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Abstract. – *We consider an energy functional on measures in $\mathbb{R}^2$ arising in superconductivity as a limit case of the well-known Ginzburg Landau functionals. We study its gradient flow with respect to the Wasserstein metric of probability measures, whose corresponding time evolutive problem can be seen as a mean field model for the evolution of vortex densities. Improving the analysis made in [AS], we obtain a new existence and uniqueness result for the evolution problem.*

1. – Introduction.

Let Ω be a bounded open connected region in $\mathbb{R}^2$ with smooth boundary, and denote with $P(\overline{\Omega})$ the space of probability measures over $\overline{\Omega}$. We are concerned with the following evolution problem:

$$\frac{d}{dt}\,\mu(t) - \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)}\mu(t)) = 0 \quad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2) \tag{1}$$

with the initial datum $\mu(0) = \mu_0 \in P(\overline{\Omega}) \cap H^{-1}(\Omega)$. We look for a solution $\mu(t)$ which is a measure in $P(\overline{\Omega}) \cap H^{-1}(\Omega)$. For every t the velocity field $-\chi_\Omega \nabla h_\mu$ and μ are coupled by

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ \qquad\quad h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Clearly, $H^{-1}(\Omega)$ is the natural ambient space for the problem, so we are working with measures on $\overline{\Omega}$ in order to treat masses in Ω which vary during the evolution. Masses on $\overline{\Omega}$ are also normalized to 1 without loss of generality.

Let $\mathcal{M}_+(\Omega)$ be the space of nonnegative measures on Ω, and consider, for $\mu \in \mathcal{M}_+(\Omega)$, the functionals

$$\Phi_\lambda(\mu) = \frac{\lambda}{2}\,\mu(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \quad \lambda \geq 0. \tag{3}$$

For measures μ on $\overline{\Omega}$ we will write $\mu = \hat{\mu} + \widetilde{\mu}$, where $\hat{\mu} = \chi_\Omega\,\mu$ and $\widetilde{\mu} = \chi_{\partial\Omega}\,\mu$. Functionals (3), defined in $\mathcal{M}_+(\Omega)$, will be understood to be defined as $\Phi_\lambda(\hat{\mu})$ for $\mu \in P(\overline{\Omega})$. So, they depend only on the internal part of the measures.

It is shown in [AS] that equation (1), with the coupling described by (2), can be viewed as a *gradient flow* of functionals (3) with respect to the structure induced on $P(\overline{\Omega})$ by the 2-Wasserstein distance $W_2(\cdot,\cdot)$ (see Section 2 below). So in [AS] the problem is studied exploiting the techniques of gradient flows in metric spaces developed in [AGS], and a global existence result is proved making use of the following, classical time discretization: given μ_τ^k, μ_τ^{k+1} is chosen among the minimizers of

$$\min_{\nu\in P(\overline{\Omega})} \Phi_\lambda(\nu) + \frac{1}{2\tau} W_2^2(\mu_\tau^k,\nu). \tag{4}$$

Here $\mu_\tau^0 = \mu_0$ and τ stands for the step of the scheme (see [JKO, AGS]).

In particular, once the sequence of minimizers of the discrete scheme is found, then a family of measures $\mu(t)$ is built as the limit of some subsequence of interpolations (it is a generalized minimizing movement, see [AGS, Chapter 2]). The general theory of gradient flows ensures that this limit satisfies a continuity equation with a suitable velocity field. Finally, this velocity field is shown to be the same as in problem (1), by means of suitable Euler-Lagrange equations associated to problem (4).

In [AS], thanks the introduction of some "entropies" which are shown to decrease along the flow, a regularity result is also obtained, that is, if the initial datum μ_0 is such that $\hat{\mu}_0 \in L^p(\Omega)$, $p \geq 4/3$, then there exist a global solution $\mu(t)$ such that $\|\hat{\mu}(t)\|_p$ is uniformly controlled by the L^p norm of $\hat{\mu}_0$.

Finally, in the case $p = +\infty$, a short time uniqueness theorem is established in [AS, Theorem 3.6]. The argument therein cannot deal with the presence of mass on $\partial\Omega$, so that it holds until some mass reaches the boundary during evolution, preventing the result to be global in time.

Main theorem.

In this paper we study further properties of minimizers of (4), in order to obtain global uniqueness for measures with L^∞ interior part and complete a well-posedness picture. Our main result, which will be proven in the last section (see Theorem 4.2), reads as follows:

let Ω be convex, $\hat{\mu}_0 \in L^\infty(\Omega)$ and $T > 0$. Problem (1)-(2) *possesses a unique solution satisfying $\|\hat{\mu}(t)\|_\infty \in L^\infty(0,T)$ and, for $t \in (0,T]$,*

$$\langle \nabla h_{\mu(t)}(x), y - x\rangle \geq 0 \ \textit{for all}\ (x,y) \in \operatorname{supp}(\tilde{\mu}(t)) \times \overline{\Omega}. \tag{5}$$

We stress that we allow $\mu(t)$ to have a nonzero boundary part.

Concerning the new condition (5), we will show later in Section 3 that it is a byproduct of our Wasserstein variational approach. In Theorem 3.1 we will indeed prove the analogous property for discrete minimizers of (4), in the case $\lambda = 0$.

Notice that, since the domain is supposed to be convex, (5) can be interpreted as follows: the gradient of $h_{\mu(t)}$ on the boundary (whenever some mass is there present) points towards the interior of the domain. This is in fact reminiscent of the nondecreasing boundary mass condition appearing in [AS, Definition 3.1], which is meaningful since a gradient flow of Φ_λ, at least for $\lambda > 0$, is expected to enjoy such a behavior (see the energy comparison argument in [AS, Section 3]).

Plan of the paper.

In Section 2 we briefly discuss the physical relevance of the functionals. We then recall some definitions and already known properties, also in connection with the Wasserstein structure, that we introduce. Moreover, we formally show that equation (1) represents the gradient flow of Φ_0. In Section 3 we perform our variational argument, which allows us to obtain the discrete version of (5). In Section 4 we prove the existence of solutions satisfying (5) and the main uniqueness result.

2. – The functionals.

The well known Ginzburg-Landau energy functional is

$$J(u,A) = \frac{1}{2}\int\limits_{\Omega} |\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2, \tag{6}$$

where $\Omega \subset \mathbb{R}^2$ is the section of the superconductor, h_{ex} represents the intensity of an external magnetic field, constant and orthogonal to the section, A is the potential vector of the magnetic field h induced in the material ($h = \nabla \times A$ and $\nabla_A = \nabla - iA$), and ε is a parameter depending on the material. The function u takes complex values and its modulus ($|u| \leq 1$) accounts for the density of superconducting electron pairs, so that a value close to 1 indicates a significant presence of the superconducting phenomenon.

Different behaviors are observed for different values of the applied magnetic field intensity h_{ex} with respect to the parameter ε. Let, as in [SS1],

$$\lambda = \lim_{\varepsilon\to 0} \frac{|\log\varepsilon|}{h_{ex}(\varepsilon)}. \tag{7}$$

When λ is finite and positive or zero (in the latter case with a not too large magnetic field, that is $h_{ex} \ll \varepsilon^{-2}$), we are in the so called 'mixed phase', characterized by the classical vortex structure.

It is shown in [SS1] (see also [SS2]) that our functional (3), with λ as in (7), is the Γ-limit as $\varepsilon \to 0$ of the Ginzburg Landau functional defined by (6), and the measure μ represents the density of vortices, whereas h_μ is the induced magnetic field. So, this is the physical interest of this kind of energy functionals.

Inequalities about the functional.

Now we introduce some basic results that will often be useful in the sequel.

LEMMA 2.1. – *For all $\mu, \nu \in \mathcal{M}_+(\Omega)$ there hold*

$$\Phi_\lambda(\mu) - \frac{\lambda}{2}\mu(\Omega) \geq \Phi_\lambda(\nu) - \frac{\lambda}{2}\nu(\Omega) + \int_\Omega (h_\nu - 1)\, d(\mu - \nu) \tag{8}$$

and

$$\Phi_\lambda(\mu) - \Phi_\lambda(\nu) = \left(\frac{\lambda}{2} - 1\right)(\mu(\Omega) - \nu(\Omega)) + \frac{1}{2}\int_\Omega (h_\mu + h_\nu) d(\mu - \nu) \tag{9}$$

PROOF. – See Proposition 2.2 and (28) in Proposition 3.1 of [AS]. $\square$

Moreover, we have

LEMMA 2.2. – *For all $\mu, \nu \in P(\overline{\Omega})$ there hold*

$$\Phi_\lambda(\nu) - \Phi_\lambda(\mu) \geq \frac{\lambda}{2}(\hat{\nu}(\Omega) - \hat{\mu}(\Omega)) + \int_{\overline{\Omega}} h_\mu d(\nu - \mu) \tag{10}$$

and

$$\Phi_\lambda(\mu) - \Phi_\lambda(\nu) = \left(\frac{\lambda}{2} - 1\right)(\hat{\mu}(\Omega) - \hat{\nu}(\Omega)) + \frac{1}{2}\int_\Omega (h_\mu + h_\nu) d(\hat{\mu} - \hat{\nu}). \tag{11}$$

PROOF. – These are straightforward consequences of Lemma 2.1, taking into account that, since the solution of problem (2) does not depend on the boundary part of μ, we have $h_{\hat{\mu}} = h_\mu$ and that $h_\mu|_{\partial\Omega} = 1$. $\square$

The Wasserstein structure.

We now recall some definitions about the Wasserstein structure, which has proved to be an important tool for studying different evolution problems (see for instance [O1, AGS, VI]). For $\mu, \nu \in P(\overline{\Omega})$ let $\Gamma(\mu, \nu)$ denote the set of *transport plans* between them, i.e. measures $\gamma \in P(\overline{\Omega} \times \overline{\Omega})$ whose first and second marginals are respectively μ and ν. We let $P(\overline{\Omega})$ be endowed with the *Wasserstein distance*, defined by

$$W_2(\mu,\nu) := \left(\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\, d\gamma(x,y) \right)^{1/2}. \tag{12}$$

Here the infimum can be shown to be a minimum, and we let $\Gamma_0(\mu,\nu)$ be the class of *optimal plans*, where this minimum is attained. A transport plan is a generalization of a *transport map* from μ to ν, that is, a map $\boldsymbol{t}$ such that $\boldsymbol{t}_{\#}\mu = \nu$ (i.e. $\mu(\boldsymbol{t}^{-1}(A)) = \nu(A)$, for A Borel). Indeed, to any transport map $\boldsymbol{t}$ we can associate the transport plan $\gamma = (I, \boldsymbol{t})_{\#}\mu$.

Formal gradient flow.

Here we relate the functionals (3) to a time evolutive problem (the Chapman-Rubinstein-Schatzman mean-field model for superconductors. See [CRS]). We can show that such a problem is the formal gradient flow of Φ_0 with respect to the Wasserstein structure, that is, ∇h_μ is the gradient of Φ_0 at μ along transport maps. The Wasserstein (sub)gradient $\nabla^W\Phi(\mu)$ is a vector $\xi\in L^2(\mu;\mathbb{R}^2)$ defined by the subdifferential relation

$$\Phi(\boldsymbol{s}_{\#}\mu) - \Phi(\mu) \geq \int \xi\cdot(\boldsymbol{s}-I)\,d\mu + o(\|\boldsymbol{s}-I\|^2_{L^2(\mu)}).$$

Now consider the functional (3), and by the representation (see [AS, Proposition 2.1])

$$\Phi_\lambda(\mu) = \frac{1}{2}(\lambda\mu(\Omega)+|\Omega|) + \sup_{h-1\in H^1_0(\Omega)} \left\{ \int_\Omega (h-1)\,d\mu - \frac{1}{2}\int_\Omega |\nabla h|^2 + |h|^2 \right\}, \tag{13}$$

being the supremum attained for $h = h_\mu$, we are led to

$$\begin{aligned}\Phi_\lambda(\boldsymbol{s}_{\#}\mu) - \Phi_\lambda(\mu) &\geq \frac{\lambda}{2}\big(s_{\#}\mu(\Omega) - \mu(\Omega)\big) + \int_{\overline{\Omega}} (h_\mu - 1)d(\boldsymbol{s}_{\#}\mu - \mu)\\ &= \frac{\lambda}{2}\big(s_{\#}\mu(\Omega) - \mu(\Omega)\big) + \int_{\overline{\Omega}} (h_\mu(\boldsymbol{s}(x)) - h_\mu(x))\,d\mu.\end{aligned}$$

Since

$$\int_{\overline{\Omega}} (h_\mu(\boldsymbol{s}(x)) - h_\mu(x))\,d\mu \sim \int_{\overline{\Omega}} \nabla h_\mu(x)\cdot(\boldsymbol{s}(x)-x)\,d\mu$$

as $\|\boldsymbol{s} - I\|_{L^2(\mu)} \to 0$, if $\lambda = 0$, the formal Wasserstein gradient of Φ_λ at μ (if $\mu = \hat{\mu}$) is $\chi_\Omega \nabla h_\mu$. The argument works also with $\lambda > 0$ if we consider transports which do not increase the mass on $\partial\Omega$.

3. – Variation.

Consider the discrete problem

$$\min_{\nu \in P(\overline{\Omega})} \Phi_\lambda(\nu) + \frac{1}{2\tau} W_2^2(\mu, \nu). \tag{14}$$

Recalling Proposition 5.4 of [AS], we have

LEMMA 3.1. – *Let* $p \in (1, \infty]$, $\mu \in P(\overline{\Omega})$ *with* $\hat{\mu} \in L^p(\Omega)$. *Then there is a minimizer* μ_τ *of problem* (14) *such that* $\hat{\mu}_\tau \in L^p(\Omega)$. *Moreover, the* L^p *norm of* $\hat{\mu}_\tau$ *is uniformly bounded in* τ. *In particular, if* $p = \infty$ *we have* $\|\hat{\mu}_\tau\|_\infty \leq \max\{1, \|\hat{\mu}\|_\infty\}$.

Now we state the result about minimizers of (14) in the case $\lambda = 0$. Mind that, by Lemma 3.1, if $\hat{\mu} \in L^p(\Omega)$ in (14), a minimizer can be found with L^p interior as well.

THEOREM 3.1. – *Let* $\nu = \mu_\tau$ *be a minimizer of* (14), *with* $\lambda = 0$, *such that* $\hat{\nu} \in L^4(\Omega)$. *Let* Ω *be convex. Then*

$$\langle \nabla h_\nu(x), y - x \rangle \geq 0 \quad \forall (x, y) \in \operatorname{supp}(\widetilde{\nu}) \times \overline{\Omega}. \tag{15}$$

We need a measure theoretic lemma before proceeding with the proof. We recall also that, given two measures μ and ν in $\mathcal{M}_+(\mathbb{R}^2)$ with same mass, if μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^2$, then there exists a unique optimal transport plan between μ and ν (for which the infimum in (12) is achieved), and such plan is induced by a transport map (see [AGS, Section 6]).

LEMMA 3.2. – *Let* $\mu, \nu \in P(\overline{\Omega})$, $\sigma \ll \mathcal{L}^2 \llcorner \Omega$, *with* $\sigma(\Omega) = \nu(\partial\Omega)$, *and let* T *be the optimal transport map between* σ *and* $\widetilde{\nu}$.

Then there exist $\gamma \in \Gamma_0(\nu, \mu)$, $\gamma_T \in \Gamma(\sigma, \mu_1)$, *where* μ_1 *is the second marginal of* $\chi_{\partial\Omega \times \overline{\Omega}}\, \gamma$, *such that*

$$W_2^2(\nu_S, \mu) - W_2^2(\nu, \mu) \leq \int\limits_{\Omega \times \overline{\Omega}} \left[|y - S(x)|^2 - |y - T(x)|^2 \right] d\gamma_T(x, y)$$

for all $S : \Omega \mapsto \Omega$, *where* $\nu_S = \hat{\nu} + S_\# \sigma$.

PROOF. – Let us introduce a sequence of auxiliary measures $\widetilde{\nu}_n$, with equi-compact supports contained in $\mathbb{R}^2\backslash\Omega$, such that $\widetilde{\nu}_n(\mathbb{R}^2\backslash\Omega) = \sigma(\Omega)$, $\widetilde{\nu}_n \ll \mathcal{L}^2$ and $\widetilde{\nu}_n \rightharpoonup \widetilde{\nu}$ as $n \to \infty$. Let T_n be the optimal transport maps between σ and $\widetilde{\nu}_n$. Moreover, let γ_n be optimal transport plans between ν_n and μ, where $\nu_n = \hat{\nu} + \widetilde{\nu}_n$. As an optimal transport map between absolutely continuous measures, T_n is essentially invertible for every n (i.e. its restriction to the complement of a σ-negligible set in Ω is injective, see [AGS, Remark 6.2.11]). So we can define

$$\overline{\gamma}_n = (S \circ T_n^{-1}, I)_{\#}\chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma_n + \chi_{\Omega\times\overline{\Omega}}\,\gamma_n,$$

$$\gamma_{T_n} = (T_n^{-1}, I)_{\#}\chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma_n.$$

Clearly, $\overline{\gamma}_n \in \Gamma(\nu_S, \mu)$ and $\gamma_{T_n} \in \Gamma(\sigma, \mu_n)$ for every n, where we introduced μ_n as the second marginal of $\chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma_n$. So with the change of variables $z = T_n^{-1}(x)$, for every n we have

$$\begin{aligned} W_2^2(\nu_S, \mu) &\le \int_{\overline{\Omega}\times\overline{\Omega}} |y-x|^2 d\overline{\gamma}_n \\ &= \int_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}} |y-x|^2 d((S \circ T_n^{-1}, I)_{\#}\gamma_n) + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n \\ &= \int_{\Omega\times\overline{\Omega}} |y-S(z)|^2 d\gamma_{T_n}(z,y) + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n, \end{aligned}$$

and

$$\begin{aligned} W_2^2(\nu_n, \mu) &= \int_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}} |y-x|^2 d\gamma_n + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n \\ &= \int_{\Omega\times\overline{\Omega}} |y-T_n(z)|^2 d\gamma_{T_n}(z,y) + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n. \end{aligned} \tag{16}$$

We get, for every n,

$$W_2^2(\nu_S, \mu) - W_2^2(\nu_n, \mu) \le \int_{\Omega\times\overline{\Omega}} \left[|y-S(x)|^2 - |y-T_n(x)|^2\right] d\gamma_{T_n}(x,y). \tag{17}$$

Now we have to pass to the limit as $n \to \infty$. As $\widetilde{\nu}_n \rightharpoonup \widetilde{\nu}$, for the stability property of optimal transport maps, we have that $T_n \to T$ strongly in $L^p_\sigma(\Omega)$, $1 \le p < \infty$, where T is the optimal transport map between σ and $\widetilde{\nu}$. Moreover, γ_n has a weak limit point in $P(\mathbb{R}^2 \times \overline{\Omega})$ which is an optimal plan $\gamma \in \Gamma_0(\nu, \mu)$ (see [AG, Lemma 3.3]). We will not relabel the sequence for simplicity.

We can also show that

$$\chi_{\Omega\times\overline{\Omega}}\,\gamma_n \rightharpoonup \chi_{\Omega\times\overline{\Omega}}\,\gamma. \tag{18}$$

In fact, let $\eta(x)$ be a smooth cutoff function approximating χ_Ω, with $\eta(x) \equiv 0$ on $\mathbb{R}^2 \backslash \Omega$ and

$$\int_\Omega |\eta(x) - 1|\, d\hat{\nu} < \varepsilon.$$

Let $f \in C^0(\mathbb{R}^2 \times \overline{\Omega})$, with $M = \|f\|_\infty$ finite. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\overline{\Omega}} f(x,y)\chi_{\Omega\times\overline{\Omega}}(x,y)\,d(\gamma_n-\gamma)(x,y)\\
&= \int_{\mathbb{R}^2\times\overline{\Omega}} f(x,y)[\chi_\Omega(x)-\eta(x)+\eta(x)]\,d(\gamma_n-\gamma)(x,y)\\
&\le \int_{\mathbb{R}^2\times\overline{\Omega}} f(x,y)\eta(x)\,d(\gamma_n-\gamma)(x,y)\\
&\quad + M\int_{\mathbb{R}^2\times\overline{\Omega}} |\chi_\Omega(x)-\eta(x)|\,d(\gamma_n+\gamma)(x,y)\\
&= \int_{\mathbb{R}^2\times\overline{\Omega}} f(x,y)\eta(x)\,d(\gamma_n-\gamma)(x,y) + 2M\int_\Omega |1-\eta(x)|\,d\hat{\nu}\\
&\le \int_{\mathbb{R}^2\times\overline{\Omega}} f(x,y)\eta(x)\,d(\gamma_n-\gamma)(x,y) + 2M\varepsilon.
\end{aligned}$$

Now the first integral tends to zero, since $f\eta$ is continuous, and by arbitrariness of ε we get the convergence. Here we used the fact that the measures $\chi_{\Omega\times\overline{\Omega}}\,\gamma_n$ and $\chi_{\Omega\times\overline{\Omega}}\,\gamma$ have $\hat{\nu}$ as first marginal. In the same way one can prove that $\chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma_n \rightharpoonup \chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma$. This implies $\mu_n \rightharpoonup \mu_1$, since $\mu_n := \pi^2_\#(\chi_{(\mathbb{R}^2\backslash\Omega)\times\overline{\Omega}}\,\gamma_n)$. Besides, μ_n is also the second marginal of γ_{T_n}, which by tightness has a limit point γ_T (again we avoid relabeling the sequence). The first marginal of γ_{T_n} is σ for every n, and as a consequence $\gamma_T \in \Gamma(\sigma, \mu_1)$.

Now consider the first integral in the second member of (17). We have the weak convergence of γ_{T_n} to γ_T, and we can pass to the limit even though the integrand is not continuous. Indeed, reasoning exactly as in the proof of (18), we can approximate it with continuous functions (in the Lusin sense) and use the fact that both the first marginal of γ_{T_n} and of γ_T are equal to the absolutely continuous measure σ.

Finally, consider the last term in (17). We have

$$\begin{aligned}\int_{\Omega\times\overline{\Omega}} |y-T_n|^2 d\gamma_{T_n} &= \int_{\Omega\times\overline{\Omega}} \Big[|y-T|^2 + |y-T_n|^2 - |y-T|^2\Big] d\gamma_{T_n} \\ &\leq \int_{\Omega\times\overline{\Omega}} |y-T|^2 d\gamma_{T_n} \\ &\quad + K\int_{\Omega\times\overline{\Omega}} |T_n(x)-T(x)| d\gamma_{T_n} \\ &\leq \int_{\Omega\times\overline{\Omega}} |y-T|^2 d\gamma_{T_n} + K\int_{\Omega} |T_n(x)-T(x)| d\sigma,\end{aligned}$$

with K being a suitable positive constant depending on Ω. Now the second term goes to zero for the strong convergence of T_n, and the first one can be treated as before and shown to converge to

$$\int_{\Omega\times\overline{\Omega}} |y-T(x)|^2 d\gamma_T(x,y).$$

We have all what is needed to pass to the limit in (16) and (17) and obtain

$$W_2^2(\nu_S,\mu) - W_2^2(\nu,\mu) \leq \int_{\Omega\times\overline{\Omega}} \Big[|y-S(x)|^2 - |y-T(x)|^2\Big] d\gamma_T(x,y) \tag{19}$$

as desired. □

We also state a slight generalization of the previous lemma.

LEMMA 3.3. – *Let μ, ν, σ and T be as in* Lemma 3.2. *Let $S : \Omega \mapsto \Omega$, $\theta \in [0,1]$ and*

$$\nu_S = \hat{\nu} + S_\#(\theta\sigma) + (1-\theta)\widetilde{\nu}.$$

Then there exist $\gamma \in \Gamma_0(\nu,\mu)$, $\gamma_T \in \Gamma(\sigma,\mu_1)$, μ_1 being the second marginal of $\chi_{\partial\Omega\times\overline{\Omega}}\gamma$, such that

$$W_2^2(\nu_S,\mu) - W_2^2(\nu,\mu) \leq \theta\int_{\Omega\times\overline{\Omega}} \big[|y-S(x)|^2 - |y-T(x)|^2\big] d\gamma_T(x,y).$$

PROOF. – The case $\theta = 0$ is trivial. Otherwise, define $\widetilde{\nu}_n, \nu_n, T_n, \gamma_n$ and γ_{T_n} as in the proof of Lemma 3.2. Moreover, let $\nu_S^n = \hat{\nu} + S_\#(\theta\sigma) + (1-\theta)\widetilde{\nu}_n$ and introduce transport plans $\overline{\overline{\gamma}}_n \in \Gamma(\nu_S^n,\mu)$ as follows:

$$\overline{\overline{\gamma}}_n = \theta(S\circ T_n^{-1}, I)_\#\chi_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}}\gamma_n + \chi_{\Omega\times\overline{\Omega}}\gamma_n + (1-\theta)\chi_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}}\gamma_n.$$

Then, with the change of variables $z = T_n^{-1}(x)$, we have

$$
\begin{aligned}
W_2^2(\nu_S^n, \mu) &\le \int_{\overline{\Omega}\times\overline{\Omega}} |y-x|^2 d\overline{\overline{\gamma}}_n \\
&= \theta \int_{\Omega\times\overline{\Omega}} |y - S(z)|^2 d\gamma_{T_n}(z,y) + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n \\
&\quad + (1-\theta) \int_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}} |y-x|^2 d\gamma_n.
\end{aligned}
$$

We can rewrite (16) as

$$
\begin{aligned}
W_2^2(\nu_n, \mu) &= \theta \int_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}} |y-x|^2 d\gamma_n + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n \\
&\quad + (1-\theta) \int_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}} |y-x|^2 d\gamma_n \\
&= \theta \int_{\Omega\times\overline{\Omega}} |y - T_n(z)|^2 d\gamma_{T_n}(z,y) + \int_{\Omega\times\overline{\Omega}} |y-x|^2 d\gamma_n \\
&\quad + (1-\theta) \int_{(\mathbb{R}^2\setminus\Omega)\times\overline{\Omega}} |y-x|^2 d\gamma_n.
\end{aligned}
$$

This way, it is clear that

$$
W_2^2(\nu_S^n, \mu) - W_2^2(\nu_n, \mu) \le \theta \int_{\Omega\times\overline{\Omega}} \left[|y - S(x)|^2 - |y - T_n(x)|^2\right] d\gamma_{T_n}(x,y).
$$

Here we can pass to the limit in n exactly as done for (17), so we refer to the proof of Lemma 3.2 for concluding. The only element to add is the lower semicontinuity of W_2 for treating the first term, so that

$$
W_2(\nu_S, \mu) \le \liminf_{n\to\infty} W_2(\nu_S^n, \mu)
$$

as $\nu_S^n \rightharpoonup \nu_S$. □

REMARK 3.1. – With minor modifications one can also obtain the same result for the case

$$
\nu_S = \hat{\nu} + S_\#(\theta\sigma) + (1-\theta)\chi_A\widetilde{\nu} + \chi_{\partial\Omega\setminus A}\widetilde{\nu},
$$

where A is an arc contained in $\partial\Omega$. In this case we have $\gamma \in \Gamma_0(\nu, \mu)$, $\sigma \ll \mathcal{L}^2 \llcorner \Omega$, $\sigma(\Omega) = \widetilde{\nu}(A)$, $(I, T)_\#\sigma \in \Gamma_0(\sigma, \chi_A\widetilde{\nu})$. μ_1 will be a suitable measure such that $\mu_1 \le \pi^2_\#(\chi_{\partial\Omega\times\overline{\Omega}}\,\gamma)$.

We are now ready for the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Let $\sigma \ll \mathcal{L}^2 \llcorner \Omega$ have a bounded density, and let $\sigma(\Omega) = \widetilde{\nu}(\overline{\Omega})$. Let moreover T be the optimal transport map between σ and $\widetilde{\nu}$, and

$$T_\varepsilon = (1-\varepsilon)I + \varepsilon T, \quad \varepsilon \in [0,1].$$

We introduce the following perturbed measure

$$\nu_\varepsilon := \hat{\nu} + T_{\varepsilon\#}(a^2\sigma) + (1-a^2)\widetilde{\nu},$$

where $a = (1-\varepsilon)^2$.

Now we apply Lemma 3.3, with T_ε in the role of S: there exist a transport plan $\gamma \in \Gamma_0(\nu, \mu)$ and a transport plan $\gamma_T \in \Gamma(\sigma, \mu_1)$, where μ_1 is the second marginal of $\chi_{\partial\Omega\times\overline{\Omega}}\,\gamma$, such that

$$W_2^2(\nu_\varepsilon,\mu) - W_2^2(\nu,\mu) \le a^2 \int\limits_{\Omega\times\overline{\Omega}} \left[|y - T_\varepsilon(x)|^2 - |y - T(x)|^2\right] d\gamma_T(x,y). \tag{20}$$

Next, we apply (11) to ν_ε and ν, and we find

$$\Phi_0(\nu_\varepsilon) - \Phi_0(\nu) = -\left(\hat{\nu}_\varepsilon(\Omega) - \hat{\nu}(\Omega)\right) + \frac{1}{2}\int\limits_{\Omega} (h_{\nu_\varepsilon} + h_\nu)\, d(\hat{\nu}_\varepsilon - \hat{\nu}),$$

so that

$$\Phi_0(\nu_\varepsilon) - \Phi_0(\nu) = -a^2\widetilde{\nu}(\partial\Omega) + \frac{1}{2}a^2\int\limits_{\Omega} (h_{\nu_\varepsilon} + h_\nu)\, d(T_{\varepsilon\#}\sigma). \tag{21}$$

Since ν is a minimizer, there holds

$$\Phi_0(\nu_\varepsilon) - \Phi_0(\nu) + \frac{1}{2\tau}\left(W_2^2(\nu_\varepsilon,\mu) - W_2^2(\nu,\mu)\right) \ge 0,$$

for all $\mu \in P(\overline{\Omega})$. Substituting (20) e (21) in this inequality, we obtain:

$$\begin{aligned} &\frac{a^2}{2\tau}\int\limits_{\Omega\times\overline{\Omega}} \left[|y - T_\varepsilon(x)|^2 - |y - T(x)|^2\right] d\gamma_T \\ &\quad - a^2\widetilde{\nu}(\partial\Omega) + \frac{1}{2}a^2\int\limits_{\Omega} (h_{\nu_\varepsilon} + h_\nu)\, d(T_{\varepsilon\#}\sigma) \ge 0. \end{aligned} \tag{22}$$

Since $T_\varepsilon = T + (1-\varepsilon)(I - T)$, we obtain the following expansion (of the first order centered in $\varepsilon = 1$)

$$|y - T_\varepsilon(x)|^2 = |y - T(x)|^2 + 2(\varepsilon - 1)\langle y - T(x), x - T(x)\rangle + o(\varepsilon - 1).$$

Of course the remainder is uniformly bounded with respect to $x \in \overline{\Omega}$. For treating the second integral in (22), notice that, as $\hat{\nu} \in L^4(\Omega)$, $h_\nu \in W^{2,4}(\Omega)$, and by Sobolev embedding $h_\nu \in C^1(\overline{\Omega})$ (since Ω has smooth boundary). So we can perform the expansion

$$
\begin{aligned}
h_\nu \circ T_\varepsilon = {} & h_\nu \circ T + (\varepsilon - 1)\langle \nabla h_\nu \circ T, T - I\rangle \\
& + (\varepsilon - 1)\langle (\nabla h_\nu \circ T_\theta - \nabla h_\nu \circ T), T - I\rangle,
\end{aligned} \tag{23}
$$

for a suitable $\theta \in (\varepsilon, 1)$. If $K = \sup_{x \in \overline{\Omega}} |T(x) - x|$, the last term is bounded by $K(\varepsilon - 1)\,\omega(|T_\theta(x) - T(x)|)$, $\omega(\delta)$ being the modulus of continuity of ∇h_ν, which, as $\delta \to 0$, goes to zero uniformly with respect to $x \in \overline{\Omega}$, since $\nabla h_\nu \in C^0(\overline{\Omega})$. So there holds

$$
h_\nu \circ T_\varepsilon = h_\nu \circ T + (\varepsilon - 1)\langle \nabla h_\nu \circ T, T - I\rangle + o(\varepsilon - 1), \tag{24}
$$

and the remainder is uniform in x.

Finally, since

$$
h_{\nu_\varepsilon} \circ T_\varepsilon = h_\nu \circ T_\varepsilon + (h_{\nu_\varepsilon} - h_\nu) \circ T_\varepsilon, \tag{25}
$$

we have to estimate $h_{\nu_\varepsilon} - h_\nu$. This quantity is solution of the problem

$$
\begin{cases}
-\Delta u + u = a^2 T_{\varepsilon\#}(\sigma) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Hence we can write

$$
\sup_{x \in \overline{\Omega}} |h_{\nu_\varepsilon}(x) - h_\nu(x)| = a^2 \sup_{x \in \overline{\Omega}} |\varphi_\varepsilon|, \tag{26}
$$

where φ_ε satisfies

$$
\begin{cases}
-\Delta \varphi_\varepsilon + \varphi_\varepsilon = T_{\varepsilon\#}(\sigma) & \text{in } \Omega \\
\varphi_\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases}
$$

But $aT_{\varepsilon\#}(\sigma)$ converges to 0 in $L^4(\Omega)$, since for $\varepsilon \in (0,1)$ there holds $|\det(JT_\varepsilon)| \geq (1-\varepsilon)^2$ and we have

$$
\begin{aligned}
\int_\Omega |aT_{\varepsilon\#}(\sigma)|^4 &= a^4 \int_\Omega \left(\frac{\sigma}{|\det(JT_\varepsilon)|} \right)^4 |\det(JT_\varepsilon)| \\
&\leq \frac{a^4}{(1-\varepsilon)^6} \int_\Omega |\sigma|^4 = (1-\varepsilon)^2 \int_\Omega |\sigma|^4.
\end{aligned}
$$

This implies the $W^{2,4}(\Omega)$ convergence and the $C^1(\overline{\Omega})$ convergence of $a\varphi_\varepsilon$ as $\varepsilon \to 1$. So there exists a constant C which bounds $a\varphi_\varepsilon$ uniformly in x and ε, and from (26) we get

$$
\sup_{x \in \overline{\Omega}} |h_{\nu_\varepsilon}(x) - h_\nu(x)| \leq Ca = C(1-\varepsilon)^2. \tag{27}
$$

Making use of (24) and (27), from (25) we find

$$h_{\nu_\varepsilon} \circ T_\varepsilon = h_\nu \circ T + (\varepsilon - 1)\langle \nabla h_\nu \circ T, T - I\rangle + o(\varepsilon - 1), \tag{28}$$

where the remainder is again uniformly bounded in x.

Now, dividing by α^2, we expand to the first order in (22) with respect to $\varepsilon \to 1$, and with τ fixed, to find

$$\frac{1-\varepsilon}{\tau} \int_{\Omega \times \overline{\Omega}} \langle y - T(x), T(x) - x\rangle d\gamma_T - \widetilde{\nu}(\partial\Omega) + \int_\Omega h_\nu(T(x)) d\sigma$$

$$+ (1-\varepsilon)\int_\Omega \langle \nabla h_\nu(T(x)), x - T(x)\rangle d\sigma + o(1-\varepsilon) \geq 0.$$

As a consequence, since $\sigma(\Omega) = \widetilde{\nu}(\partial\Omega)$ and $h_\nu = 1$ on $\partial\Omega$, upon dividing by $(1-\varepsilon)$ we get

$$\frac{1}{\tau}\int_{\Omega \times \overline{\Omega}} \langle y - T(x), T(x) - x\rangle d\gamma_T + \int_\Omega \langle \nabla h_\nu(T(x)), x - T(x)\rangle d\sigma \geq 0.$$

As $T(x) \in \operatorname{supp}(\widetilde{\nu})$, in the first integral the scalar product is nonpositive for geometric reasons (we are working with a convex domain). It follows that

$$\int_\Omega \langle \nabla h_\nu(T(x)), x - T(x)\rangle d\sigma \geq 0. \tag{29}$$

Let $A \subset \partial\Omega$ be an arc such that $\widetilde{\nu}(A) > 0$. We point out that, redefining ν_ε as $\hat{\nu} + \alpha^2 T_{\varepsilon\#}\sigma + (1-\alpha^2)\chi_A\widetilde{\nu} + \chi_{\partial\Omega\setminus A}\widetilde{\nu}$, with $T_\varepsilon = (1-\varepsilon)I + \varepsilon T$ and T now being the optimal transport map between an absolutely continuous σ and $\chi_A\widetilde{\nu}$, this proof works in the same way. Indeed, in view of Remark 3.1, inequality (20) still holds for some $\gamma_T \in \Gamma(\sigma, \mu_1)$, where $\mu_1 \leq \pi^2_\#(\chi_{\partial\Omega\times\overline{\Omega}}\gamma)$. So we obtain (29) with $T(x)$ taking values in $\operatorname{supp}(\widetilde{\nu}) \cap A$. Now, suppose by contradiction that

$$\langle \nabla h_\nu(\bar{z}), \bar{y} - \bar{z}\rangle < 0$$

for some $(\bar{z}, \bar{y}) \in \operatorname{supp}(\widetilde{\nu}) \times \overline{\Omega}$. Then, recalling that $\nabla h_\nu \in C^0(\overline{\Omega})$, there exist an arc $I \subset \partial\Omega$ containing $\bar{z}$ and a neighborhood Q of $\bar{y}$ such that the same inequality holds whenever $(z, y) \in I \times Q$. Because of the arbitrariness of A and σ, we can choose σ supported in $\Omega \cap Q$ and $A \subset I$. Since T transports σ to $\chi_A\widetilde{\nu}$, this implies $\langle \nabla h_\nu(T(x)), x - T(x)\rangle < 0$ for all $x \in \operatorname{supp}(\sigma)$, against (29). □

4. – Uniqueness of the gradient flow.

We now consider the problem of uniqueness of solutions for (1)-(2) in the case of measures with L^∞ internal part. Taking into account the result of Theorem 3.1, we focus on the following class of solutions.

DEFINITION 4.1 (Regular gradient flow). – *Let $T > 0$. A solution of problem* (1)-(2) *is a regular gradient flow if*

i) $\|\hat{\mu}(t)\|_\infty \in L^\infty(0, T)$,
ii) $\langle \nabla h_{\mu(t)}(x), y - x\rangle \geq 0$ for all $(x, y) \in \operatorname{supp}(\widetilde{\mu}(t)) \times \overline{\Omega}$ and $t \in (0, T]$.

REMARK 4.1. – Condition *ii)* is related, as already noticed in the introduction, to the one appearing in [AS, Definition 3.1], that is, $t \mapsto \widetilde{\mu}(t)$ is nondecreasing as a measure valued map. In fact, if the negative gradient at the boundary (that is the limit of velocities in Ω) is directed towards the exterior of the domain, we expect that no mass can move from $\partial\Omega$ to Ω during the evolution. Such a behavior was argued in [AS] in the case $\lambda > 0$ by means of direct energy arguments, which do not extend for $\lambda = 0$. Actually, condition *ii)*, obtained in Theorem 3.1 only for $\lambda = 0$, will allow us to obtain a stronger uniqueness result.

THEOREM 4.1 (Construction of a regular gradient flow). – *Let Ω be convex. Let $\mu_0 \in P(\overline{\Omega})$, with $\hat{\mu}_0 \in L^\infty(\Omega)$. Then there exists a solution to problem* (1)-(2) *which is a regular gradient flow.*

PROOF. – Let $\mu_\tau^0 := \mu_0$. We find μ_τ^{k+1} solving (4) with $\lambda = 0$ recursively. We then define

$$\bar{\mu}_\tau(t) := \mu_\tau^{k+1} \qquad \text{if } t \in \ (k\tau, (k+1)\tau], \tag{30}$$

and for $\tau \downarrow 0$ we can find limit points, that is, we can find sequences $\tau_n \downarrow 0$ such that in the sense of measures

$$\lim_{n\to\infty} \bar{\mu}_{\tau_n}(t) = \mu(t) \qquad \forall t \geq 0. \tag{31}$$

So, there exists a solution constructed in this way (see [AS, Section 6] for more details). Thanks to Lemma 3.1, the interior parts of all the discrete minimizers will belong to L^∞. Letting $T > 0$, and passing to the limit in τ, we will have $\hat{\mu}(t) \in L^\infty((0, T); L^\infty(\Omega))$. Moreover, after Theorem 3.1, the discrete minimizers can also be chosen to satisfy (15), which, passing again to the limit in τ, becomes condition *ii)* of Definition 4.1. In fact, as a consequence of (31), $h_{\bar{\mu}_{\tau_n}(t)} \to h_{\mu(t)}$ in $C^1(\overline{\Omega})$ for every $t \in [0, T]$. In conclusion, there exists a regular gradient flow as in such definition. $\square$

The next inequality prepares the proof of the uniqueness theorem.

LEMMA 4.1. – *Let $\mu, \nu \in P(\overline{\Omega})$, with $\hat{\mu}, \hat{\nu} \in L^\infty(\Omega)$ and $W_2^2(\mu,\nu) \le e^{-3}$. Then there holds*

$$
\begin{aligned}
\Phi_\lambda(\nu) - \Phi_\lambda(\mu) \ge \frac{\lambda}{2}(\hat{\nu}(\Omega) - \hat{\mu}(\Omega)) \\
+ \int\limits_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} \langle \nabla h_\mu(x), y - x\rangle d\gamma(x,y) - \omega(W_2^2(\mu,\nu)),
\end{aligned}
\tag{32}
$$

where $\omega(t) = \widetilde{K} t|\log t|$, $\widetilde{K}$ being a suitable nonnegative constant depending only on Ω, $\|\hat{\nu}\|_\infty$ and $\|\hat{\mu}\|_\infty$.

PROOF. – We shall estimate the last term of inequality (10). For all $\gamma \in \Gamma_0(\mu,\nu)$ we have

$$
\int\limits_{\overline{\Omega}} h_\mu d(\nu - \mu) = \int\limits_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} (h_\mu(y) - h_\mu(x)) d\gamma(x,y)
$$

and a Taylor expansion (with remainder in integral form) yields

$$
\begin{aligned}
\int\limits_{\overline{\Omega}} h_\mu d(\nu - \mu) = \int\limits_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} \langle \nabla h_\mu(x), y - x\rangle d\gamma(x,y) \\
+ \frac{1}{2}\int\limits_0^1 \int\limits_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} \langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y-x), y-x\rangle d\gamma(x,y) d\theta.
\end{aligned}
\tag{33}
$$

In order to treat the remainder, we split it in two terms:

$$
\begin{aligned}
&\frac{1}{2}\int\limits_0^1 \int\limits_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} |\langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y-x), y-x\rangle| d\gamma(x,y) d\theta \\
&\le \frac{1}{2}\int\limits_0^1 \int\limits_{\Omega\times\overline{\Omega}} |\langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y-x), y-x\rangle| d\gamma(x,y) d\theta \\
&+ \frac{1}{2}\int\limits_0^1 \int\limits_{\overline{\Omega}\times\Omega} |\langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y-x), y-x\rangle| d\gamma(x,y) d\theta.
\end{aligned}
$$

First term:

the measure $\chi_{\Omega\times\overline{\Omega}}\gamma$ is a transport plan between $\hat{\mu}$ and σ_1 for a suitable $\sigma_1 \le \nu$, then it is induced by a transport map T. Let

$$
T_\theta = (1-\theta)I + \theta T, \quad \mu_\theta = T_{\theta\#}\hat{\mu}.
$$

It follows that

$$\begin{aligned}
&\int_0^1 \int_{\Omega\times\overline{\Omega}} |\langle \nabla^2 h_\mu((1-\theta)x+\theta y)(y-x), y-x\rangle| d\gamma(x,y)d\theta \\
&= \int_0^1 \frac{1}{\theta^2}\int_\Omega |\langle \nabla^2 h_\mu(T_\theta(x))(T_\theta(x)-x), T_\theta(x)-x\rangle| d\hat{\mu}(x)d\theta \\
&= \int_0^1 \frac{1}{\theta^2}\int_\Omega |\langle \nabla^2 h_\mu(x)(x-T_\theta^{-1}(x)), x-T_\theta^{-1}(x)\rangle| d\mu_\theta(x)d\theta \qquad (34)\\
&\le \int_0^1 \frac{1}{\theta^2}\int_\Omega |\nabla^2 h_\mu(x)||x-T_\theta^{-1}(x)|^2 d\mu_\theta(x)d\theta \\
&\le \int_0^1 \frac{1}{\theta^2}\left(\int_\Omega |\nabla^2 h_\mu(x)|^p d\mu_\theta(x)\right)^{1/p}\left(\int_\Omega |x-T_\theta^{-1}(x)|^{2p'} d\mu_\theta(x)\right)^{1/p'} d\theta,
\end{aligned}$$

where $p>1$ and p' are conjugate exponents. Let ρ and ρ_θ be the densities of $\hat{\mu}$ and μ_θ respectively. The change of variables formula gives

$$\begin{aligned}
&\int_\Omega |\nabla^2 h_\mu|^p d\mu_\theta = \int_\Omega |\nabla^2 h_\mu|^p \rho_\theta \, d\mathcal{L}^2 \\
&\le \left(\int_\Omega |\nabla^2 h_\mu|^{2p} d\mathcal{L}^2\right)^{1/2}\left(\int_\Omega |\rho_\theta|^2 d\mathcal{L}^2\right)^{1/2} \\
&\le \left(\int_\Omega |\nabla^2 h_\mu|^{2p} d\mathcal{L}^2\right)^{1/2}\left(\int_\Omega \left(\frac{\rho}{|\det(JT_\theta)|}\right)^2 |\det(JT_\theta)| d\mathcal{L}^2\right)^{1/2} \\
&\le M\left(\int_\Omega |\nabla^2 h_\mu|^{2p} d\mathcal{L}^2\right)^{1/2}\left(\int_\Omega \frac{d\mathcal{L}^2}{|\det(JT_\theta)|}\right)^{1/2}.
\end{aligned}$$

But for $\theta\in(0,1)$, there holds $|\det(J((1-\theta)I+\theta T))|\ge(1-\theta)^2$, yielding

$$\begin{aligned}
\int_\Omega |\nabla^2 h_\mu|^p d\mu_\theta &\le M\left(\int_\Omega |\nabla^2 h_\mu|^{2p} d\mathcal{L}^2\right)^{1/2}\left(\int_\Omega \frac{d\mathcal{L}^2}{(1-\theta)^2}\right)^{1/2} \\
&\le M\|\nabla^2 h_\mu\|^p_{L^{2p}(\Omega)}|\Omega|^{1/2}(1-\theta)^{-1}.
\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\Omega} |I - T_\theta^{-1}|^{2p'} d\mu_\theta &= \int_{\Omega} |I - T_\theta^{-1}|^2 |I - T_\theta^{-1}|^{2p'-2} d\mu_\theta \\ &\leq (\operatorname{diam}\Omega)^{2(p'-1)} \int_{\Omega} |I - T_\theta^{-1}|^2 d\mu_\theta \\ &= (\operatorname{diam}\Omega)^{2(p'-1)} \theta^2 \int_{\Omega} |T - I|^2 d\hat{\mu} \\ &= \theta^2 (\operatorname{diam}\Omega)^{2(p'-1)} W_2^2(\mu, \nu).\end{aligned}$$

Substituting in (34), we get

$$\begin{aligned}&\int_0^1 \int_{\Omega\times\overline{\Omega}} |\langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y - x), y - x\rangle| d\gamma(x, y) d\theta \\ &\leq M^{1/p} |\Omega|^{1/(2p)} (\operatorname{diam}\Omega)^{2(p'-1)/p'} \|\nabla^2 h_\mu\|_{L^{2p}(\Omega)} W_2^{2/p'}(\mu, \nu) \\ &\quad \int_0^1 \frac{1}{\theta^2} (1-\theta)^{-1/p} \theta^{2/p'} d\theta.\end{aligned}$$

Now, for p sufficiently large (for example $p \geq 3$), the integral in the last term is finite and uniformly bounded in p. Moreover by elliptic regularity we have $\|\nabla^2 h_\mu\|_{L^{2p}(\Omega)} \leq c\, p \|\mu\|_\infty$, so that

$$\int_0^1 \int_{\Omega\times\overline{\Omega}} |\langle \nabla^2 h_\mu((1-\theta)x + \theta y)(y - x), y - x\rangle| d\gamma(x, y) d\theta \leq C p W_2^{2/p'}(\mu, \nu).$$

As done by Yudovitch in the study of Euler equations in two dimensions (see [YU1, YU2]), we minimize in p and, since $W_2^2(\mu, \nu) \leq e^{-3}$, we find

$$\min_{p\geq 3} p W_2^{2/p'}(\mu, \nu) = e W_2^2(\mu, \nu) |\log (W_2^2(\mu, \nu))|.$$

This is the desired logarithmic bound.

Second term:

it can be treated in the same way: for example we can consider $\chi_{\overline{\Omega}\times\Omega}\gamma \in \Gamma(\sigma_2, \hat{\nu})$, where σ_2 is a suitable measure with $\sigma_2 \leq \mu$. Now there exists a transport map s such that $s_{\#}\hat{\nu} = \sigma_2$. Letting $s_\theta = (1-\theta)s + \theta I$, we get

$$\int_0^1 \int_{\overline{\Omega}\times\Omega} |\langle \nabla^2 h_\mu((1-\theta)x+\theta y)(y-x), y-x\rangle| d\gamma(x,y)d\theta$$

$$\leq \int_0^1 \frac{1}{(1-\theta)^2} \int_\Omega |\nabla^2 h_\mu||s_\theta^{-1} - I|\, d(s_{\theta\#}\hat{\nu})d\theta.$$

The calculation is now analogous, taking into account that $|\det(Js_\theta)| \geq \theta^2$ and that $\int_\Omega |I-s|^2 d\hat{\nu} \leq W_2^2(\mu,\nu)$.

Thanks to the logarithmic bound on the remainder of (33), from (10) we obtain (32). $\square$

Eventually, we are going to state and prove our main result. The procedure is analogous to the one of [AS, Theorem 3.2], but here we can show that uniqueness holds also if some mass is present on the boundary of Ω during the evolution. Even if the initial datum is not supported in Ω, this guarantees a global uniqueness result.

THEOREM 4.2 (Uniqueness of the regular gradient flow). – *Let Ω be convex. Let μ^1, μ^2 be solutions of* (1)-(2) *satisfying the conditions of* Definition 4.1. *Then $\mu^1(0) = \mu^2(0)$ implies $\mu^1(t) = \mu^2(t)$ for all $t \in [0,T]$.*

PROOF. – Let $\mu(t)$ be a regular gradient flow as in Definition 4.1 (it is coupled with the velocity field $-\nabla h_{\mu(t)}\chi_\Omega$), $\gamma_t \in \Gamma_0(\mu(t),\nu)$ and $\nu \in P(\overline{\Omega})$. Applying (32) we find

$$\Phi_\lambda(\nu) - \Phi_\lambda(\mu(t)) \geq \frac{\lambda}{2}(\hat{\nu}(\Omega) - \hat{\mu}(t)(\Omega))$$

$$+ \int_{(\overline{\Omega}\times\overline{\Omega})\setminus(\partial\Omega\times\partial\Omega)} \langle \nabla h_{\mu(t)}(x), y-x\rangle d\gamma_t(x,y) - \omega(W_2^2(\mu(t),\nu))$$

whenever $W_2^2(\mu(t),\nu) \leq e^{-3}$. Since $\mu(t)$ satisfies the continuity equation, for almost every t there holds (see [AGS, Theorem 8.4.7])

$$\frac{1}{2}\frac{d}{dt} W_2^2(\mu(t),\nu) = \int_{\overline{\Omega}\times\overline{\Omega}} \langle \chi_\Omega(x)\nabla h_{\mu(t)}(x), y-x\rangle d\gamma_t(x,y).$$

Substituting in the previous relation we get (for $W_2^2(\mu(t), \nu) \leq e^{-3}$)

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu(t), \nu) \leq \Phi_\lambda(\nu) - \Phi_\lambda(\mu(t)) - \frac{\lambda}{2}(\hat{\nu}(\Omega) - \hat{\mu}(t)(\Omega))$$
$$+ \omega(W_2^2(\mu(t), \nu) - \int\limits_{\partial\Omega\times\Omega} \langle \nabla h_{\mu(t)}(x), y - x\rangle d\gamma_t(x, y).$$

On supp $(\widetilde{\mu}(t))$, $\nabla h_{\mu(t)}$ points towards the interior of the convex domain, then the last term is non positive, and so, for $W_2^2(\mu(t), \nu) \leq e^{-3}$,

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu(t), \nu) \leq \Phi_\lambda(\nu) - \Phi_\lambda(\mu(t)) - \frac{\lambda}{2}(\hat{\nu}(\Omega) - \hat{\mu}(t)(\Omega)) + \omega(W_2^2(\mu(t), \nu)). \tag{35}$$

Applying (35) first to $\mu = \mu^1(t)$, with $\nu = \mu^2(s)$, and then reversing the roles of μ^1 and μ^2, we can sum the corresponding inequalities as done in [AS, Theorem 3.2] (for a rigorous argument, see [AGS, Lemma 4.3.4]) and we get

$$\frac{d}{dt}W_2^2(\mu^1(t), \mu^2(t)) \leq 4\omega(W_2^2(\mu^1(t), \mu^2(t)))$$

for almost every t such that $W_2^2(\mu^1(t), \mu^2(t)) \leq e^{-3}$. Now we make use of the logarithmic bound, that yields $\int_0^1 1/\omega(s)\,ds = \infty$. So Gronwall's lemma entails $\mu^1(t) = \mu^2(t)$ for all $t \in [0, T]$. □

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