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Algebraic Surfaces and Their Moduli Spaces: Real, Differentiable and Symplectic Structures

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Abstract. – *The theory of algebraic surfaces, according to Federigo Enriques, revealed 'riposte armonie' (hidden harmonies) who the mathematicians to undertook their investigation. Purpose of this article is to show that this holds still nowadays; and point out, while reviewing recent progress and unexpected new results, the many faceted connections of the theory, among others, with algebra (Galois group of the rational numbers), with real geometry, and with differential and symplectic topology of* 4 *manifolds.*

1. – Riposte armonie?

At the onset of the theory of algebraic surfaces (ca. 1870-1895) Noether, Enriques and Castelnuovo said that while algebraic curves had been made by God, algebraic surfaces were made by the devil.

But Enriques, in the final page 464 of his famous book 'Le Superficie Algebriche' ([45], published posthumously in 1949), says that really God created for algebraic surfaces a higher level order of 'hidden harmonies', where an incredible beauty shines. The richness and beauty of their properties, which were only after a long time and hard work revealed, should inspire not only a sense of awe in the contemplation of this divine order, but also the hope for researchers that difficulties, doubts and contradictions which pave their way to discovery are eventually to fade away and uncover this divine light of harmony.

What does it mean that everything goes 'right' for curves? In order to explain this, we must first of all recall the basic definition of a complex projective variety.

But why 'in primis' do we need complex projective varieties?

The point is that in general, in 'real life' (i.e., in applied science), one wants to solve polynomial equations with real coefficients, and find real solutions. But since by the fundamental theorem of algebra every 'univariate' polynomial $P(x) \in \mathbb{C}[x]$ (a polynomial in a single variable x) of degree n has exactly n complex roots (counted with multiplicity), the simple but basic approach is to view the real numbers as the subset of the complex numbers fixed by complex conjugation, hence the first approach is to start to look at the complex solutions, and only later to look at the action of complex conjugation on the set of complex solutions.

Given moreover a system of polynomial equations in several variables $(z_1, \dots, z_n)$, in order to have some continuity of the dependence of the solutions upon the choice of the coefficients, one reduces it to a system of homogeneous equations,

$$f_i(z_0, z_1, \dots, z_n) = 0, i = 1, \dots, r$$

defining an algebraic set or a projective Variety

$$X \subset \mathbb{P}^n_{\mathbb{C}}, X := \{z := (z_0, z_1, \dots, z_n) | f_i(z) = 0 \ \forall i = 1, \dots, r\}$$

(one finds then the solutions originally sought for by setting $z_0 = 1$).

We assume throughout here that X is a smooth complex projective variety, i.e., that X is a smooth compact submanifold of $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ of complex dimension d, and that X is connected.

Observe that X is a compact oriented real manifold of real dimension $2d$, so, for instance, a complex surface gives rise to a real 4-manifold.

For $d = 1$, X is a complex algebraic curve (a real surface, called also Riemann surface) and its basic invariant is the genus $g = g(X)$.

The genus g is defined as the dimension of the vector space $H^0(\Omega^1_X)$ of rational 1-forms

$$\eta = \sum_i \phi_i(z) dz_i$$

which are homogeneous of degree zero and are regular, i.e., do not have poles on X (the $\phi_i(z)$'s are here rational functions of z).

It turns out that the genus determines the topological and the differentiable manifold underlying X: its intuitive meaning is the 'number of handles' that one must add to a sphere in order to obtain X as a topological space.

Actually, as conjectured by Mordell and proven by Faltings ([46], [47], see also [17] for an 'elementary' proof), it also governs the arithmetic aspects of X: if the coefficients of the polynomial equations defining X belong to $\mathbb{Q}$, or more generally to a number field k (a finite extension of $\mathbb{Q}$), then the number of solutions with coordinates in $\mathbb{Q}$ (respectively: with coordinates in k) is finite if the genus $g = g(X) \geq 2$.

The rough classification of curves is the following:

- $g = 0$: $X \cong \mathbb{P}^1_{\mathbb{C}}$, topologically X is a sphere S^2 of real dimension 2.
- $g = 1$: $X \cong \mathbb{C}/\Gamma$, with Γ a discrete subgroup $\cong \mathbb{Z}^2$: X is called an elliptic curve, and topologically we have a real 2-torus $S^1 \times S^1$.
- $g \geq 2$: then we have a 'curve of general type', and topologically we have a sphere with g handles.

Moreover, X admits a metric of constant curvature, positive if $g = 0$, negative if $g \geq 2$, zero if $g = 1$.

Finally, we have a **Moduli space** $\mathfrak{M}_g$, an open set of a complex projective variety, which parametrizes the isomorphism classes of compact complex curves of genus g. $\mathfrak{M}_g$ is connected, and it has complex dimension $(3g-3)$ for $g \geq 2$.

Things do not seem so devilish when one learns that also for algebraic surfaces of general type there exist similar moduli spaces $\mathfrak{M}_{x,y}$ (by the results of [16], [54]).

Here, $\mathfrak{M}_{x,y}$ parametrizes isomorphism classes of minimal (smooth projective) surfaces of general type S such that $\chi(S) = x, K_S^2 = y$.

Again, these two numbers are determined algebraically, through the dimensions of certain vector spaces of differential forms without poles, namely, we have $\chi(S) := 1 - q(S) + p_g(S), K_S^2 := P_2(S) - \chi(S)$, where:

$$q(S) := dim_{\mathbb{C}} H^0(\Omega_S^1), p_g(S) := dim_{\mathbb{C}} H^0(\Omega_S^2), P_2(S) := dim_{\mathbb{C}} H^0(\Omega_S^{2\otimes 2}).$$

As does the genus of an algebraic curve, these numbers are determined by the topological structure of S.

BAD NEWS : WE SHALL SEE THAT THESE TWO NUMBERS DO NOT DETERMINE THE TOPOLOGY of S!

GOOD NEWS : $\mathfrak{M}_{x,y}$ HAS FINITELY MANY CONNECTED COMPONENTS!

The above finiteness statement is good news because the connected components of $\mathfrak{M}_{x,y}$ parametrize

Deformation classes of surfaces of general type, and, by a classical theorem of Ehresmann ([44]), deformation equivalent varieties are diffeomorphic.

Hence, fixed the two numerical invariants $\chi(S) = x, K_S^2 = y$, which are determined by the topology of S (indeed, by the Betti numbers of S), we have a finite number of differentiable types.

In the next section we shall accept the bad news, trying to learn something from them.

2. – Field automorphisms, the absolute Galois group and conjugate varieties.

Let $\phi : \mathbb{C} \to \mathbb{C}$ be a field automorphism: then, since

$$\phi(x) = \phi(1 \cdot x) = \phi(1) \cdot \phi(x),$$

it follows that $\phi(1) = 1$, therefore $\phi(n) = n \ \forall n \in \mathbb{N}$, and also $\phi|_{\mathbb{Q}} = Id_{\mathbb{Q}}$.

We recall the first algebra exercises, quite surprising for first semester students: the real numbers have no other automorphism than the identity, while the complex numbers have 'too many'.

LEMMA 2.1. – $Aut(\mathbb{R}) = \{Id\}$

PROOF. – For each choice of $x, a \in \mathbb{R}$, $\phi(x + a^2) = \phi(x) + \phi(a)^2$, thus ϕ of a square is a square, ϕ carries the set of squares $\mathbb{R}_+$ to itself, ϕ is increasing. But ϕ equals the identity on $\mathbb{Q}$: thus ϕ is the identity. *Q.E.D.*

On the other hand, the theory of transcendence bases and the theorem of Steiniz (any bijection between two transcendence bases $\mathcal{B}_1$ and $\mathcal{B}_2$ is realized a suitable automorphism) tell us:

LEMMA 2.2. – $|Aut(\mathbb{C})| = 2^{2^{\aleph_0}}$

REMARK 2.3. – Paolo Maroscia informed me, after the lecture, that the above result is the content of a short note of Beniamino Segre ([92]) published 60 years ago.

Observe now that the only continuous automorphisms of $\mathbb{C}$ are the identity and the complex conjugation σ, such that $\sigma(z) := \bar{z} = x - iy$. All the others are impossible (just very hard?) to visualize!

2) The field of algebraic numbers $\overline{\mathbb{Q}}$ is the subfield of $\mathbb{C}$, $\overline{\mathbb{Q}} := \{z \in \mathbb{C} | \exists P \in \mathbb{Q}[x] s.t.\ P(z) = 0\}$. It is carried to itself by any field automorphism of $\mathbb{C}$.

The fact that $Aut(\mathbb{C})$ is so large is essentially due to the fact that the kernel of $Aut(\mathbb{C}) \to Aut(\overline{\mathbb{Q}})$ is very large.

The group $Aut(\overline{\mathbb{Q}})$ is called the **absolute Galois group** and denoted by $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$. It is one of the most interesting objects of investigation in algebra and arithmetic.

Even if we have a presentation of this group, still our information about it is quite scarce. A presentation of a group G often does not even answer the question: is the group G nontrivial? The solution to this question is often gotten if we have a **representation** of the group G, for instance, an action of G on a set M that can be very well described.

For instance, M could be here a moduli space.

To explain how $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ acts on several moduli spaces M we come now to the crucial notion of a conjugate variety.

REMARK 2.4. – 1) $\phi \in Aut(\mathbb{C})$ acts on $\mathbb{C}[z_0, \dots, z_n]$, by sending $P(z) = \sum_{i=0}^{n} a_i z^i \mapsto \phi(P)(z) := \sum_{i=0}^{n} \phi(a_i) z^i$.

2) Let X be as above a projective variety

$$X \subset \mathbb{P}^n_{\mathbb{C}}, X := \{z | f_i(z) = 0 \forall i\}.$$

The action of ϕ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries X to another variety, denoted X^ϕ, and called the **conjugate variety**. Since $f_i(z) = 0$ implies

$\phi(f_i)(\phi(z)) = 0$, we see that

$$X^{\phi} = \{w | \phi(f_i)(w) = 0 \forall i\}.$$

If ϕ is complex conjugation, then it is clear that the variety X^{ϕ} that we obtain is diffeomorphic to X, but in general, what happens when ϕ is not continuous?

For curves, since in general the dimensions of spaces of differential forms of a fixed degree and without poles are the same for X^{ϕ} and X, we shall obtain a curve of the same genus, hence X^{ϕ} and X are diffeomorphic.

But for higher dimensional varieties this breaks down, as discovered by Jean Pierre Serre in the 60's ([93]), who proved the existence of a field automorphism $\phi \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^{ϕ} have non isomorphic fundamental groups.

In work in progress in collaboration with Ingrid Bauer and Fritz Grunewald ([14]) we discovered wide classes of algebraic surfaces for which the same phenomenon holds.

Shortly said, our method should more generally yield a way to transform the bad news into good news:

CONJECTURE 2.5 – *Assume that $\phi \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is different from the identity and from complex conjugation. Then there is a minimal surface of general type S such that S and S^{ϕ} have non isomorphic fundamental groups. In particular, S and S^{ϕ} are not homeomorphic.*

Hence the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli space of minimal surfaces of general type,

$$\mathfrak{M} := \cup_{x,y \geq 1} \mathfrak{M}_{x,y}.$$

Concerning the above two statements, we should observe that, while the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of connected components of $\mathfrak{M}$, it does not act on the set of isomorphism classes of fundamental groups of surfaces of general type: this means that, given two varieties X, X' with isomorphic fundamental groups, their conjugate varieties X^{ϕ}, X'^{ϕ} do not need to have isomorphic fundamental groups. Else, not only complex conjugation would not change the isomorphism class of the fundamental group, but also the minimal normal subgroup containing it (which is very large) would do the same.

Let me end this section giving a few hints about the main ideas and methods for our proposed approach, which depends on a single general conjecture about faithfulness of the action of the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the isomorphism classes of unmarked triangle curves.

An elementary but key lemma describes our candidate triangle curves for the above conjecture.

Fix a positive integer $g \in \mathbb{N}$, $g \geq 3$, and define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g-1\}$, C_a as the hyperelliptic curve of genus g given by

the equation

$$w^2 = (z-a)(z+2g)\prod_{i=0}^{2g-1}(z-i).$$

We have then:

LEMMA 2.6. – *Consider two complex numbers a, b such that $a \in \mathbb{C} \setminus \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.*

Through the above lemma algebraic numbers are therefore encoded into isomorphism class of curves.

We use then the method of proof of the well known Belyi theorem:

THEOREM 2.7. – **(BELYI)** *An algebraic curve C can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a holomorphic map $f : C \to \mathbb{P}^1_{\mathbb{C}}$ with branch set (set of critical values) equal to $\{0, 1, \infty\}$.*

Assume now that a is algebraic, i.e., that $a \in \overline{\mathbb{Q}}$: take a Belyi function for C_a (i.e., $f_a : C_a \to \mathbb{P}^1_{\mathbb{C}}$ with branch set $\{0, 1, \infty\}$) and its normal closure $D_a \to \mathbb{P}^1_{\mathbb{C}}$. We have then constructed a triangle curve D_a according to the following

DEFINITION 2.8. – *D is a* **TRIANGLE CURVE** *if there is a finite group G acting effectively on D and with the property that $D/G \cong \mathbb{P}^1_{\mathbb{C}}$, and the quotient map $f : D \to \mathbb{P}^1_{\mathbb{C}} \cong D/G$ has $\{0, 1, \infty\}$ as branch set.*

A **MARKED TRIANGLE CURVE** *is a triple (D, G, i) where D, G are as above, and where we have fixed an embedding $i : G \to Aut(D)$.*

Two marked triangle curves (D, G, i), (D', G', i) are isomorphic iff there exists isomorphisms $D \cong D'$, $G \cong G'$ which transform i into i'.

Let us explain now the basic idea which lies behind our new results: the theory of surfaces isogenous to a product, introduced in [25] (see also [26]), and which holds more generally also for higher dimensional varieties.

DEFINITION 2.9. – 1) *A surface* **isogenous to a (higher) product** *is a compact complex surface S which is a quotient $S = (C_1 \times C_2)/G$ of a product of curves of resp. genera ≥ 2 by the free action of a finite group G.*

2) *A* **Beauville surface** *is a surface isogenous to a (higher) product which is* **rigid***, i.e., it has no nontrivial deformation. This amounts to the condition, in the case where the two curves C_1 and C_2 are nonisomorphic, that (C_i, G) is a triangle curve.*

For surfaces isogenous to a product holds the following ([25], [26]):

THEOREM 2.10. – *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding subset of the moduli space $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$, corresponding to surfaces homeomorphic, resp, diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

If S is a Beauville surface this implies: $X \cong S$ or $X \cong \bar{S}$. It follows also that a Beauville surface is defined over $\overline{\mathbb{Q}}$, whence the Galois group $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ operates on the discrete subset of the moduli space $\mathfrak{M}$ corresponding to Beauville surfaces.

Work in progress with the same coauthors (Ingrid Bauer and Fritz Grunewald) aims at proving also the following

CONJECTURE 2.11 – *The absolute Galois group $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ operates faithfully on the discrete subset of the moduli space $\mathfrak{M}$ corresponding to Beauville surfaces.*

Already established is the following ([14])

THEOREM 2.12. – *Beauville surfaces yield explicit examples of conjugate surfaces with nonisomorphic fundamental groups whose completions are isomorphic (the completion of a group G is the inverse limit*

$$\hat{G} = lim_{K<G \text{ normal of finite index}} \ (G/K)$$

Our candidate triangle curves C_a determine now a family $\mathfrak{N}_a$ consisting of all the possible surfaces isogenous to a product of the form $S := (D_a \times D')/G$, where the genus of D' is fixed, and G acts without fixed points on D'.

Using the theory of surfaces **isogenous to a product**, referred to above, follows easily that:

1) $\mathfrak{N}_a$ is a union of connected components of $\mathfrak{M}$

2) $\phi(\mathfrak{N}_a) = \mathfrak{N}_{\phi(a)}$.

Assume that for each $\phi \in Aut(\overline{\mathbb{Q}})$ which is nontrivial we can find a such that, setting $b := \phi(a)$:

3) $a \neq b$ and $\mathfrak{N}_a$ and $\mathfrak{N}_b$ do not intersect.

The desired conclusion would then be that, since $\phi(\mathfrak{N}_a)$ and $\mathfrak{N}_a$ do not intersect by 2), 3), hence ϕ acts nontrivially on the set of connected components of $\mathfrak{M}$.

The condition that $\mathfrak{N}_a$ and $\mathfrak{N}_b$ intersect easily implies, by the structure theorem for surfaces isogenous to a product, that the two triangle curves D_a and D_b are isomorphic. There is thus an isomorphism $F : D_a \to D_b$ which

transforms the action of G_a on D_a into the action of G_b on D_b. Identifying G_a with G_b under the transformation ϕ, one sees however that F is only 'twisted' equivariant. This means that there is an isomorphism $\psi \in Aut(G)$ such that $F(g(x)) = \psi(g)(x)$.

If (D_a, G, i_a) and (D_b, G, i_b) are isomorphic as marked triangle curves (for instance, if ψ is inner), then it follows that C_a is isomorphic to C_b, and we derive a contradiction, that $a = b$. In other words, our previous lemma shows that the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of marked triangle curves.

The main point is to find such an $a \neq b = \phi(a)$ with the above property that the group G has only inner automorphisms.

Indeed, the only crucial property which should be proven amounts to the following

CONJECTURE 2.13 – *The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of (unmarked) triangle curves.*

There are other interesting open problems:

QUESTION 1. – Existence and classification of Beauville surfaces, i.e.,

a) which finite groups G can occur?

b) classify all possible Beauville surfaces for a given finite group G.

We have made a substantial progress ([11], [12]) on question a) which leads for instance to substantial evidence in the direction of the following

CONJECTURE 2.14 – *Every finite nonabelian simple group occurs except $\mathfrak{U}_5$.*

3. – Surfaces of general type, DEF= DIFF? and beyond.

Let S be a minimal surface of general type: then we saw that to S we attach two positive integers ≥ 1

$$x = \chi(\mathcal{O}_S), \ y = K_S^2$$

which are invariants of the oriented topological type of S (they are determined by the Euler number $e(S)$ and by the signature $\tau(S)$).

The **moduli space** $\mathfrak{M}_{x,y}$ of the surfaces with invariants (x, y) is a quasi-projective variety defined over the integers, and it has a finite number of irreducible components.

For fixed (x, y) we have thus a finite number of possible differentiable types, and a fortiori a finite number of topological types.

Michael Freedman's big Theorem of 1982 ([48]) shows that there are indeed at most two topological structures if moreover the surface S is assumed to be simply connected (i.e., with trivial fundamental group).

Topologically, our 4-manifold is then obtained from very simple building blocks, one of them being the K3 surface, where:

DEFINITION 3.1. – *A* **K3** *surface is a smooth surface of degree* 4 *in* $\mathbb{P}^3_{\mathbb{C}}$.

Observe moreover that a complex manifold carries a natural orientation corresponding to the complex structure, and, in general, given an oriented differentiable manifold M, M^{opp} denotes the same manifold, but endowed with the opposite orientation. This said, we can explain the corollary of Freedman's theorem for the topological manifolds underlying simply connected (compact) complex surfaces.

There are two cases which are distinguished as follows:

- S is EVEN, i.e., its intersection form on $H^2(S, \mathbb{Z})$ is even: then S is a connected sum of copies of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and of a $K3$ surface if the signature $\tau(S)$ is negative, and of copies of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and of a $K3$ surface with reversed orientation if the signature is positive.
- S is ODD : then S is a connected sum of copies of $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}^{2\ opp}_{\mathbb{C}}$.

We recall that the connected sum is the operation which, from two oriented manifolds M_1 and M_2, glues together the complements of two open balls $B_i \subset M_i$, having a differentiable (resp. : tame) boundary, by identifying together the two boundary spheres via an orientation reversing diffeomorphism.

Kodaira and Spencer defined quite generally ([68]) for compact complex manifolds X, X' the equivalence relation called **deformation equivalence**: this, for surfaces of general type, means that the corresponding isomorphism classes yield points in the same connected component of the moduli space $\mathfrak{M}$. The cited theorem of Ehresmann guarantees that DEF $\Rightarrow$ DIFF:

REMARK 3.2. – Deformation equivalence implies the existence of a diffeomorphism carrying the canonical class K_X to the canonical class $K_{X'}$.

In the 80' s, groundbreaking work of Simon Donaldson ([36], [37], [38], [39], see also [43]) showed that homeomorphism and diffeomorphism differ drastically for projective surfaces.

REMARK 3.3. – A refinement of Donaldson's theory, made by Seiberg and Witten (see [98], [40], [90]), showed then more easily that a diffeomorphism $\phi : S \to S'$ between minimal surfaces of general type satisfies $\phi^*(K_{S'}) = \pm K_S$.

Based on the successes of gauge theory, the following conjecture was made

(I had been writing five years before the opposite conjecture, in [65], but almost no one believed in it):

FRIEDMAN-MORGAN'S SPECULATION (1987) (see [51]): DEF = DIFF (Differentiable equivalence and deformation equivalence coincide for surfaces).

However, finally the question was answered negatively in every possible way ([83], [66], [26], [28], [11], see also [31] for a rather comprehensive survey):

THEOREM 3.4. – *(Manetti '98, Kharlamov -Kulikov 2001, C. 2001, C. - Wajnryb 2004, Bauer- C. - Grunewald 2005)*

The Friedman- Morgan speculation does not hold true.

- (1) *Manetti used* $(\mathbb{Z}/2)^r$*-covers of blow ups of the quadric* $Q := \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$*, his surfaces have* $b_1 = 0$*, but are not simply connected.*
- (2) *Kharlamov and Kulikov used quotients* S *of the unit ball in* $\mathbb{C}^2$*: the surfaces they use are rigid but with infinite fundamental group.*
- (3) *I used non rigid surfaces isogenous to a product* $S = (C_1 \times C_2)/G$*, thus with* $b_1 > 0$ *and a fortiori the surfaces have infinite fundamental group.*
- (4) *The examples given with Bauer and Grunewald are Beauville surfaces, again the surfaces are rigid, thus they have* $b_1 = 0$ *but infinite fundamental group.*
- (5) *The examples obtained with Wajnryb are instead simply connected, i.e., they have trivial fundamental group.*

Common feature of (2), (3) and (4) : we take S **and the conjugate surface** $\bar{S}$ **(thus** $\bar{S}, S$ **are diffeomorphic), and if** $\bar{S}$ **and** S **were deformation equivalent, there would be a self-diffeomorphism** ψ **of** S **with** $\psi^*(K_S) = -K_S$**. If** ψ **exists, it should be antiholomorphic (by general properties of these surfaces). The technical heart of the proof is to construct examples where this cannot happen, and the fundamental group is heavily used for this issue.**

After the first counterexamples were found, the following weaker conjectures were posed:

4. – Weakenings of the conjecture by Friedman and Morgan.

- (I) require a diffeomorphism $\phi : S \to S'$ with $\phi^*(K_{S'}) = K_S$.
- (II) require the surfaces to be simply connected (1-connected).

Even these weaker conjectures were disproven in my joint work with Bronek Wajnryb ([28]), which I will now briefly describe.

The simply connected examples we used, called **abc-surfaces**, are a special case of a class of surfaces which I introduced in 1982 ([19]), namely, bidouble covers of the quadric and their natural deformations.

Bidouble covers of the quadric are smooth projective complex surfaces S endowed with a (finite) Galois covering $\pi : S \to Q := \mathbb{P}^1 \times \mathbb{P}^1$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$.

More concretely, they are defined by a single pair of equations

$$z^2 = f_{(2a,2b)}(x_0, x_1; y_0, y_1)$$
$$w^2 = g_{(2c,2d)}(x_0, x_1; y_0, y_1)$$

where $a, b, c, d \in \mathbb{N}^{\geq 3}$ and the notation $f_{(2a,2b)}$ denotes that f is a bihomogeneous polynomial, homogeneous of degree $2a$ in the variables x, and of degree $2b$ in the variables y.

These surfaces are simply connected and minimal of general type, and they were introduced in [19] in order to show that the moduli spaces $\mathfrak{M}_{\chi,K^2}$ of smooth minimal surfaces of general type S with $K_S^2 = K^2, \chi(S) := \chi(\mathcal{O}_S) = \chi$ need not be equidimensional or irreducible (and indeed the same holds for the open and closed subsets $\mathfrak{M}^{00}_{\chi,K^2}$ corresponding to simply connected surfaces).

Given in fact our four integers $a, b, c, d \in \mathbb{N}^{\geq 3}$, considering the socalled natural deformations of these bidouble covers, defined by equations [1]

$$z^2 = f_{(2a,2b)}(x_0, x_1; y_0, y_1) + w\ \Phi_{(2a-c,2b-d)}(x_0, x_1; y_0, y_1)$$
$$w^2 = g_{(2c,2d)}(x_0, x_1; y_0, y_1) + z\ \Psi_{(2c-a,2d-b)}(x_0, x_1; y_0, y_1)$$

one defines a bigger open subset $\mathcal{N}'_{a,b,c,d}$ of the moduli space, whose closure $\overline{\mathcal{N}'}_{a,b,c,d}$ is an irreducible component of $\mathfrak{M}_{\chi,K^2}$, where $\chi = 1 + (a-1)(b-1) + (c-1)(d-1) + (a+c-1)(b+d-1)$, and $K^2 = 8(a+c-2)(b+d-2)$.

The **abc-surfaces** are obtained as the special case where $b = d$, and the upshot is that, once the values of the integers b and $a + c$ are fixed, one obtains diffeomorphic surfaces.

In other more technical words the **abc-surfaces** are the **natural deformations** of $(\mathbb{Z}/2)^2$-covers of $(\mathbb{P}^1 \times \mathbb{P}^1)$, of *simple type* $(2a, 2b), (2c, 2b)$, which means that they are defined by equations

$$(***) \qquad z_{a,b}^2 = f_{2a,2b}(x, y) + w_{c,b}\phi_{2a-c,b}(x, y)$$
$$w_{c,b}^2 = g_{2c,2b}(x, y) + z_{a,b}\psi_{2c-a,b}(x, y)$$

where f,g ,ϕ, ψ, are bihomogeneous polynomials, belonging to respective vector spaces of sections of line bundles:

$f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a, 2b))$,
$\phi \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a - c, b))$ and
$g \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2c, 2d))$,
$\psi \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2c - a, b))$.

[1] In the following formula, a polynomial of negative degree is identically zero.

The main new result of [28] is the following

THEOREM 4.1. – **(C. -Wajnryb)** *Let S be an (a,b,c) - surface and S' be an $(a+1,b,c-1)$-surface. Moreover, assume that $a,b,c-1 \geq 2$. Then S and S' are diffeomorphic.*

This result couples then with a more technical result:

THEOREM 4.2. – **(C. -Wajnryb)** *Let S, S' be simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types ((2a, 2b),(2c, 2b), and (2a + 2k, 2b),(2c − 2k, 2b), and assume*

- (I) *a,b,c,k are strictly positive even integers with $a,b,c-k \geq 4$*
- (II) *$a \geq 2c+1$,*
- (III) *$b \geq c+2$ and either*
- (IV1) *$b \geq 2a+2k-1$ or*
  (IV2) *$a \geq b+2$*

Then S and S' are not deformation equivalent.

The second theorem uses techniques which have been developed in a long series of papers by the author and by Marco Manetti in a period of almost 20 years.

They use essentially

i) the local deformation theory á la Kuranishi, but for the canonical models,

ii) normal degenerations of smooth surfaces and a study of quotient singularities of rational double points and of their smoothings.

A detailed expositions for both theorems can be found in the Lecture Notes of the C.I.M.E. courses 'Algebraic surfaces and symplectic 4-manifolds' (see especially [31]).

The result in differential topology obtained with Wajnryb is based instead on a refinement of Lefschetz theory obtained by Kas ([64]).

This refinement allows us to encode the differential topology of a 4-manifold X Lefschetz fibred over $\mathbb{P}^1_{\mathbb{C}}$ (i.e., $f : X \to \mathbb{P}^1_{\mathbb{C}}$ has the property that all the fibres are smooth and connected, except for a finite number which have a nodal singularity) into an equivalence class of factorization of the identity in the Mapping class group $\mathcal{M}ap_g$ of a compact curve C of genus g.

The mapping class group, introduced by Max Dehn ([35]) in the 30's, is defined for each manifold M as

$$\mathcal{M}ap(M) := Diff(M)/Diff^0(M),$$

where $Diff^0(M)$ is the connected component of the identity, the so called subgroup of the diffeomorphisms which are **isotopic to the identity**.

A major advance in our knowledge of $\mathcal{M}ap_g$ was made by Hatcher and

Thurston ([61]), and the simplest known presentation of this group is due to Wajnryb ([95], [96]).

Verifying isotopy of diffeomorphisms is a difficult and very geometric task, which is accomplished in [28] by constructing chains of loops in the complex curve C, which lead to a dissection of C into open cells. One has then to choose several associate Coxeter elements to express a given diffeomorphism, used for glueing two manifolds with boundary M_1 and M_2 in two different ways, as a product of certain Dehn twists. This expression shows that this diffeomorphism extends to the interior of M_1, hence that the results of the two glueing operations yield diffeomorphic 4-manifolds.

I would like to finish in the next section commenting on some very interesting open problems.

To discuss them, I need to explain the connection with the theory of symplectic manifolds.

5. – Symplectic manifolds.

DEFINITION 5.1. – *A pair (X, ω) of a real manifold X, and of a real differential 2-form ω is called a* **Symplectic pair** *if*

i) *ω is a symplectic form, i.e., ω is closed ($d\omega = 0$) and ω is nondegenerate at each point (thus X has even real dimension).*

A symplectic pair (X, ω) is said to be **integral** *iff the De Rham cohomology class of ω comes from $H^2(X, \mathbb{Z})$, or equivalently, there is a complex line bundle L on X such that ω is a first Chern form of L.*

An **almost complex structure** *J on X is a differentiable endomorphism of the real tangent bundle of X satisfying $J^2 = -1$. It is said to be*

ii) **compatible with** *ω if*

$$\omega(Jv, Jw) = \omega(v, w)$$

iii) **tame** *if the quadratic form $\omega(v, Jv)$ is strictly positive definite.*

Finally, a symplectic manifold is a manifold admitting a symplectic form ω.

For long time (before the celebrated examples of Kodaira and Thurston, [72], [94]) the basic examples of symplectic manifolds were given by symplectic submanifolds of the flat space $\mathbb{C}^n$ and of Kähler manifolds, in particular of the projective space $\mathbb{P}^N_{\mathbb{C}}$, which possesses the Fubini-Study form

DEFINITION 5.2. – *The Fubini-Study form is the differential 2-form*

$$\omega_{FS} := \frac{i}{2\pi} \partial\overline{\partial} \log |z|^2,$$

where z is the homogeneous coordinate vector representing a point of $\mathbb{P}^N$.

In fact the above 2-form on $\mathbb{C}^{N+1} \setminus \{0\}$ is invariant

1) *for the action of $U(N+1, \mathbb{C})$ on homogeneous coordinate vectors*

2) *for multiplication of the vector z by a nonzero holomorphic scalar function $f(z)$ (z and $f(z)z$ represent the same point in $\mathbb{P}^N$), hence*

3) *ω_{FS} descends to a differential form on $\mathbb{P}^N$, being $\mathbb{C}^*$-invariant.*

I recently observed ([27], [30]):

THEOREM 5.3. – *A minimal surface of general type S has a symplectic structure (S, ω), unique up to symplectomorphism, and invariant for smooth deformation, with class$(\omega) = K_S$. This symplectic structure is called the* **canonical symplectic structure**. *A similar result holds for higher dimensional complex projective manifolds with ample canonical divisor K_X.*

The above result is, in the case where K_S is ample, a rather direct consequence of the famous Moser's lemma ([91]).

LEMMA 5.4. – *Let $f : \Xi \to T$ a proper submersion of differentiable manifolds, with T connected, and let (ω) be a 2-form on Ξ whose fibre restriction $\omega_t := \omega|_{X_t}$ makes each X_t a symplectic manifold.*

If the class of (ω_t) in De Rham cohomology is constant, then the (X_t, ω_t)'s are all symplectomorphic.

In the above theorem, when K_S is ample, it suffices to pull-back $1/m$ of the Fubini-Study metric by an m-canonical embedding.

In the case where K_S is not ample, the proof is more involved and uses techniques from the following symplectic approximation theorem

THEOREM 5.5. – *Let $X_0 \subset \mathbb{P}^N$ be a projective variety with isolated singularities each admitting a smoothing.*

Assume that for each singular point $x_h \in X$, we choose a smoothing component $T_{j(h)}$ in the basis of the semiuniversal deformation of the germ (X, x_h).

Then (obtaining different results for each such choice) X can be approximated by symplectic submanifolds W_t of $\mathbb{P}^N$, which are diffeomorphic to the glueing of the 'exterior' of X_0 (the complement to the union $B = \cup_h B_h$ of suitable (Milnor) balls around the singular points) with the Milnor fibres $\mathcal{M}_h$, glued along the singularity links $\mathcal{K}_{h,0}$.

An important consequence is the following theorem ([27], [30], see also [31] for a survey including the basics concerning the construction of the Manetti surfaces)

THEOREM 5.6. – *Manetti's surfaces yield examples of surfaces of general type which are not deformation equivalent but are canonically symplectomorphic.*

QUESTIONS: 1) Are there (minimal) surfaces of general type which are orientedly diffeomorphic through a diffeomorphism carrying the canonical class to the canonical class, but, endowed with their canonical symplectic structure, are not canonically symplectomorphic?

2) Are there such simply connected examples ?

3) Are the diffeomorphic abc-surfaces canonically symplectomorphic (thus yielding a counterexample to Can. Sympl = Def also in the simply connected case)?

I am currently working with Wajnryb and Lönne on the very difficult problem of understanding the canonical symplectic structures of abc-surfaces ([32]).

To explain our result, let us go back to our equations

$$(***) \qquad z_{a,b}^2 = f_{2a,2b}(x,y) + w_{c,b}\phi_{2a-c,b}(x,y)$$
$$w_{c,b}^2 = g_{2c,2b}(x,y) + z_{a,b}\psi_{2c-a,b}(x,y)$$

where f, g are bihomogeneous polynomials as before, and instead we allow ϕ, ψ, in the case where for instance the degree relative to x is negative, to be an antiholomorphic polynomial in x. In other words, we allow ϕ, ψ to be sections of certain line bundles which are dianalytic (holomorphic or antiholomorphic) in each variable x, y.

In this way we obtain a symplectic 4-manifold which (we call a dianalytic perturbation and) is canonically symplectomorphic to the bidouble cover we started with. But now we have gained that, for general choice of f, g, ϕ, ψ, the projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is **generic** and its branch curve B (the locus of the critical values) is a dianalytic curve with nodes and cusps as only singularities.

The only price one has to pay is to allow also negative nodes, i.e., nodes which in local holomorphic coordinates are defined by the equation

$$(y - \bar{x})(y + \bar{x}) = 0.$$

Now, projection onto the first factor $\mathbb{P}^1$ gives a movement of n points in a fibre $\mathbb{P}^1$, which is encoded in the so called **vertical braid monodromy factorization**.

The first result that we have achieved is the computation of this vertical braid monodromy factorization of the above branch curve $B \subset \mathbb{P}^1 \times \mathbb{P}^1$.

The second very interesting result that we have obtained, and which is too complicated to explain here in detail, is that certain invariants of these vertical braid monodromy factorizations allow to reconstruct all the three numbers a, b, c and not only the numbers $b, a + c$, which determine the diffeomorphism type.

This result represent the first positive step towards the realization of a more general program set up by Moishezon ([86], [87]) in order to produce braid

monodromy invariants which should distinguish connected components of the moduli spaces $\mathfrak{M}_{\chi,K^2}$.

Moishezon's program is based on the consideration (assume here for simplicity that K_S is ample) of a general projection $\psi_m : S \to \mathbb{P}^2$ of a pluricanonical embedding $\psi_m : S \to \mathbb{P}^{P_m-1}$, and of the braid monodromy factorization corresponding to the (cuspidal) branch curve Δ_m of ψ_m.

An invariant of the connected component is here given by the equivalence class (for Hurwitz equivalence plus simultaneous conjugation) of this braid monodromy factorization. Moishezon, and later Moishezon-Teicher calculated a coarser invariant, namely the fundamental group $\pi_1(\mathbb{P}^2 - B_m)$. This group turned out to be not overly complicated, and in fact, as shown in many cases in [8], it tends to give no extra information beyond the one given by the topological invariants of S (such as χ, K^2).

Auroux and Katzarkov showed instead ([5]) that, for $m \gg 0$, a more general equivalence class (called m-equivalence class, and allowing creation of a pair of neighbouring nodes, one positive and one negative) of the above braid monodromy factorization determines the canonical symplectomorphism class of S, and conversely.

The work by Auroux, Katzarkov adapted Donaldson's techniques for proving the existence of symplectic Lefschetz fibrations ([41], [42]) in order to show that each symplectic 4-manifold is in a natural way 'asymptotically' realized by a generic symplectic covering of $\mathbb{P}^2_{\mathbb{C}}$, given by almost holomorphic sections of a high multiple $L^{\otimes m}$ of a complex line bundle L whose class is the one of the given integral symplectic form.

The methods of Donaldson on one side, Auroux and Katzarkov on the other, use algebro geometric methods in order to produce invariants of symplectic 4-manifolds.

For instance, in the case of a generic symplectic covering of the plane, we get a corresponding branch curve Δ_m which is a symplectic submanifold with singularities only nodes and cusps.

To Δ_m corresponds then a factorization in the braid group, called m-th braid monodromy factorization: it contains only factors which are conjugates of σ_1^j, not only with $j = 1, 2, 3$ as in the complex algebraic case, but also with $j = -2$ (here σ_1 is a standard half twist on a segment connecting two roots, the first of the standard Artin generators of the braid group).

Although the factorization is not unique (because it may happen that a pair of two consecutive nodes, one positive and one negative, may be created, or disappear) one considers its m-equivalence class, and the authors show that this class, for $m \gg 1$, is an invariant of the integral symplectic manifold.

In the case of abc-surfaces, consider now again the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$, and let $p : S \to \mathbb{P}^2$ be the morphism obtained as the composition of $\pi : S \to Q$ with the standard (Segre) embedding $Q \hookrightarrow \mathbb{P}^3$ and with a general projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$.

In the special case of those particular abc-surfaces such that $a + c = 2b$, the m-th pluricanonical mapping $\psi_m : S \to \mathbb{P}^{P_m - 1}$ has a (non generic) projection given by the composition of p with a Fermat type map $v_r : \mathbb{P}^2 \to \mathbb{P}^2$ (given by $v_r(x_0, x_1, x_2) = (x_0^r, x_1^r, x_2^r)$ in a suitable linear coordinate system), where $r := m(a + c - 2)$.

Let B'' be the branch curve of a generic perturbation of p: then the braid monodromy factorization corresponding to B'' can be calculated from the vertical and horizontal braid monodromies put together.

The problem of calculating the braid monodromy factorization corresponding instead to the (cuspidal) branch curve Δ_m starting from the braid monodromy factorization of B'' has been addressed, in greater generality but in the special case $m = 2$, by Auroux and Katzarkov ([9]). Iteration of their formulae should lead to the calculation of the braid monodromy factorization corresponding to the (cuspidal) branch curve Δ_m in the case, sufficient for applications, where m is a sufficiently large power of 2.

Whether these formidable calculations will yield factorizations whose m-equivalence is for us decidable is still an open question: but in both directions the result would be extremely interesting, leading either to

i) a counterexample to the speculation DEF = CAN. SYMPL also in the simply connected case, or to

ii) examples of diffeomorphic but not canonically symplectomorphic simply connected algebraic surfaces.

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