x \in \Omega_0.$$

PROOF. – Let $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$, take (as in [4]) $\dfrac{\phi}{(u_n+\frac{1}{n})^{\frac{b}{a}}}$ as test function in (2.1) to obtain

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi \frac{1}{(u_n+\frac{1}{n})^{\frac{b}{a}}} - \int_\Omega f_n \frac{\phi}{(u_n+\frac{1}{n})^{\frac{b}{a}}}$$

$$= \frac{b}{a}\int_\Omega M(x,u_n)\nabla u_n \nabla u_n \frac{\phi}{(u_n+\frac{1}{n})^{\frac{b}{a}+1}} - \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{(u_n+\frac{1}{n})^{\frac{b}{a}+2}}\phi.$$

Use (1.2) and (1.3) to get

$$\frac{b}{a}\int_\Omega M(x,u_n)\nabla u_n \nabla u_n \frac{\phi}{(u_n+\frac{1}{n})^{\frac{b}{a}+1}} - \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{(u_n+\frac{1}{n})^{\frac{b}{a}+2}}\phi \ge 0$$

and consequently

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi \frac{1}{(u_n+\frac{1}{n})^{\frac{b}{a}}} \ge \int_\Omega f_n \frac{\phi}{(u_n+\frac{1}{n})^{\frac{b}{a}}}. \tag{2.2}$$

We fix $L > 0$ such that the Lebesgue measure of the level set $\{x \in \Omega : u(x) = L\}$ is zero. (Observe that the values L for which this property is false is at most countable). Thus, thanks to the choice of L, and since $u_n(x) \to u(x)$ a.e. $x \in \Omega$, it follows that $\chi_{\{u_n \le L\}} \to \chi_{\{u \le L\}}$ a.e. $x \in \Omega$. Therefore, we have

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi \frac{1}{(u_n+\frac{1}{n})^{\frac{b}{a}}} \ge \int_\Omega \chi_{\{u_n \le L\}} f_1 \frac{\phi}{(L+1)^{\frac{b}{a}}}.$$

We consider also $P_n(s) = \int_0^s \frac{1}{(t+\frac{1}{n})^{\frac{b}{a}}}\,dt$ and $w_n(x) = P_n(u_n(x))$. Therefore we can rewrite the previous inequality in the form

$$\int_\Omega M(x,u_n)\nabla w_n \nabla \phi \geq \int_\Omega \chi_{\{u_n\leq L\}} f_1 \frac{\phi}{(L+1)^{\frac{b}{a}}}.$$

The comparison principle in $H_0^1(\Omega)$ implies that $w_n(x) \geq z_n(x)$, where $z_n \in H_0^1(\Omega)$ is the bounded weak solution of

$$-\mathrm{div}\,(M(x,u_n)\nabla z_n) = \frac{\chi_{\{u_n\leq L\}}}{(L+1)^{\frac{b}{a}}} f_1. \tag{2.3}$$

It is easy to see that z_n converges strongly in $H_0^1(\Omega)$ to z, the solution of

$$\begin{aligned} &-\mathrm{div}\,(M(x,u)\nabla z) = \frac{\chi_{\{u\leq L\}}}{(L+1)^{\frac{b}{a}}} f_1, \\ &z \in H_0^1(\Omega). \end{aligned}$$

The strong maximum principle for weak solutions (see [11]) implies $z > 0$ in Ω (recall that $f \geq 0$ and $f \not\equiv 0$ in Ω and so also f_1).

We claim that the sequence z_n is equi-continuous in Ω. Indeed, by using $T_m(G_k(z_n))$, with $m > k$, as test function in (2.3), it is easy to see that $z_n \in L^\infty(\Omega)$ (for classical lines we refer the reader to [15]). The main idea of the proof is to take $\zeta \in C^\infty(\Omega)$ with $0 \leq \zeta(x) \leq 1$ (this construction is adapted from the proof of Theorem 1.1 of Chapter 4 in [12]), for every $x \in \Omega$ and compact support in a ball B_ρ of radius $\rho > 0$. Let us denote by $A_{k,\rho} = \{x \in B_\rho \cap \Omega : z_n(x) > k\}$. Choose $\phi = \zeta^2 G_k(z_n)$ as test function in (2.3), and we consider $q > N/2$ to conclude from (1.2) and Hölder's inequality that

$$a\int_{A_{k,\rho}} |\nabla z_n|^2\zeta^2 \leq \frac{\|f_1\|_{L^q(\Omega)}\|z_n\|_{L^\infty(\Omega)}}{(L+1)^{\frac{b}{a}}}|A_{k,\rho}|^{1-\frac{1}{q}} + 2\beta\int_{A_{k,\rho}} |\nabla z_n||\nabla\zeta|\zeta G_k(z_n).$$

Here, to set a bound for the second term, we use Young's inequality and we have

$$\int_{A_{k,\rho}} |\nabla z_n|^2\zeta^2 \leq \frac{\|f_1\|_{L^q(\Omega)}\|z_n\|_{L^\infty(\Omega)}}{a(L+1)^{\frac{b}{a}}}|A_{k,\rho}|^{1-\frac{1}{q}} + \frac{4\beta}{a^2}\int_{A_{k,\rho}} |\nabla\zeta|^2 G_k^2(z_n).$$

Now, if we take the function ζ such that it is constantly equal to 1 in the ball $B_{\rho-\sigma\rho}$ of radius $\rho - \sigma\rho$, where $\sigma \in (0,1)$ that is concentric with the ball B_ρ in such a way that $|\nabla\zeta| < \frac{1}{\sigma\rho}$, we obtain

$$\int_{A_{k,\rho}-\sigma\rho} |\nabla z_n|^2 \leq \gamma\left(1 + \frac{1}{\sigma^2\rho^{2(1-\frac{N}{2q})}}\max_{A_{k\rho}}(z_n-k)^2\right)|A_{k\rho}|^{1-\frac{1}{q}},$$

where $\gamma = \max\left\{\dfrac{2\|f_1\|_{L^q(\Omega)}\|z_n\|_{L^\infty(\Omega)}}{a}, \dfrac{4\beta}{a^2}\omega_N^{\frac{1}{q}}\right\}$ with ω_N denoting the measure of the unit ball of $\mathbb{R}^N$. This means that for $\delta > 0$ small enough the function z_n belongs to the De Giorgi class $\mathcal{B}_2\left(\Omega, M, \gamma, \delta, \dfrac{1}{2q}\right)$ with $2q > N$ (see [12], pag. 81). Therefore, applying Theorem 6.1 of [12] we obtain our claim.

Hence, since the sequence $\{z_n\}$ is equi-bounded and equi-continuous, by the Ascoli-Arzelà Theorem, $C^\lambda(\overline{\Omega_0})$ is compactly embedded into $C(\overline{\Omega_0})$ for every $\Omega_0 \subset\subset \Omega$, we deduce that the sequence $\{z_n\}$ has a subsequence (supposed to be itself) that converges uniformly to some z in $C(\overline{\Omega_0})$. Thanks to that z is continuous and $z > 0$ in Ω, given $\Omega_0 \subset\subset \Omega$ there exists $l_{\Omega_0} > 0$ such that $z \geq l_{\Omega_0} > 0$ for $x \in \Omega_0$. This clearly forces

$$w_n \geq \frac{1}{2}\, l_{\Omega_0}, \quad \forall x \in \Omega_0, \quad \forall n >> 0.$$

Observe that the assumption $a > b$ implies that $P(s) = \int_0^s \frac{1}{t^{\frac{b}{a}}}\, dt$ is well-defined. Since the real functions $P_n(s)$ and $P(s)$ are strictly increasing and $P_n < P$, then $P_n^{-1} > P^{-1}$ and we get that

$$u_n \geq P_n^{-1}\left(\frac{1}{2} l_{\Omega_0}\right) > P^{-1}\left(\frac{1}{2} l_{\Omega_0}\right) := c_{\Omega_0} > 0, \quad \forall \Omega_0 \subset\subset \Omega.$$

□

Let us prove that, up to a subsequence, the sequence $\{u_n\}$ converges to a positive solution of (1.1). We divide the rest of the proof in three steps.

STEP 1. For every $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$,

$$\lim_{n\to+\infty} \int_\Omega |\nabla(T_k(u_n) - T_k(u))|^2 \phi = 0, \quad \forall k > 0. \tag{2.4}$$

STEP 2. For every $\Omega_0 \subset\subset \Omega$, $\{u_n\}$ converges in $H^1(\Omega_0)$ to u.

STEP 3. u is a solution of (1.1).

STEP 1. Consider $0 \leq \phi \in C_0^\infty(\Omega)$ and $\Omega_0 \subset\subset \Omega$ such that $\operatorname{supp}\phi \subset \Omega_0$. Given $k > 0$, we define $\varphi_\lambda(s) = se^{\lambda s^2}$, where the positive constant $\lambda > \left(\dfrac{bk}{ac_{\Omega_0}}\right)^2$. We will denote by $\varepsilon(n)$ any quantity that tends to 0 as n diverges. Following [7], take $\varphi_\lambda(T_k(u_n) - T_k(u))\phi$ as test function in (2.1) to obtain

$$
\begin{aligned}
&\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(T_k(u_n)-T_k(u))\phi\\
&+\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\,\varphi_\lambda(T_k(u_n)-T_k(u))\\
&+\int_\Omega \frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}\varphi_\lambda(T_k(u_n)-T_k(u))\phi\\
&=\int_\Omega f_n\,\varphi_\lambda(T_k(u_n)-T_k(u))\phi\,.
\end{aligned}
\tag{2.5}
$$

By Proposition 2.1 and (1.3), we derive that

$$
\frac{Q(x,u_n)\nabla u_n\nabla u_n}{u_n+\frac{1}{n}}\le\frac{b|\nabla u_n|^2}{c_{\Omega_0}},\quad \forall x\in\Omega_0. \tag{2.6}
$$

We get from this inequality, and by the positivity of both terms $\frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}$ and $\varphi_\lambda(k-T_k(u))$, that

$$
\begin{aligned}
&\int_\Omega \frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}\varphi_\lambda(T_k(u_n)-T_k(u))\phi\\
&=\int_{\{u_n\le k\}} \frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}\varphi_\lambda(T_k(u_n)-T_k(u))\phi\\
&+\int_{\{u_n\ge k\}} \frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}\varphi_\lambda(T_k(u_n)-T_k(u))\phi\\
&\ge-\frac{bk}{c_{\Omega_0}}\int_\Omega|\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\,.
\end{aligned}
$$

Since $T_k(u_n)\to T_k(u)$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, it follows that

$$
\int_\Omega f_n\,\varphi_\lambda(T_k(u_n)-T_k(u))\phi-\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\,\varphi_\lambda(T_k(u_n)-T_k(u))=\varepsilon(n),
$$

and as a consequence of the above inequality and (2.5), we have

$$
\begin{aligned}
&\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(T_k(u_n)-T_k(u))\phi\\
&-\frac{bk}{c_{\Omega_0}}\int_\Omega|\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\le\varepsilon(n).
\end{aligned}
\tag{2.7}
$$

Note that

$$\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(T_k(u_n)-T_k(u))\phi\chi_{\{u_n\geq k\}}$$
$$=-\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla T_k(u)\varphi_\lambda'(k-T_k(u))\phi\chi_{\{u_n\geq k\}}\;=\varepsilon(n).$$

Adding the quantity

$$-\int_\Omega M(x,u_n)\nabla T_k(u)\cdot\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(T_k(u_n)-T_k(u))\phi\;=\varepsilon(n)$$

in both sides of (2.7), since

$$\begin{aligned}&\int_\Omega|\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\\&\leq 2\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\\&+2\int_\Omega|\nabla T_k(u)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\\&=2\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\;+\varepsilon(n),\end{aligned}$$

using (1.2) and that $\lambda>\left(\dfrac{bk}{ac_{\Omega_0}}\right)^2$ shows that $a\varphi_\lambda'(s)-2\,\dfrac{bk}{c_{\Omega_0}}|\varphi_\lambda(s)|\geq\dfrac{a}{2}$, we obtain

$$\frac{a}{2}\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2\phi\leq\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2\Big[a\varphi_\lambda'(T_k(u_n)-T_k(u))$$
$$-2\frac{bk}{c_{\Omega_0}}|\varphi_\lambda(T_k(u_n)-T_k(u))|\Big]\phi\leq\varepsilon(n)$$

which establishes that (2.4) holds.

STEP 2. Let us choose $G_k(u_n)$ as test function in (2.1) to obtain

$$\int_\Omega M(x,u_n)\nabla u_n\nabla G_k(u_n)\;+\int_\Omega\frac{u_nQ(x,u_n)\nabla u_n\nabla u_n}{\left(u_n+\frac{1}{n}\right)^2}G_k(u_n)=\int_\Omega f_nG_k(u_n).$$

Using that the term that involves the lower order term is positive (see (1.3)) and taking into account (1.2), and Hölder and Sobolev inequalities, we have

$$\int_\Omega|\nabla G_k(u_n)|^2\;\leq\frac{\mathcal{S}^2}{a^2}\left(\int_{\{u_n\geq k\}}f^{\frac{2N}{N+2}}\right)^{1+\frac{2}{N}}.\tag{2.8}$$

Since meas$(\{x \in \Omega : u_n \geq k\})$ converges to zero, uniformly with respect to n, when k goes to $+\infty$ we obtain that the last integral in the above inequality tends to zero as k goes to $+\infty$. Therefore, for all $\varepsilon > 0$ there exists k_0 such that

$$\int_\Omega |\nabla G_{k_0}(u_n)|^2 \leq \frac{\varepsilon}{2}.$$

Taking into account that $T_{k_0}(u_n)$ is strongly compact in $H^1_{\mathrm{loc}}(\Omega)$, it follows that ∇u_n is equiintegrable in $(L^2_{\mathrm{loc}}(\Omega))^N$. Hence, by Vitali theorem

$$u_n \to u \qquad \text{in } H^1_{\mathrm{loc}}(\Omega). \tag{2.9}$$

STEP 3. The procedure is to pass to the limit in the equation satisfied by the approximated solutions u_n, i.e., in

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi + \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2}\phi = \int_\Omega f_n \phi, \quad \forall \phi \in C_0^\infty(\Omega).$$

First of all, the weak convergence of u_n to u and the $*$-weak convergence of $M(x,u_n)$ to $M(x,u)$ in $L^\infty(\Omega)$ implies that

$$\lim_{n\to+\infty} \int_\Omega M(x,u_n)\nabla u_n \nabla\phi = \int_\Omega M(x,u)\nabla u \nabla\phi, \quad \forall \phi \in C_0^\infty(\Omega).$$

On the other hand, if we fix $\Omega_0 \subset\subset \Omega$ and we consider $E \subset\subset \Omega_0$, we deduce, using (2.6), that

$$\begin{aligned}
&\int_E \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \\
&\leq \int_{E\cap\{u_n\leq k\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} + \int_{E\cap\{u_n\geq k\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \\
&\leq \frac{b}{c_{\Omega_0}} \int_{E\cap\{u_n\leq k\}} |\nabla T_k(u_n)|^2 + \int_{\{u_n\geq k\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2}.
\end{aligned} \tag{2.10}$$

Let $\varepsilon > 0$ be fixed. Observe that if, for $k > 1$, we use $T_1(G_{k-1}(u_n))$ as test function in (2.1) and drop positive terms, it follows that

$$\int_{\{u_n\geq k\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \leq \int_{\{u_n\geq k-1\}} f_n \leq \int_{\{u_n\geq k-1\}} f.$$

Thus, since the right hand side tends to 0 uniformly in n as k diverges, we obtain the existence of $k_0 > 1$ such that

$$\int_{\{u_n \geq k\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \leq \frac{\varepsilon}{2}, \quad \forall k \geq k_0, \ \forall n \in \mathbb{N}.$$

Moreover, since $T_{k_0}(u_n)$ is strongly compact in $H^1_{\mathrm{loc}}(\Omega)$, there exist n_ε, δ_ε such that for every $E \subset\subset \Omega$ with $\mathrm{meas}\,(E) < \delta_\varepsilon$ we have

$$\int_{E \cap \{u_n \leq k_0\}} |\nabla T_{k_0}(u_n)|^2 < \frac{\varepsilon c_{\Omega_0}}{2b}, \quad \forall n \geq n_\varepsilon.$$

In conclusion, by (2.10), taking $k \geq k_0$ we see that $\mathrm{meas}\,(E) < \delta_\varepsilon$ implies

$$\int_E \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \leq \varepsilon, \quad \forall n \geq n_\varepsilon,$$

i.e., the sequence $\dfrac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2}$ is equiintegrable. This, together with its a.e. convergence to $\dfrac{Q(x,u)\nabla u \nabla u}{u}$, implies by Vitali theorem that

$$\lim_{n \to +\infty} \int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \phi = \int_\Omega \frac{Q(x,u)\nabla u \nabla u}{u} \phi.$$

It follows that, passing to the limit as n goes to infinity in the equation satisfied by u_n we deduce that

$$\int_\Omega M(x,u)\nabla u \nabla \phi + \int_\Omega \frac{Q(x,u)\nabla u \nabla u}{u} \phi = \int_\Omega f\phi, \quad \forall \phi \in C_0^\infty(\Omega),$$

i.e. $u \in H_0^1(\Omega)$ is a solution of

$$-\mathrm{div}\,(M(x,u)\nabla u) + \frac{Q(x,u)\nabla u \nabla u}{u} = f \quad \text{in } \Omega.$$

□

REMARK 2.2. – Using $v = T_k(u_n)/k$ as test function in (2.1), taking into account that $f_n \leq f$ in Ω, we have

$$\int_{\{u_n > 0\}} \frac{T_k(u_n)}{k} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} = \int_\Omega \frac{T_k(u_n)}{k} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \leq \int_\Omega f.$$

If we take the limit as k tends to zero, and we use that $u_n > 0$ in Ω, we get

$$\int_\Omega \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} = \int_{\{u_n > 0\}} \frac{u_n Q(x,u_n)\nabla u_n \nabla u_n}{\left(u_n + \frac{1}{n}\right)^2} \leq \int_\Omega f(x).$$

By applying Fatou lemma in the above inequality it follows that

$$\int_\Omega \frac{Q(x,u)\nabla u \nabla u}{u} \leq \int_\Omega f(x).$$

REMARK 2.3. – Now, we analyse the role of the parameter $a > 0$. For this, consider the model problem (1.4) and the function

$$h(s) = \begin{cases} a\dfrac{s^{\frac{a}{a-1}}}{a-1}, & a \neq 1, \\ \log(s), & a = 1. \end{cases}$$

Making the change of variables $w(x) = h(u(x))$, it is proved in Section 5.1 of [4] that u satisfies the differential equation in (1.4) if and only if w is a solution of $-\Delta w = f(x) g_a(w)$ in Ω, where

$$g_a(w) = \begin{cases} \dfrac{1}{a}\left(\dfrac{|a-1|}{a}\right)^{\frac{1}{1-a}} |w|^{\frac{1}{1-a}}, & a \neq 1, \\ e^{-w}, & a = 1. \end{cases}$$

Observe that in the case $a > 1$, the boundary condition in (1.4) means that $w = 0$ on $\partial\Omega$. Therefore (1.4) is equivalent to the b.v.p.

$$\begin{cases} w > 0 & \text{in } \Omega, \\ -\Delta w = f(x)\dfrac{1}{a}\left(\dfrac{(a-1)}{a}\right)^{\frac{1}{1-a}} \dfrac{1}{w^{\frac{1}{a-1}}}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

which has been studied at least with bounded f (see [9] and [13]). Remark explicitly that from this point of view the assumption $a \geq 2$ (observe that the hypothesis $a > 2$ is crucial for the existence result in [4]) implies that the above problem can be seen as the Euler-Lagrange equation of the coercive functional

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{1}{a}\left(\frac{(a-1)}{a}\right)^{\frac{1}{1-a}} \int_\Omega f(x) v^{\frac{a-2}{a-1}}, \quad f(x) \geq 0.$$

However, if $1 < a < 2$ (remind that Theorem 1.1 handle this case) $J(v)$ is not well-defined in $H_0^1(\Omega)$.

We also point out that if $a < 1$, formally the boundary condition becomes $\dfrac{a}{a-1} u^{\frac{a-1}{a}}(x) = w(x) \to -\infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$. This explains that the nature of problem (1.4) changes depending whether $a > 1$ or $a \leq 1$.

3. – A more general lower order term without sign condition.

In this section, combining the above ideas with those in [1], we extend Theorem 1.1 to cover the general problem (1.5) with a nonlinearity g which can be negative or changing of sign. Specifically, we assume that the function $g(x,s)$ verifies

$$-\mu/s \leq g(x,s) \leq h(s), \quad \forall s > 0, \quad \text{a.e. } x \in \Omega, \tag{3.1}$$

where $h : (0,+\infty) \longrightarrow (0,+\infty)$ is a function such that $sh(s)$ is increasing an

$$\lim_{s\to 0^+} \int_0^s \mathrm{e}^{-\frac{b}{a}\int_1^t h(r)dr} dt < +\infty \tag{3.2}$$

and $\mu > 0$. Since $sh(s)$ may be every nondecreasing function, we remark that no condition on the growth of $g(x,s)$ as s tends to infinity is imposed. Notice that (3.2) is a condition about the behavior of h near to 0. Consequently, if we take $h(s) = \frac{1}{s^\gamma}$, then (3.2) holds if and only if $\gamma < 1$ or if $\gamma = 1$ and $a > b$. Therefore, if we assume that $g(x,s)$ is bounded in $\Omega \times [\varepsilon, M]$ for every $M > \varepsilon > 0$, then a simple example in which (3.1) and (3.2) are satisfied is that, for $R > 0$ and $a > Rb$, the condition

$$-\frac{\mu}{s} \leq g(x,s) \leq \frac{R}{s}, \quad \text{for } s \text{ in a neighborhood of } 0 \quad \text{a.e. } x \in \Omega,$$

holds. We are thus led to the following strengthening of Theorem 1.1.

THEOREM 3.1. – *Let* $0 \leq f \in L^m(\Omega)$ *for some* $m \geq \frac{2N}{N+2}$ *with* $f \not\equiv 0$ *in* Ω *and assume that* (1.2), (1.3), (3.1) *and* (3.2) *hold. If* $a > a\mu$, *then there exists a solution* $u \in H_0^1(\Omega)$ *of* (1.5) *i.e.* u *satisfies* $u > 0$ *in* Ω, $g(x,u)Q(x,u)\nabla u \nabla u \in L^1_{\mathrm{loc}}(\Omega)$, *and*

$$\int_\Omega M(x,u)\nabla u \nabla \phi + \int_\Omega g(x,u)Q(x,u)\nabla u \nabla u \phi = \int_\Omega f\phi,$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

REMARK 3.2. – Observe that we improve the result in [1] because:

1)We do not assume that f is strictly positive in every compact subset of Ω.

2) A more general class of operators (not only linear like in [1]) is considered in the principal part of the equation and we deal with slightly more general lower order terms (in [1] it is assumed that Q is the identity matrix).

Outline of the proof of the Theorem 3.1. We approximate the function g by continuous functions $g_n\colon \Omega\times(0,+\infty)\to\mathbb{R}$ ($n\in\mathbb{N}$) defined by

$$g_n(x,s)=\begin{cases}0 & \text{if } s\le 0,\\[4pt] \dfrac{s^2 g(x,s)}{\left(s+\frac{1}{n}\right)^2} & \text{if } 0<s.\end{cases}$$

Observe that g_n verifies $g_n(x,s)\overset{n\to+\infty}{\longrightarrow} g(x,s)$, and, by (3.1), we have

$$g_n(x,s)s+\mu\ge 0 \quad \text{a.e. } x\in\Omega,\quad \forall s\in\mathbb{R}, \tag{3.3}$$

for every $n\in\mathbb{N}$.

We consider

$$\begin{aligned} -\operatorname{div}(M(x,u_n)\nabla u_n)+g_n(x,u_n)Q(x,u_n)\nabla u_n\nabla u_n &= f_n \quad \text{in } \Omega,\\ u_n &\in H_0^1(\Omega). \end{aligned} \tag{3.4}$$

If $f_n=T_n(f)$, using [14] there exists a solution $u_n\in H_0^1(\Omega)$ of (3.4) that belongs to $L^\infty(\Omega)$ (see [15]).

Let us take u_n as test function in (3.4) to conclude

$$\int_\Omega M(x,u_n)|\nabla u_n|^2+\int_\Omega g_n(x,u_n)Q(x,u_n)\nabla u_n\nabla u_n u_n=\int_\Omega f_n u_n.$$

Hence, by (1.2) and (1.3) we get

$$a\int_\Omega |\nabla u_n|^2+\int_\Omega a g_n(x,u_n)|\nabla u_n|^2 u_n\le\int_\Omega f_n u_n,$$

or, equivalently,

$$(a-a\mu)\int_\Omega |\nabla u_n|^2+\int_\Omega\Big[a g_n(x,u_n)|\nabla u_n|^2 u_n+a\mu|\nabla u_n|^2\Big]\le\int_\Omega f_n u_n.$$

Observing that, by (3.3), $s g_n(x,s)|\xi|^2+\mu|\xi|^2\ge 0$, a.e. $x\in\Omega$, for every $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$ we have

$$a s g_n(x,s)|\xi|^2+a\mu|\xi|^2\ge 0. \tag{3.5}$$

According to (3.5) and by the definition of f_n we have

$$(a-a\mu)\|u_n\|^2\le\int_\Omega f_n u_n\le\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|u_n\|_{L^{2^*}(\Omega)}\le S\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}.$$

Since $a>a\mu$ we obtain that the sequence u_n is bounded in $H_0^1(\Omega)$. Therefore, up to a subsequence, we have that $u_n\rightharpoonup u$ in $H_0^1(\Omega)$.

On the other hand, taking $u_n^- \equiv \min\{u_n, 0\}$ as test function in (3.4) and using the same ideas as before (thanks to that $a > a\mu$) we obtain that $u_n \geq 0$.

Taking into account (3.1) we can proceed analogously to the proof of Proposition 2.1 to show that u_n are uniformly away from zero in every compact set in Ω. Indeed, let $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. Take $\mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr}\phi$ as test function in (3.4), to get

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi \mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr} - \int_\Omega f_n \mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr}\phi$$
$$= \frac{b}{a}\int_\Omega M(x,u_n)\nabla u_n \nabla u_n h\left(u_n + \frac{1}{n}\right)\mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr}\phi$$
$$-\int_\Omega g_n(x,u_n)Q(x,u_n)\nabla u_n \nabla u_n \mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr}\phi.$$

Using (1.2) and (3.1), we have

$$\int_\Omega M(x,u_n)\nabla u_n \nabla\phi \mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr} \geq \int_\Omega f_n \mathrm{e}^{-\frac{b}{a}\int_1^{u_n+\frac{1}{n}} h(r)dr}\phi,$$

which is (2.2) with $h(s) = \dfrac{1}{s}$.

From (3.2), we deduce that the function $P(s) = \int_0^s \mathrm{e}^{-\frac{b}{a}\int_1^t h(r)dr} dt$ is well-defined. Consequently, if we fix $L > 0$ such that $\chi_{\{u_n \leq L\}} \to \chi_{\{u \leq L\}}$ a.e. $x \in \Omega$, we can follow the arguments of the Proposition 2.1 to conclude that u_n are uniformly away from zero in every compact set in Ω.

We proceed to show that, up to a subsequence, the sequence $\{u_n\}$ converges to a positive solution of (1.5) by following the ideas of the proof of Theorem 1.1. The main difference consists in proving a similar inequality to (2.8) without using the sign condition (1.6). To make that, let us choose $G_k(u_n)$ as test function in (3.4) to obtain

$$\int_\Omega M(x,u_n)\nabla u_n \nabla G_k(u_n) + \int_\Omega g_n(x,u_n)Q(x,u_n)\nabla u_n \nabla u_n G_k(u_n) = \int_\Omega f_n G_k(u_n).$$

Using (1.2), (1.3) and adding and substracting $\int_\Omega a\mu|\nabla G_k(u_n)|^2$ we have

$$(a - a\mu)\int_\Omega |\nabla G_k(u_n)|^2 + \int_{\{u_n \geq k\}} a g_n(x,u_n)|\nabla G_k(u_n)|^2 G_k(u_n) + a\mu|\nabla G_k(u_n)|^2$$
$$\leq \int_\Omega f_n G_k(u_n).$$

Thanks to (3.3) we deduce that $ag_n(x, u_n)|\nabla G_k(u_n)|^2 G_k(u_n) + a\mu|\nabla G_k(u_n)|^2 \geq 0$, and therefore we get

$$(a - a\mu)\int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega f_n G_k(u_n).$$

Since $a > a\mu$ we derive from the Hölder and Sobolev inequalities that

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq \frac{\mathcal{S}^2}{(a - a\mu)^2} \left(\int_{\{u_n \geq k\}} f^{\frac{2N}{N+2}} \right)^{1+\frac{2}{N}}$$

which plays the role of (2.8). Therefore, $|\nabla G_k(u_n)|^2$ is equiintegrable. Moreover, since

$$-\mathrm{div}(M(x, u_n)\nabla u_n) = f_n - g_n(x, u_n)Q(x, u_n)\nabla u_n \nabla u_n$$

and the right hand side is bounded in $L^1_{\mathrm{loc}}(\Omega)$, we can apply Lemma 1 of [5] to deduce that, up to (not relabeled) subsequences, ∇u_n converges to ∇u a.e. in Ω. Hence, by Vitali theorem $G_k(u_n) \to G_k(u)$ in $H^1_0(\Omega)$. Now, the convergence in $H^1_{\mathrm{loc}}(\Omega)$ of $T_k(u_n)$ to $T_k(u)$ is proved in a similar way to Step 1 of Theorem 1.1. Finally, we conclude the proof as in Step 3 of Theorem 1.1. $\square$

REMARK 3.3. – If $N = 2$ (which implies $2N/(N + 2) = 1$), then the results are also true provided that we strength the assumption $f \in L^{\frac{2N}{N+2}}(\Omega)$ by assuming $f \in L^m(\Omega)$ for $m > 1$.

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