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Abstract. – *Let Δ be thick affine building of type $\widetilde{G}_2$. We prove that any eigenfunction of the Laplace operators of Δ, associated with a pair (γ_1, γ_2), is the Poisson transform of a suitable finitely additive measure on the maximal boundary Ω of Δ, by using only the combinatorial structure of Δ.*

1. – Introduction.

In his paper [5], S. Helgason conjectured that any joint-eigenfunction of all invariant differential operators on a symmetric space can be given by the Poisson integral. This problem has been solved in [9]. In [10] S. Kato proved a p-adic analogue of the Helgason's conjecture by giving necessary and sufficient conditions for the bijectivity of the Poisson integral in terms of the cyclicity of a K-fixed vector of an unramified principal series representation.

Since the affine buildings introduced by Tits [16] are the objects playing the role of symmetric spaces for semisimple matrix groups over p-adic fields , we may consider Kato's results as results on buildings.

Nonetheless the definition of affine building is strictly combinatorial. Therefore it is interesting to prove an analogue of the Helgason's conjecture for affine buildings, not necessarily arising from a linear p-adic group, by using only their combinatorial structure.

The case of affine buildings of rank 2 (homogeneous trees) has been solved in [11], by introducing the space of martingales on the Poisson boundary and studying their relation with the solution of the eigenvalue problem for the convolution operator by the radial function μ_1. See also [4].

The buildings of rank 3 may be classified, following Tits [16], into buildings of type $\widetilde{A}_2$, $\widetilde{B}_2$ and $\widetilde{G}_2$ according to the triangle tesselation operated on the Euclidean plane by the associated Coxeter reflection group W.

In [2] it has been proved that there exist affine buildings of type $\widetilde{A}_2$ which are non-linear; moreover in his paper [7] Kantor constructed two geometries having universal covers which are affine buildings of type $\widetilde{B}_2$ and $\widetilde{G}_2$ respectively which do not arise from matrix groups (see [8]).

In [12] we considered affine buildings of type $\widetilde{A}_2$ and two average operators, the Laplacians, which are, in the linear case, the generators of the Hecke algebra of the convolution operators by compactly supported bi-K-invariant functions on the vertices of the building; then we described the solution of the eigenvalue problem for these operators in terms of the Poisson transform of a unique finitely additive measure on the maximal boundary Ω.

In this paper we solve the Helgason's conjecture for affine buildings of type $\widetilde{G}_2$, so extending the results obtained in [12].

The strategy we use here is inspired from [12]: we first define the Poisson kernel and the Laplace operators $\mathcal{L}_1, \mathcal{L}_2$ on the building, then we reduce the problem to an eigenvalue problem on an abstract apartment $\mathbb{A}$ for two operators obtained by retracting on $\mathbb{A}$ the Laplace operators with respect to a chamber. Actually the characterization of the eigenfunctions of the Laplace operators is based on the property that any joint eigenspace of the retracted operators has dimension equal to 12 (i.e. the cardinality of the associated finite Coxeter reflection group) and it has a basis consisting of functions obtained by retracting the Poisson kernel for a suitable choice of boundary points. To this purpose we select on $\mathbb{A}$ a fundamental region $\mathcal{R}_0$ consisting of 12 vertices, with the property that if an eigenfunction is zero on $\mathcal{R}_0$, then it is the null function on $\mathbb{A}$, and we evaluate on the vertices of $\mathcal{R}_0$ the retraction of the Poisson kernel corresponding to twelve suitable boundary points, so forming a 12×12 matrix $\mathbb{P}$. The non-singularity of the matrix $\mathbb{P}$ implies the existence of the required basis for the eigenspace and then the theorem.

Nevertheless buildings of type $\widetilde{G}_2$ present substantial differences from those of type $\widetilde{A}_2$. First of all, the residue of a vertex depends on its type, and there is only one type of special vertices which form, on any apartment, a lattice of $\mathbb{Z}^2$. Moreover, in the linear case, the compactly supported bi-K-invariant functions on the special vertices are generated by the characteristic functions of the set of vertices at distance respectively two and four from the origin. For these reasons, Poisson kernel and Laplace operators are defined only on the special vertices and these operators are the average operators on the vertices at distance respectively 2 and 4 from a fixed special vertex. Therefore it is much harder to evaluate, in this context, the retraction (with respect to a chamber c_0 of the building) of the Poisson kernel.

To overcome these difficulties, in this paper we introduce a quick and easy to handle tool to evaluate the retraction (with respect to any c_0) of the Poisson kernel. More precisely we introduce three operators T_i which describe the retraction on $\mathbb{A}$ and we associate to each of them a 12×12 matrix. Once the problem is reduced to a matricial problem, we are able to determine the matrix $\mathbb{P}$, combining in a appropriate manner these matrices, and then to compute (by using a mathematical software) its determinant.

This method of computing the retraction by using the operators T_i easily applies to buildings of type $\widetilde{B}_2$ and it allows us to solve Helgason's conjecture also in this case [13].

In this paper we discussed also the cases when only one type of special vertices or the non-special vertices are considered. We point out that these cases differ from those when all types of special vertices are considered, because for them Helgason's conjecture fails, for there exist eigenfunctions of the Laplace operators that cannot be expressed in terms of a finitely additive measure on Ω. The same happens if we consider only one type of vertices in a building of type $\widetilde{A}_2$, as it will be shown in a forthcoming paper [14].

In Section 2 we collect all general facts we need about buildings of type $\widetilde{G}_2$ and we define the Poisson kernel.

In Section 3 we define the operators T_i, the associated matrices and we describe the technique to evaluate the retraction of the Poisson kernel in term of them.

In Section 4 we introduce the Laplace operators and we evaluate their joint eigenvalues. Moreover we retract these operators on the abstract apartment and we describe the fundamental region $\mathcal{R}_0$ on it.

In Section 5 we compute the matrix $\mathbb{P}$ and we evaluate his determinant.

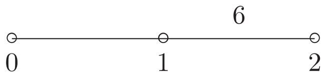
Finally in Section 6 we prove the theorem.

We would like to thank T. Steger for valuable suggestions about the matter of this paper and Donald Cartwright for helping us to manage the problem of computing the determinant of a 12×12 matrix, whose entries are polynomials in four variables of degree up to 20.

2. – Preliminaries.

2.1 – *The building.*

A thick affine building Δ of rank 3 is a simplicial complex consisting of vertices, edges and triangles (the "chambers"), realized by "gluing together" a family of subcomplexes (the "apartments"), each of which is isomorphic to the Coxeter complex $\mathbb{A}$ of a reflection group W acting on the Euclidean plane E^2. We refer to [15] and to [1] for the formal definition. The building has type $\widetilde{G}_2$ if the Coxeter graph is the following:



In this case

$$W = \langle \{r_0, r_1, r_2\} : r_0^2 = r_1^2 = r_2^2 = 1, (r_0 r_1)^3 = (r_0 r_2)^2 = (r_1 r_2)^6 = 1 \rangle.$$

We denote by $\mathcal{V}, \mathcal{E}$ and $\mathcal{C}$ the set of vertices, the set of edges and the set of chambers of Δ respectively; we also denote by d the usual graph-theoretic distance on $\mathcal{V}$. There is a functions τ from $\mathcal{E}$ (resp on $\mathcal{V}$) onto $\mathbb{Z}/3\mathbb{Z}$, called "the type", such that each chamber contains one vertex and one edge of each type. We assign the type to the vertices and to the edges of any chamber according to the notation of the Coxeter graph, as shown in Figure 1. The angle between the i-edge and the j-edge of any chamber is π/m_{ij}, where $m_{01} = 3, m_{12} = 6$ and $m_{02} = 2$.

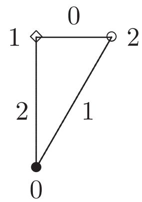


Fig. 1.

We define two chambers $i-$adjacent if they share an edge of type i. Any two chambers c, c' may be joined by a "minimal" gallery $[c, c']$ of two by two adjacent and distinct chambers $c_0 = c, \ldots, c_l = c'$; if c_{k-1} and c_k are i_k-adjacent for each k, then $\pi = (i_1, \ldots, i_l)$ is the "type" of the gallery, denoted by $\pi(c, c')$, and the number $l + 1$ is its "length". Analogously, any chamber c (resp. any vertex x) may be joined to any vertex x' by a minimal gallery $[c, x']$ (resp. $[x, x']$) and $\pi[c, x']$ (resp. $\pi[x, x']$) denotes its type.

The residue $St(x)$ of any vertex x is a spherical building, whose type depends on $\tau(x)$. Actually $St(x)$ has type G_2 if $\tau(x) = 0$, type $A_1 \times A_1$ if $\tau(x) = 1$ and type A_2 if $\tau(x) = 2$. (see [15] or [1]). In particular if $\tau(x) = 0$, the finite Coxeter group W_0 associated to $St(x)$ is the dihedral group of order 12

$$D_6 = \langle \{r_1, r_2\} : r_1^2 = r_2^2 = (r_1 r_2)^6 = 1 \rangle$$

and $W = \mathbb{Z}^2 \rtimes W_0$. Thus the vertices of type 0 are the "special vertices" of Δ.

The building Δ is assumed to be locally finite; therefore any edge belongs to finitely many chambers. We call "valency" of an edge the number of chambers sharing it, and we denote by $q_i + 1$ the valency of any edge of type i. Edges of type 0 and 1 have the same valency, but the valency of edges of type 2 is possibly different. For ease of notation we set $q_0 = q_1 = p$ and $q_2 = q$. In [3] W. Feit and G. Higmann proved that pq is a perfect square; moreover in [6] W. Haemers proved that $p \leq q^3, \; q \leq p^3$.

For any special x, the residue $St(x)$ consists of $(p+1)(p^2q^2 + pq + 1)$ vertices of type 1, $(q+1)(p^2q^2 + pq + 1)$ vertices of type 2 and $(p + 1)(q + 1)(p^2q^2 + pq + 1)$ chambers.

Let x be a special vertex. A sector Q_x based at x is a simplicial cone of vertex x determined, in any apartment containing x, by a chamber, called "base chamber" of the sector, having x as one of its vertices [15]. We call "i-wall" of Q_x, for $i = 1, 2$, the wall containing the edge of type i emanating from x. Two sectors based at a same vertex are said "i-adjacent" if they share a i-wall.

Two sectors Q_x, Q_y are said equivalent, or parallel, $Q_x \sim Q_y$, if they contain a common subsector. The set Ω of equivalence classes ω of parallel sectors is called "maximal boundary" of Δ. We denote by $Q_x(\omega)$ the sector based at x associated with ω. If we fix a special vertex e, Ω may be endowed with a totally disconnected compact Hausdorff topology, generated by the family $\mathcal{B}$ consisting of the sets

$$\Omega(c) = \{\omega \in \Omega : c \subset Q_e(\omega)\}, \quad \forall\, c \in \mathcal{C}.$$

We call "fundamental apartment" of the building the abstract apartment $\mathbb{A}$ and we denote by $\overline{\mathcal{V}}, \overline{\mathcal{E}}$ and $\overline{\mathcal{C}}$ the set of its vertices, the set of its edges and the set of its chambers respectively. The definition of retraction $r_{\omega}^{x_0}$ of Δ to $\mathbb{A}$ with respect to a boundary point ω (of initial vertex x_0) given in [12] applies also to buildings of type $\widetilde{G}_2$, provided x_0 is a special vertex. The same is true for the retraction r_c of the building to the fundamental apartment with respect to a chamber $c \in \mathcal{C}$ (see [12]). For every function f on $\mathcal{V}$, the "retraction" of f with respect to a chamber c is the following function on $\overline{\mathcal{V}}$

$$\widetilde{f}_c(X) = \frac{1}{|r_c^{-1}(X)|} \sum_{x \in r_c^{-1}(X)} f(x).$$

2.2 – *Coordinates on an apartment.*

Given an apartment $\mathcal{A}$ and a sector Q_{x_0} in it, any special vertex of $\mathcal{A}$ may be identified by an ordered pair of integer coordinates (m, n) (with respect to Q_{x_0}) in the following way. We select the line H_1 (resp. H_2) passing through x_0 and containing the 2-wall of Q_{x_0} (resp. the 2-wall of the sector Q'_{x_0} 1-adjacent to Q_{x_0}). The special vertices of H_1 are assigned coordinates $(0, n)$, $n \in \mathbb{Z}$, (resp. $(m, 0)$, $m \in \mathbb{Z}$,) assuming that $(0, 1)$ (resp $(1, 0)$) are the coordinates of the vertex of $Q_{x_0} \cap H_1$ (resp $Q'_{x_0} \cap H_2$) at distance 2 from x_0. Thus a special vertex x of $\mathbb{A}$ has coordinates (m, n) if the line parallel to H_1 (resp. H_2) intersects H_2 (resp. H_1) in the vertex of coordinates $(m, 0)$ (resp. $(0, n)$). With this assumption the special vertices of Q_{x_0} are characterized by coordinates (m, n) with $0 \le m \le n$ (see Figure 2).

If $\mathcal{A}'$ is another apartment containing x and Q_{x_0}, then the coordinates of x in $\mathcal{A}'$ are the same as those in $\mathcal{A}$. Moreover, if $x \in Q_{x_0} \cap Q'_{x_0}$, then x has the same coordinates with respect to both Q_{x_0} and Q'_{x_0}.

Each chamber c of $\mathcal{A}$ may be endowed, with respect to the sector Q_{x_0}, with a triple of integer coordinates (k, m, n), where $(m, n) \in \mathbb{Z}^2$ are the coordinates of the

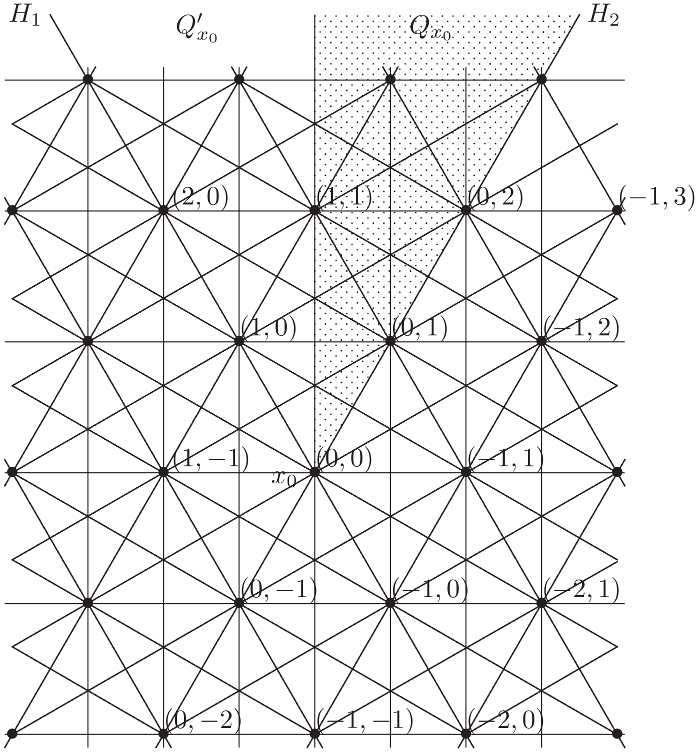


Fig. 2.

special vertex x of c (with respect to Q_{x_0}) and $k \in \{1, \cdots, 12\}$, characterizes (among the chambers of $\mathcal{A}$ sharing the vertex x) the position of c with respect to the sector $Q_x \sim Q_{x_0}$. Figure 3 exhibits the chosen numbering. Assuming that W_0 stabilizes the vertex x of $\mathcal{A}$ of coordinates (m,n), we denote by σ_k the element of W_0 mapping the base chamber of the sector Q_x onto the chamber of coordinates (k,m,n).

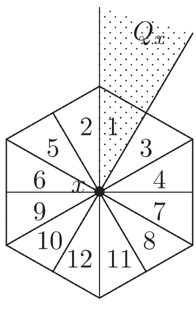


Fig. 3.

We denote by $\mathcal{U}$ (resp. $\overline{\mathcal{U}}$) the set of special vertices of the building (resp. of the fundamental apartment).

On the fundamental apartment $\mathbb{A}$ we set $X = X_{m,n}$ if the special vertex X has coordinates (m, n) (with respect to a sector $\overline{Q}$).

REMARK 2.2.1. – Since each chamber of Δ contains exactly one special vertex, a complex valued function f defined on $\mathcal{U}$ (resp. on $\overline{\mathcal{U}}$) may be interpreted as a function on $\mathcal{C}$ (resp. on $\overline{\mathcal{C}}$), which is constant on all chambers sharing a special vertex, and viceversa.

2.3 – *Poisson kernel.*

Fix a sector $\overline{Q}$ on $\mathbb{A}$.

DEFINITION 2.3.1. – *For every $\alpha, \beta \in \mathbb{C}^{\times}$, let $\phi_{\alpha,\beta} : \overline{\mathcal{U}} \to \mathbb{C}$ be the multiplicative function*

$$\phi_{\alpha,\beta}(X_{m,n}) = \alpha^m \beta^n, \qquad \forall (m, n) \in \mathbb{Z}^2,$$

with respect to the coordinate system associated to $\overline{Q}$. The Poisson kernel, of initial point x_0 and of parameter α, β, is the function

$$P^{x_0}_{\alpha,\beta}(x, \omega) = \phi_{\alpha,\beta}(r^{x_0}_{\omega}(x)), \qquad \forall x \in \mathcal{U}, \quad \forall \omega \in \Omega.$$

We simply write $P(x, \omega) = P^{x_0}_{\alpha,\beta}(x, \omega)$, whenever there is not ambiguity.

REMARK 2.3.2. – By Remark 3.1.1 we may define $P(c, \omega)$, $\forall c \in \mathcal{C}$, as

$$P(c, \omega) = P(x, \omega), \quad \text{if} \quad x \in c.$$

Poisson kernels depend on the initial point in the following way.

LEMMA 2.3.3. – *For every $x_0, y_0 \in \mathcal{U}$*

$$P^{y_0}(x, \omega) = P^{x_0}(x, \omega)(P^{x_0}(y_0, \omega))^{-1}, \qquad \forall x \in \mathcal{U}, \quad \forall \omega \in \Omega. \tag{1}$$

In particular $P^{y_0}(x_0, \omega) = (P^{x_0}(y_0, \omega))^{-1}$, $\forall \omega \in \Omega$.

PROOF. – We refer to [12], Lemma 2.8, for the proof of (1). □

3. – Retraction of the Poisson kernel to $\mathbb{A}$ with respect to a chamber.

3.1 – *The operators T_i.*

For every $C \in \overline{\mathcal{C}}$ we denote by $t_i(C)$ the chamber of $\mathbb{A}$ which is i-adjacent to C and, for every $c \in \mathcal{C}$, we denote by $\mathcal{T}_i(c)$ the set of all chambers of Δ which are i-adjacent to c. Then $|\mathcal{T}_i(c)| = q_i$ and $r_c(\mathcal{T}_i(c)) = t_i(r_c(c))$. Fix $\omega \in \Omega$, $x_0 \in \mathcal{U}$ and consider the retraction $r_\omega^{x_0}$, assuming $r_\omega^{x_0}(Q_{x_0}(\omega)) = \overline{Q}$. We give an orientation (with respect to ω) to the pair $(C, t_i(C))$ by setting

$$C \to t_i(C), \quad \text{if} \quad r_\omega^{x_0}(c) = C \quad \text{implies} \quad r_\omega^{x_0}(\mathcal{T}_i(c)) = t_i(C),$$

$$C \leftarrow t_i(C), \quad \text{otherwise.}$$

See Figure 4 for $i = 0$.

We point out that when $C \leftarrow t_i(C)$, there exists a unique $c_i \in \mathcal{T}_i(c)$ such that $r_\omega^{x_0}(c_i) = t_i(C)$, while for all $c' \in \mathcal{T}_i(c)$, $c' \neq c_i$ we have $r_\omega^{x_0}(c') = C$.

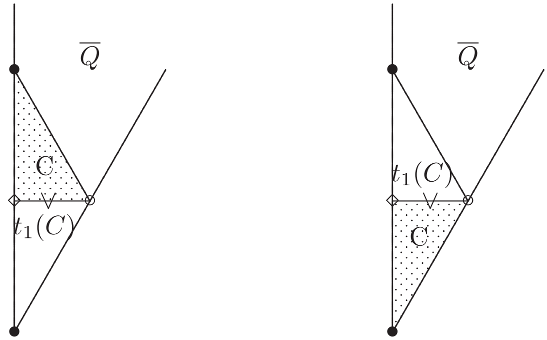


Fig. 4.

As a consequence, each edge of $\mathbb{A}$ may be endowed by an arrow, according to the previous notation, and edges lying on parallel lines have the same orientation. In Figure 5 we exhibit the orientation of the edges belonging to the chambers sharing the base vertex of the sector $\overline{Q}$.

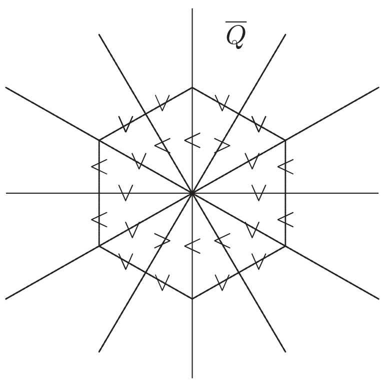


Fig. 5.

DEFINITION 3.1.1. – For every function F on $\overline{\mathcal{C}}$ and for every $C \in \overline{\mathcal{C}}$ we define

$$T_i(F)(C) = \begin{cases} q_i F(t_i(C)), & \text{if} \quad C \to t_i(C), \\ F(t_i(C)) + (q_i - 1)F(C), & \text{if} \quad C \leftarrow t_i(C). \end{cases}$$

For every $C \in \overline{\mathcal{C}}$ we denote by $t_\pi(C)$ the chamber of $\mathbb{A}$ connected to C by a minimal gallery of type $\pi = (i_1, \ldots, i_l)$.

PROPOSITION 3.1.2. – *Let $T_\pi = T_{i_1} \cdots T_{i_l}$. If $r_c(c) = C$, then, for every function F on $\overline{\mathcal{C}}$,*

$$T_\pi(F)(C) = \sum_{c' \in r_c^{-1}(t_\pi(C))} F(r_\omega^{x_0}(c')). \tag{2}$$

PROOF. – We use induction on the length $l+1$ of π.

Let $l = 1$. We recall that the assumption $r_c(c) = C$ implies $r_c^{-1}(t_i(C)) = \mathcal{T}_i(c)$; therefore if $C \to t_i(C)$, then $r_\omega^{x_0}(c') = t_i(C)$, for every $c' \in r_c^{-1}(t_i(C))$. On the other hand, if $C \leftarrow t_i(C)$, there exists a unique chamber c' in $r_c^{-1}(t_i(C))$ such that $r_\omega^{x_0}(c') = t_i(C)$, while $r_\omega^{x_0}(c') = C$ for any other chamber in $r_c^{-1}(t_i(C))$. As a straightforward consequence of the definition of the operators T_i we obtain

$$T_i(F)(C) = \sum_{c' \in r_c^{-1}(t_i(C))} F(r_\omega^{x_0}(c')).$$

We assume now (2) is true for $l-1$ and we prove it for l. Let $\pi_0 = (i_1, \cdots, i_{l-1})$ and let $t_{\pi_0}(C)$ be the chamber i_l-adjacent to $t_\pi(C)$ in the minimal gallery $[C, t_\pi(C)]$. Then

$$T_\pi(F)(C) = T_{\pi_0}(T_{i_l}(F))(C) = \sum_{c' \in r_c^{-1}(t_{\pi_0}(C))} T_{i_l}(F))(r_\omega^{x_0}(c')).$$

On the other hand, by denoting $C' = r_\omega^{x_0}(c')$, for every $c' \in r_c^{-1}(t_{\pi_0}(C))$, we have

$$T_{i_l}(F)(C') = \sum_{c'' \in r_{c'}^{-1}(t_{i_l}(C'))} F(r_\omega^{x_0}(c'')).$$

Since

$$\bigcup_{c' \in r_c^{-1}(t_{\pi_0}(C))} r_{c'}^{-1}(t_{i_l}(C')) = r_c^{-1}(t_\pi(C)),$$

then

$$\sum_{c' \in r_c^{-1}(t_{\pi_0}(C))} \left(\sum_{c'' \in r_{c'}^{-1}(t_{i_l}(C'))} F(r_\omega^{x_0}(c'')) \right) = \sum_{\tilde{c} \in r_c^{-1}(t_\pi(C))} F(r_\omega^{x_0}(\tilde{c})).$$

So (2) is proved. □

Let us consider the function $P(\cdot,\omega) = P^{x_0}(\cdot,\omega)$ on $\mathcal{U}$. The retraction of $P(\cdot,\omega)$ on $\mathbb{A}$ with respect to a chamber c_0 is the function

$$\widetilde{P}(X,\omega) = \frac{1}{|r_{c_0}^{-1}(X)|}\sum_{x\in r_{c_0}^{-1}(X)} P(x,\omega) = \frac{1}{|r_{c_0}^{-1}(X)|}\sum_{x\in r_{c_0}^{-1}(X)} \phi_{a,\beta}(r_\omega^{x_0}(x)), \qquad \forall X \in \overline{\mathcal{U}},$$

and by Remark 2.2.1 it may be written also in the following way

$$\widetilde{P}(X,\omega) = \widetilde{P}(C,\omega) = \frac{1}{|r_{c_0}^{-1}(C)|}\sum_{c\in r_{c_0}^{-1}(C)} \phi_{a,\beta}(r_\omega^{x_0}(c)), \qquad \text{if} \quad X \in C.$$

Assuming $r_{c_0}(c_0) = C_0 = r_\omega^{x_0}(c_0)$ we have the following corollary.

COROLLARY 3.1.3. – *For every $X \in \overline{\mathcal{U}}$, let C be the chamber containing X in a minimal gallery connecting C_0 to X. Let $\pi = (i_1, \cdots, i_l)$ be the type of $[C_0, C]$. Then*

$$\widetilde{P}(X,\omega) = \widetilde{P}(C,\omega) = \frac{1}{|r_{c_0}^{-1}(C)|} T_\pi(\phi_{a,\beta})(C_0).$$

3.2 – *The matrices M_i.*

Fix $(m,n) \in \mathbb{Z}^2$; for every $k \in \{1,\ldots,12\}$ we denote by C_k the chamber of coordinates (k,m,n) with respect to $\overline{Q}$. Let $c \in \mathcal{C}$ and assume $C_k = r_c(c) = r_\omega^{x_0}(c)$. For every $h,k \in \{1,\ldots,12\}$ we denote by $\rho_{hk}^{(i)}$ the number of chambers $c' \in r_c^{-1}(t_i(C_k))$ such that $r_\omega^{x_0}(c')$ has coordinates (h,m',n') for some $(m',n') \in \mathbb{Z}^2$. If $\rho_{hk}^{(i)} \neq 0$, then (m',n') is uniquely determined; so we may form the 12×12-matrix $M_i = (m_{hk}^{(i)})$ whose hk-entry is

$$m_{hk}^{(i)} = \rho_{hk}^{(i)} a^{m'-m}\beta^{n'-n}. \tag{3}$$

Representing (3) in the explicit form, we get

$$M_0 = \begin{pmatrix}
p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{\beta} & 0 & 0\\
0 & p-1 & 0 & 0 & 0 & 0 & 0 & \frac{p}{a} & 0 & 0 & 0 & 0\\
0 & 0 & p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{\beta}\\
0 & 0 & 0 & p-1 & 0 & \frac{pa}{\beta} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & p-1 & 0 & 0 & 0 & 0 & 0 & \frac{p}{a} & 0\\
0 & 0 & 0 & \frac{\beta}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & p-1 & 0 & \frac{pa}{\beta} & 0 & 0 & 0\\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta}{a} & 0 & 0 & 0 & 0 & 0\\
\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & p-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & p-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & p-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & p-1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & q-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & q-1 \end{pmatrix}.$$

These matrices do not depend on (m, n) and they give a useful algorithm for evaluating $T_i(\phi_{a,\beta})(C)$ in terms of the coordinates of C, as the following lemma states.

LEMMA 3.2.1. – *Let C be a chamber of coordinates (k, m, n) with respect to $\overline{Q}$ and let $c \in \mathcal{C}$ be such that $C = r_c(c) = r_\omega^{x_0}(c)$. Then*

$$T_i(\phi_{a,\beta})(C) = a^m \beta^n V_0 M_i e_k,$$

where V_0 is a 1×12-matrix, e_k is a 12×1-matrix such that $V_{0,h} = 1$, and $e_{h,k} = \delta_{hk}$.

PROOF. – The formula is an immediate consequence of the definitions of the operator T_i and of the matrix M_i. Actually, for every k, the chamber C coincides with the chamber C_k defined above and $T_i(\phi_{a,\beta})(C_k)$ is the product of the vector $a^m \beta^n V_0$ by the k-column of the matrix M_i. □

The previous characterization may be extended to any product T_π, $\pi = (i_1, \ldots, i_l)$.

LEMMA 3.2.2. – *Let C be a chamber of coordinates (k, m, n) with respect to $\overline{Q}$ and let $c \in \mathcal{C}$ be such that $C = r_c(c) = r_\omega^{x_0}(c)$. Then for every $\pi = (i_1, \ldots, i_l)$*

$$T_\pi(\phi_{a,\beta})(C) = a^m \beta^n V_0 M_\pi e_k, \tag{4}$$

where $M_\pi = M_{i_l} \ldots M_{i_1}$.

PROOF. – Let $l = 2$. For every $(m, n) \in \mathbb{Z}^2$ we consider the chambers $C_1, \ldots, C_{12}$ containing $X_{m,n}$. By Lemma 3.2.1 we have

$$a^m \beta^n V_0 M_{i_2} M_{i_1} = \big(T_{i_2}(\phi_{a,\beta})(C_1), \ldots, T_{i_2}(\phi_{a,\beta})(C_{12})\big) M_{i_1}.$$

On the other hand, as a straightforward consequence of definition, the product of the vector $\big(T_{i_2}(\phi_{a,\beta})(C_1), \ldots, T_{i_2}(\phi_{a,\beta})(C_{12})\big)$ by the k-column of the matrix M_{i_1} gives the value of $T_{i_1} T_{i_2}(\phi_{a,\beta})$ in the chamber C_k. By induction on l we get (4). □

Assume $r_{c_0}(c_0) = C_0 = r_\omega^{x_0}(c_0)$. Let (k, m, n) be the coordinates of C_0 with respect to the fixed sector $\overline{Q}$. We have

THEOREM 3.2.3. – *For every $X \in \overline{\mathcal{U}}$, let C be the chamber containing X in a minimal gallery connecting C_0 to X. Let $\pi = (i_1, \cdots, i_l)$ the type of $[C_0, C]$; then*

$$\widetilde{P}(X, \omega) = \widetilde{P}(C, \omega) = \frac{1}{|r_{c_0}^{-1}(C)|} a^m \beta^n V_0 M_\pi e_k.$$

PROOF. – The previous identity follows immediately from Corollary 3.1.3 and from Lemmas 3.2.1 and 3.2.2. □

4. – Eigenfunctions of the Laplace Operators.

4.1 – *Laplace operators on the building.*

For every $x \in \mathcal{U}$ we define

$$S_1(x) = \{y \in \mathcal{U} : \pi(x, y) = (0)\},$$
$$S_2(x) = \{y \in \mathcal{U} : \pi(x, y) = (0, 1, 2, 1, 0)\}.$$

Analogously, for every $X \in \overline{\mathcal{U}}$, we define (see Figure 6):

$$\overline{S}_1(X) = \{Y \in \overline{\mathcal{U}} : \pi(X, Y) = (0)\} = \{X_1, \cdots, X_6\},$$
$$\overline{S}_2(X) = \{Y \in \overline{\mathcal{U}} : \pi(X, Y) = (0, 1, 2, 1, 0)\} = \{X_7, \cdots, X_{12}\}.$$

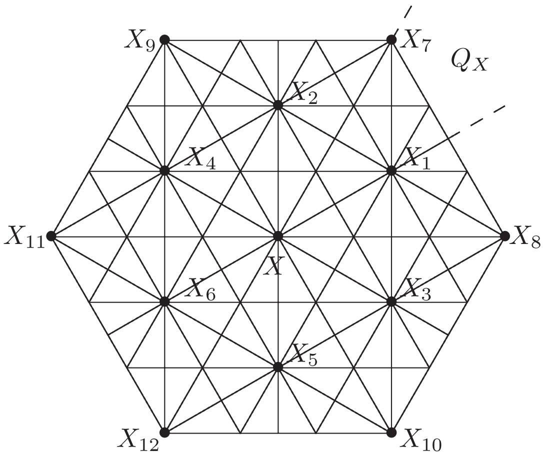


Fig. 6.

If we consider the retraction of Δ on $\mathbb{A}$ with respect to a chamber c containing the vertex x, then

$$S_i(x) = \bigcup_{Y \in \overline{S}_i(X)} r_c^{-1}(Y), \quad i = 1, 2,$$

where $X = r_c(x)$. This implies

$$|S_1(x)| = p(p+1)(p^2q^2 + pq + 1)|S_2(x)| = p^4q(q+1)(p^2q^2 + pq + 1).$$

We set $K_i = |S_i(x)|$.

DEFINITION 4.1.1. – *We define Laplace operators on Δ the linear operators*

$$\mathcal{L}_i f(x) = K_i^{-1} \sum_{y \in \mathcal{S}_i(x)} f(y), \qquad \forall x \in \mathcal{U}, \qquad i = 1, 2,$$

acting on the space of complex valued functions f on $\mathcal{U}$.

4.2 – *The eigenspace $\mathcal{S}(\gamma_1, \gamma_2)$.*

For every pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, we denote

$$\mathcal{S}(\gamma_1, \gamma_2) = \{f : \mathcal{U} \to \mathbb{C} \ / \ \mathcal{L}_i f = \gamma_i f, \ \ i = 1, 2\}.$$

PROPOSITION 4.2.1. – *Let $(\alpha, \beta) \in (\mathbb{C}^\times)^2$. For every $\omega \in \Omega$ and for every $x_0 \in \mathcal{U}$, the function $P^{x_0}_{\alpha,\beta}(\cdot, \omega)$ belongs to the eigenspace $\mathcal{S}(\gamma_1, \gamma_2)$, associated with* *eigenvalues* $\gamma_i = \gamma_i(\alpha, \beta)$ *given by*

$$\begin{aligned}\gamma_1(\alpha,\beta) &= K_1^{-1}(p\alpha + \beta + p^3q^2\alpha^{-1} + p^4q^2\beta^{-1} + p^3q\alpha\beta^{-1} + pq\alpha^{-1}\beta) + c_1,\\ \gamma_2(\alpha,\beta) &= K_2^{-1}(p-1)(q+1)(p\alpha + \beta + p^3q^2\alpha^{-1} + p^4q^2\beta^{-1} + p^3q\alpha\beta^{-1} + pq\alpha^{-1}\beta)\\ &\quad + K_2^{-1}(\alpha\beta + p^6q^4\alpha^{-1}\beta^{-1} + p^6q^3\alpha\beta^{-2} + q\alpha^{-1}\beta^2 + p^3q\alpha^2\beta^{-1} + p^3q^3\alpha^{-2}\beta)\\ &\quad + c_2,\end{aligned}$$

where

$$\begin{aligned}c_1 &= K_1^{-1}(p-1)(pq+p+1),\\ c_2 &= K_2^{-1}(p^2(q-1) + 2pq(p-1)^2 + p^2q(q-1) + p^3q(q-1) + pq(p-1)^2(q-1)).\end{aligned}$$

PROOF. – Let $P^{x_0}(\cdot,\omega) = P^{x_0}_{\alpha,\beta}(\cdot,\omega)$. For every $x \in \mathcal{U}$ and $i = 1, 2$, we have

$$\mathcal{L}_i P^{x_0}(\cdot,\omega)(x) = K_i^{-1}\sum_{y\in S_i(x)} P^{x_0}(y,\omega) = K_i^{-1}\left(\sum_{y\in S_i(x)} P^{x}(y,\omega)\right)P^{x_0}(x,\omega).$$

Let c be the base chamber of the sector $Q_x(\omega)$ and let $C = r_c(c) = r^x_\omega(c)$; then $S_i(x) = \bigcup_{Y\in\overline{S}_i(X)} r_c^{-1}(Y)$, $i = 1, 2$. So we may compute the sum on the right using the method illustrated in Section 3. For every $j \in \{1, \dots, 12\}$, we denote by C_j the chamber containing X_j in a minimal gallery connecting C to X_j, and we denote by π_j the type of the gallery $[C, C_j] = [C, X_j]$. We have

$$\begin{aligned}\pi_1 &= (0)\\ \pi_2 &= (1,0)\\ \pi_3 &= (2,1,0)\\ \pi_4 &= (1,2,1,0)\\ \pi_5 &= (2,1,2,1,0)\\ \pi_6 &= (1,2,1,2,1,0)\\ \pi_7 &= (0,1,2,1,0)\\ \pi_8 &= (0,2,1,2,1,0)\\ \pi_9 &= (1,0,2,1,2,1,0)\\ \pi_{10} &= (2,1,0,2,1,2,1,0)\\ \pi_{11} &= (1,2,1,0,2,1,2,1,0)\\ \pi_{12} &= (2,1,2,1,0,2,1,2,1,0).\end{aligned}$$

Since the coordinates of C with respect to the sector $Q_X = r^x_\omega(Q_x(\omega))$ are $(1, 0, 0)$, we have

$$\sum_{y\in S_1(x)} P^x(y,\omega) = \sum_{j=1}^{6} V_0 M_{\pi_j} e_1,$$

$$\sum_{y\in S_1(x)} P^x(y,\omega) = \sum_{j=7}^{12} V_0 M_{\pi_j} e_1.$$

Therefore the two sums on the left are independent on x and give the joint eigenvalues $\gamma_1(a,\beta), \gamma_2(a,\beta)$. The required formulas for these eigenvalues follow from an explicit computation of $V_0 M_{\pi_j} e_1$, for every j. □

DEFINITION 4.2.2. – *The pairs (a,β) and (a',β') are said "equivalent", $(a,\beta) \sim (a',\beta')$, if $\gamma_i(a,\beta) = \gamma_i(a',\beta')$, $i = 1,2$.*

Let us assume that the finite Coxeter group W_0 stabilizes the base vertex $\overline{X}$ of the sector $\overline{Q}$. Thus each element $\sigma \in W_0$ acts on the multiplicative functions on $\overline{\mathcal{U}}$ (with respect to $\overline{Q}$): for every multiplicative ϕ, the function

$$\sigma(\phi)(X_{m,n}) = \phi(\sigma(X_{m,n})), \quad \forall (m,n) \in \mathbb{Z}^2,$$

is multiplicative; for every pair $(\xi,\eta) \in (\mathbb{C}^\times)^2$ we denote by (ξ_σ,η_σ) the pair such that

$$\sigma(\phi_{\xi,\eta}) = \phi_{\xi_\sigma,\eta_\sigma}.$$

LEMMA 4.2.3. – *For every $(a,\beta) \in (\mathbb{C}^\times)^2$, let $\xi = \dfrac{a}{pq}$ and $\eta = \dfrac{\beta}{p^2 q}$. Then, for every $\sigma \in W_0$,*

$$(a,\beta) \sim \sigma(a,\beta),$$

where $\sigma(a,\beta) = (pq\xi_\sigma, p^2 q\eta_\sigma)$.

PROOF. – Setting $a = pq\xi$ and $\beta = p^2 q\eta$, we obtain the following expression for $\Gamma_i(\xi,\eta) = \gamma_i(pq\xi, p^2 q\eta)$ $(i = 1,2)$:

$$\begin{aligned}
\Gamma_1(\xi,\eta) &= K_1^{-1} p^2 q(\xi + \xi^{-1} + \eta + \eta^{-1} + \xi\eta^{-1} + \xi^{-1}\eta) + c_1,\\
\Gamma_2(\xi,\eta) &= K_2^{-1} p^2 q(p-1)(q+1)(\xi + \xi^{-1} + \eta + \eta^{-1} + \xi\eta^{-1} + \xi^{-1}\eta)\\
&\quad + K_2^{-1} p^3 q^2 (\xi\eta + \xi^{-1}\eta^{-1} + \xi\eta^{-2} + \xi^{-1}\eta^2 + \xi^2\eta^{-1} + \xi^{-2}\eta)\\
&\quad + c_2.
\end{aligned}$$

These formulas show that $\Gamma_1(\xi,\eta)$ and $\Gamma_2(\xi,\eta)$ are W_0-invariant. Therefore, for every (a,β) and for every $\sigma \in W_0$,

$$\gamma_i(a,\beta) = \gamma_i(pq\xi_\sigma, p^2 q\eta_\sigma), \; i = 1,2.$$

□

REMARK 4.2.4. – For every pair $(\gamma_1,\gamma_2) \in \mathbb{C}^2$, there exists $(a,\beta) \in (\mathbb{C}^\times)^2$ such that $\gamma_i = \gamma_i(a,\beta)$, $i = 1,2$. Actually, by setting $a_1 = \xi + \eta$, $\;a_2 = \xi^{-1} + \eta^{-1}$ and then $b_1 = a_1 + a_2$, $\;b_2 = a_1 a_2$, we may easily prove that, for every pair $(A_1, A_2) \in \mathbb{C}^2$, there exists (ξ,η) such that

$$\begin{aligned}
A_1 &= \xi + \xi^{-1} + \eta + \eta^{-1} + \xi\eta^{-1} + \xi^{-1}\eta),\\
A_2 &= \xi\eta + \xi^{-1}\eta^{-1} + \xi\eta^{-2} + \xi^{-1}\eta^2 + \xi^2\eta^{-1} + \xi^{-2}\eta).
\end{aligned}$$

COROLLARY 4.2.5. – *Let* $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$. *Then*

$$P^{x_0}_{\sigma(\alpha,\beta)}(\cdot, \omega) \in \mathcal{S}(\gamma_1, \gamma_2), \quad \forall \omega \in \Omega, \quad \forall \sigma \in W_0.$$

4.3 – *Retraction of the Laplace operators.*

Given a chamber c_0, we prove that, as for buildings of type $\widetilde{A}_2$, the Laplace operators on Δ retract to a pair of linear operators on $\mathbb{A}$.

PROPOSITION 4.3.1. – *Let* $c_0 \in \mathcal{C}$; *there exist non negative* $\chi_i = \chi_i(X, Y)$, $i = 1, 2$, *such that*

$$(\widetilde{\mathcal{L}_i f})_{c_0}(X) = \sum_{Y \in \overline{S}_i^{\sharp}(X)} \chi_i(X, Y) \widetilde{f}_{c_0}(Y), \qquad \forall X \in \overline{\mathcal{U}}, \quad i = 1, 2, \tag{5}$$

where $\overline{S}_1^{\sharp}(X) = \overline{S}_1(X) \cup \{X\}$, $\overline{S}_2^{\sharp}(X) = \overline{S}_2(X) \cup \overline{S}_1(X) \cup \{X\}$. *Moreover* $\chi_i(X, Y) > 0$ *if* $Y \in \overline{S}_i(X)$.

PROOF. – Let $X \in \overline{\mathcal{U}}$; we observe that, for every $x \in r_{c_0}^{-1}(X)$,

$$\mathcal{S}_i(x) = \bigcup_{Y \in \overline{S}_i^{\sharp}(X)} \mathcal{S}_i(x) \cap r_{c_0}^{-1}(Y)$$

as a disjoint union. This implies that, for every X,

$$\widetilde{L_i f}_{c_0}(X) = \frac{K_i^{-1}}{|r_{c_0}^{-1}(X)|} \sum_{Y \in \overline{S}_i^{\sharp}(X)} \left(\sum_{x \in r_{c_0}^{-1}(X)} \left(\sum_{y \in \mathcal{S}_i(x) \cap r_{c_0}^{-1}(Y)} f(y) \right) \right).$$

On the other hand, for every $Y \in \overline{S}_i^{\sharp}(X)$,

$$\bigcup_{x \in r_{c_0}^{-1}(X)} \mathcal{S}_i(x) \cap r_{c_0}^{-1}(Y) = r_{c_0}^{-1}(Y),$$

and, for every $y \in r_{c_0}^{-1}(Y)$,

$$\{x \in r_{c_0}^{-1}(X) : \ y \in S_i(x)\} = \mathcal{S}_i(y) \cap r_{c_0}^{-1}(X),$$

as $y \in S_i(x)$ if and only if $x \in S_i(y)$. Since the cardinality of $\mathcal{S}_i(y) \cap r_{c_0}^{-1}(X)$ does not depend on the choice of y in $r_{c_0}^{-1}(Y)$, and, for every X, the set $\mathcal{S}_i(y) \cap r_{c_0}^{-1}(X)$ is not empty if $Y \in \overline{S}_i(X)$, the proposition holds. $\square$

From now on we denote by $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ the linear operators on $\mathbb{A}$ obtained by retracting $\mathcal{L}_1$ and $\mathcal{L}_2$ with respect to a chamber c_0:

$$\widetilde{\mathcal{L}}_i F(X) = \sum_{Y \in \overline{S}_i^{\sharp}(X)} \chi_i(X, Y) F(Y), \quad \forall X \in \overline{\mathcal{U}}, \ i = 1, 2.$$

Actually $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ are independent of the choice of the chamber c_0, but they depend only on the choice of the chamber C_0 on $\mathbb{A}$ such that $r_{c_0}(c_0) = C_0$. We point out that the coefficients of $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ really depend on the vertex X.

For every pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, we denote

$$\widetilde{\mathcal{S}}(\gamma_1, \gamma_2) = \{F : \overline{\mathcal{U}} \to \mathbb{C} : \widetilde{\mathcal{L}}_i F = \gamma_i F,\ i = 1, 2\}.$$

As a straightforward consequence of definition, $\widetilde{f}_{c_0} \in \widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, for every function f in $\mathcal{S}(\gamma_1, \gamma_2)$.

4.4 – *Fundamental region on* $\mathbb{A}$.

As for a building of type $\widetilde{A}_2$, we may choose on $\mathbb{A}$ a particular region $\mathcal{R}_0$ characterized by the property that knowing the values of a function of $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$ on the special vertices of this region allows to reconstruct the whole function on $\overline{\mathcal{U}}$.

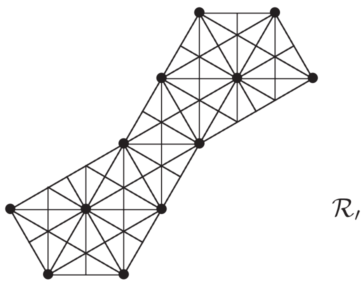


Fig. 7.

DEFINITION 4.4.1. – *We call "fundamental region" $\mathcal{R}_0$ of $\mathbb{A}$ any region obtained applying any element $w \in W$ to the region pictured in Figure* 7.

PROPOSITION 4.4.2. – *Let $F \in \widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$; then F is uniquely determined by its values on the special vertices of $\mathcal{R}_0$.*

PROOF. – We choose on $\mathbb{A}$ the fundamental region consisting of the vertices

$$X_{0,0},\ X_{1,0},\ X_{0,1},\ X_{-1,0},\ X_{-1,1},\ X_{2,0},\ X_{1,1},\ X_{-2,1},\ X_{-1,-1},\ X_{-2,0},\ X_{2,1},\ X_{1,2},$$

with respect to the sector $\overline{Q}$. For every $n \geq 1$ consider, as shown in Figure 8, the hexagon $\mathcal{R}_n$ of base vertices:

$$X_{n,1},\ X_{1,n},\ X_{-n,n},\ X_{-n,0},\ X_{-n+1,-n+1},\ X_{n,-n+1}.$$

As in [12, Proposition 3.5], we use induction on n to prove that a function F of $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, which is zero on the vertices of $\mathcal{R}_0$, is the null function. □

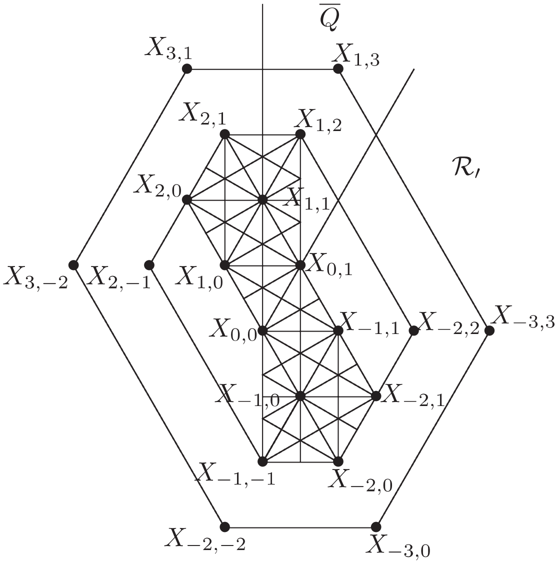


Fig. 8.

COROLLARY 4.4.3. – *For every pair* $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, $\dim \widetilde{\mathcal{S}}(\gamma_1, \gamma_2) \leq 12$.

PROOF. – Let $\mathcal{R}_0$ be a fundamental region of $\mathbb{A}$ and $Y_1, \dots, Y_{12}$ its vertices. The map $F \longmapsto \{F(Y_1), \dots, F(Y_{12})\}$ is a linear injection from $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$ to $\mathbb{C}^{12}$. □

4.5 – *A basis for* $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$.

Let $(\alpha, \beta) \in (\mathbb{C}^\times)^2$ and $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$; for every chamber c_0, the retraction $\widetilde{P}(\cdot, \omega)$ of $P^{x_0}_{\alpha,\beta}(\cdot, \omega)$ belongs to $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, for every $\omega \in \Omega$ and $x_0 \in \mathcal{U}$.

Fix a chamber c_0 and assume $x_0 \in c_0$. Our aim is to check that it is possible to choose twelve different boundary points $\omega_1, \dots, \omega_{12}$ in such a way that the corresponding functions $\widetilde{P}(\cdot, \omega_1), \cdots, \widetilde{P}(\cdot, \omega_{12})$ are linearly independent and therefore they may be chosen as a basis for the eigenspace.

DEFINITION 4.5.1. – *We denote by* $\Omega_k = \Omega_k(c_0)$ *the set of all boundary points* ω *such that the base chamber of the sector* $Q_{x_0}(\omega)$ *has coordinates* $(k, 0, 0)$ *with respect to the sector based at* c_0 *in any apartment containing both sectors.*

As immediate consequence of definition we have $\Omega = \bigcup_{k=1}^{12} \Omega_k$ and $\Omega_k \cap \Omega_{k'} = \emptyset$, if $k \neq k'$. Moreover there exists a unique value of k depending on c_0, say k_{c_0}, such that $\Omega_{k_{c_0}} = \Omega(c_0)$. In this sense the sets $\Omega_1, \ldots, \Omega_{12}$ play the role of the sets $\Omega_{C,j}$, $j = 1, \ldots, 6$, defined in [12, Definition 3.8].

REMARK 4.5.2. – On the fundamental apartment $\mathbb{A}$ we consider the sector $\overline{Q}$ emanating from the vertex $\overline{X}$ and we denote by $\overline{C}$ its base chamber. For $k = 1, \ldots, 12$ we consider the chamber C_k containing $\overline{X}$ and having coordinates $(k, 0, 0)$ with respect to $\overline{Q}$; moreover we denote by Q_k and $\widetilde{\omega}_k$ the sector based at C_k and the corresponding boundary point (see Figure 9).

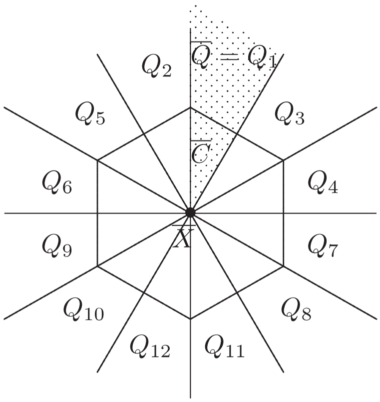


Fig. 9.

If r_{c_0} maps c_0 onto $\overline{C}$, then

$$\Omega_k = \{\omega \in \Omega : \ r_{c_0}(Q_{x_0}(\omega)) = Q_k\},$$

and therefore $r_{c_0}(\omega) = \widetilde{\omega}_k$, $\forall \omega \in \Omega_k$.

For every $\omega \in \Omega_k$, let assume that $r_\omega^{x_0}$ maps $Q_{x_0}(\omega)$ onto Q_k. Then we have:

PROPOSITION 4.5.3. – *Let $(k', 0, 0)$ be the coordinates of the chamber $\sigma_k^{-1}(\overline{C})$ (with respect to $\overline{Q}$). For every $\omega \in \Omega_k$ and for every $X \in \overline{\mathcal{U}}$*

$$\widetilde{P}(X, \omega) = \frac{1}{|r_{c_0}^{-1}(X)|} V_0 M_\pi e_{k'}, \tag{6}$$

if π is the type of a minimal gallery connecting $\overline{C}$ to X.

PROOF. – For every k we have $\sigma_k(\overline{Q}) = Q_k$. Thus $\overline{C}$ has the same coordinates with respect to Q_k as $\sigma_k^{-1}(\overline{C})$ has with respect to $\overline{Q}$. Therefore (6) follows from Theorem 3.2.3. □

REMARK 4.5.4. – Let ω_0 be a boundary point such that $Q_{x_0}(\omega_0)$ is based at c_0, and let $c_k = \sigma_k(c_0)$ be a chamber containing x_0 and having coordinates $(k, 0, 0)$ with respect to $Q_{x_0}(\omega_0)$; then $r^{x_0}_{\omega_0}(c_k) = C_k$, if $r^{x_0}_{\omega_0}$ maps $Q_{x_0}(\omega_0)$ onto $\overline{Q}$. We simply denote by r_k the retraction of the building on $\mathbb{A}$ mapping $\sigma_k^{-1}(c_0)$ onto $\sigma_k^{-1}(\overline{C})$ and by $\widetilde{P}_k(\cdot, \omega_0)$ the retraction of $P^{x_0}(\cdot, \omega_0)$ with respect to r_k. Then, for every $X \in \overline{\mathcal{U}}$,

$$\widetilde{P}(X, \omega_k) = \widetilde{P}_k(\sigma_k^{-1}(X), \omega_0).$$

Actually for every $X \in \overline{\mathcal{U}}$ we have $\pi(\overline{C}, X) = \pi(\sigma_k^{-1}(\overline{C}), \sigma_k^{-1}(X))$ and $|r_{c_0}^{-1}(X)| = |r_k^{-1}(\sigma_k^{-1}(X))|$. So the required identity follows from Theorem 3.2.3.

We pick a point ω_k in Ω_k, for each $k = 1, \dots, 12$, and we consider $\widetilde{P}(\cdot, \omega_k)$. We fix a fundamental region $\mathcal{R}_0$ containing $\overline{X}$ and $\overline{C}$ as shown in Figure 10, denoting by $Y_1, \dots, Y_{12}$ its vertices, and we consider the 12×12 matrix $\mathbb{P} = (P_{j,k})$, whose entry $P_{j,k}$ is the value $\widetilde{P}(Y_j, \omega_k)$.

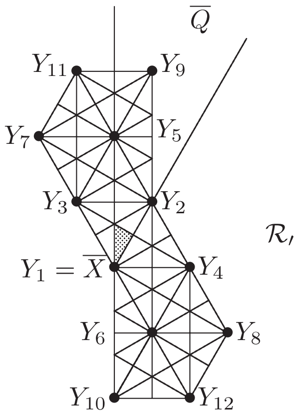


Fig. 10.

DEFINITION 4.5.5. – *A pair* (α, β) *is said "singular" if* det $\mathbb{P} = 0$.

Since any function $\widetilde{P}(\cdot, \omega_k)$ is completely determined by its values on the vertices of the region $\mathcal{R}_\prime$, the functions $\widetilde{P}(\cdot, \omega_1), \dots, \widetilde{P}(\cdot, \omega_{12})$ are linearly dependent if and only if (α, β) is singular. The following proposition exhibits the singular pairs.

PROPOSITION 4.5.6. – *The pair* (α, β) *is singular if and only if* (α, β) *satisfies at least one of the following relation:*

$$\begin{array}{ll} \alpha = p, & \beta = pq, \\ \beta = \alpha, & \alpha\beta = p^3 q, \\ \beta = \alpha^2, & \beta^2 = p^3 \alpha. \end{array}$$

PROOF. – By Proposition 4.5.3, in order to determine $P_{j,k}$ we only have to determine the type of a minimal gallery connecting $\overline{C}$ to Y_j and then to apply (6). We have

$$\begin{aligned}
\pi(\overline{C}, Y_2) &= (0)\\
\pi(\overline{C}, Y_3) &= (0,1)\\
\pi(\overline{C}, Y_4) &= (0,1,2)\\
\pi(\overline{C}, Y_5) &= (0,1,2,1,0)\\
\pi(\overline{C}, Y_6) &= (0,1,2,1,2)\\
\pi(\overline{C}, Y_7) &= (0,1,2,1,2,1,0,1)\\
\pi(\overline{C}, Y_8) &= (0,1,2,1,0,2,1,2)\\
\pi(\overline{C}, Y_9) &= (0,1,2,1,0,2,1,2,1,0)\\
\pi(\overline{C}, Y_{10}) &= (0,1,2,1,0,2,1,2,1,2)\\
\pi(\overline{C}, Y_{11}) &= (0,1,2,1,0,2,1,2,1,0,1)\\
\pi(\overline{C}, Y_{12}) &= (0,1,2,1,2,1,0,1,2,1,2).
\end{aligned}$$

The computation of the determinant of the matrix $\mathbb{P}$ is obtained operating iterate reductions of the matrix and using mathematical software "Mathematica 2". We get, for every $(\alpha,\beta) \in (\mathbb{C}^\times)^2$,

$$\det \mathbb{P} = M\alpha^{-18}\beta^{-18}(\alpha-\beta)^6(\alpha^2-\beta)^6(\alpha-p)^6(p^3\alpha-\beta^2)^6(\alpha\beta-p^3q)^6(\beta-pq)^6,$$

where M is a constant depending on p, q. This proves the proposition. □

COROLLARY 4.5.7. – *The functions $\widetilde{P}(\cdot,\omega_1),\ldots,\widetilde{P}(\cdot,\omega_{12})$ are linearly dependent if and only if (α,β) satisfies at least one of the relations:*

$$\begin{aligned}
\alpha &= p, & \beta &= pq,\\
\beta &= \alpha, & \alpha\beta &= p^3q,\\
\beta &= \alpha^2, & \beta^2 &= p^3\alpha.
\end{aligned}$$

PROPOSITION 4.5.8. – *For every singular pair (α,β) there exists an equivalent pair which is non-singular.*

PROOF. – Setting $\alpha = pq\xi$, $\beta = p^2q\eta$, the pair (α,β) is singular if and only if (ξ,η) satisfies one of the following relations:

$$\begin{aligned}
\xi &= q^{-1}, & \eta &= p^{-1},\\
\eta &= p^{-1}\xi, & \xi\eta &= q^{-1},\\
\eta &= q\xi^2, & \xi &= q\eta^2.
\end{aligned}\tag{7}$$

We shall prove that for every (ξ,η) there exists $\sigma \in W_0$ such that (ξ_σ,η_σ) does not satisfy any of the previous relations.

We may restrict to consider

$$1 \le |\xi| \le |\eta|. \tag{8}$$

In fact if $|\xi| > |\eta|$ then the pair $(\xi_\sigma, \eta_\sigma) = (\eta, \xi)$ satisfies $|\xi_\sigma| < |\eta_\sigma|$. Moreover if $|\xi| \le 1 \le |\eta|$ (resp. $|\xi| \le |\eta| \le 1$), then the pair $(\xi_\sigma, \eta_\sigma) = (\eta, \xi^{-1}\eta)$ (resp. $(\xi^{-1}\eta, \xi^{-1})$) satisfies $1 \le |\xi_\sigma| \le |\eta_\sigma|$.

If (8) holds, the only possible relation between (7) is

$$\eta = q\xi^2. \tag{9}$$

But if the pair (ξ, η) satisfies (8) and (9), then the pair $(\xi_\sigma, \eta_\sigma) = (\xi^{-1}\eta, \eta)$ again satisfies (8) but not (9). This proves the proposition. □

THEOREM 4.5.9. – *For every* $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, *there exists a non-singular pair* (α, β) *satisfying* $\gamma_i = \gamma_i(\alpha, \beta), i = 1, 2$, *such that* $\{\widetilde{P}(\cdot, \omega_1), \dots, \widetilde{P}(\cdot, \omega_{12})\}$ *is a basis for* $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$.

PROOF. – It is an immediate consequence of Proposition 4.5.8 and Remark 4.2.4. □

REMARK 4.5.10. – Let (α, β) be a non-singular pair and let $\mathbb{P}$ be the corresponding matrix. For every $f \in \widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, with $\gamma_i = \gamma_i(\alpha, \beta), i = 1, 2$, and for every chamber $c \in \mathcal{C}$, the linear system

$$\sum_{k=1}^{12} P_{j,k}\mu_k = \widetilde{f}_c(Y_j), \quad j = 1, \dots, 12, \tag{10}$$

is non-singular; we denote its unique solution by

$$(\mu_1(f, c), \dots, \mu_{12}(f, c)). \tag{11}$$

Formulas (10) and (11) generalize (9) and (10) of [12].

5. – The main theorem.

5.1 – *The Poisson transform.*

Let $H(\Omega)$ be the linear space of all locally constant functions on Ω and let $H'(\Omega)$ be its dual, consisting of all finitely additive measures defined on the algebra generated by the open sets of Ω. As for a building of type $\widetilde{A}_2$, for every $x_0, x \in \mathcal{U}$ the function $P^{x_0}(x, \cdot)$ belongs to $H(\Omega)$.

For every pair (α, β) and every $x_0 \in \mathcal{U}$, we define the Poisson transform (of initial point x_0 and parameters (α, β)) of a finitely additive measure $\nu \in H'(\Omega)$ as the function

$$\mathcal{P}^{x_0}_{(a,\beta)}\nu(x) = \int_{\Omega} P^{x_0}_{(a,\beta)}(x,\omega)d\nu(x), \quad x \in \mathcal{U}.$$

For ease of notation, we simply denote by $\mathcal{P}^{x_0}\nu$ this function, when (a,β) is fixed. As a direct consequence of Proposition 4.2.1, $\mathcal{P}^{x_0}\nu$ belongs to the eigenspace $\mathcal{S}(\gamma_1,\gamma_2)$, if $\gamma_i = \gamma(a,\beta)$, $i = 1,2$. Moreover Proposition 4.1 of [12] extends to a building of type $\widetilde{G}_2$, as a consequence of Theorem 4.5.9 and Remark 4.5.10.

PROPOSITION 5.1.1. – *Let (a,β) be a non-singular pair. If $f = \mathcal{P}^e\nu$, for $\nu \in H'(\Omega)$, then for every chamber $c \in \mathcal{C}$*

$$\nu(\Omega(c)) = \mu_{k_c}(1,f,c)P^e(x,\Omega(C))^{-1},$$

where x is the special vertex of c and $P^e_{a,\beta}(x,\Omega(C))$ denotes the value that the function $P^e_{a,\beta}(x,\cdot)$ assumes for every $\omega \in \Omega(C)$.

PROOF. – See proof of [12, Proposition 4.1], with the obvious change of notation. □

COROLLARY 5.1.2. – *For every non-singular pair (a,β) and for every $x_0 \in \mathcal{U}$, the Poisson transform $\mathcal{P}^{x_0}_{a,\beta}$ is a linear injection from $H'(\Omega)$ to $\mathcal{S}(\gamma_1,\gamma_2)$, where $\gamma_i = \gamma_i(a,\beta)$, $i = 1,2$.*

5.2 – *Surjectivity of the Poisson transform.*

The surjectivity of the Poisson transform, for non-singular parameters, may be proved using the same machinery used in case $\widetilde{A}_2$. First of all, we solve the problem for quasi-isotropic measures with respect to a chamber $c \in \mathcal{C}$. The definition of quasi-isotropic measures may be stated as in [12], referring to notation of section 4.

DEFINITION 5.2.1. – *A measure $\nu \in H'(\Omega)$ is quasi-isotropic with respect to a chamber $c_0 \in \mathcal{C}$ if, for every $k = 1,\ldots,12$ and for every chamber $C \subset Q_k$, the measure ν assumes the same value on all the sets $\Omega(c)$, such that*

$$\Omega(c) \subset \Omega_k \quad \text{and} \quad r_{c_0}(c) = C.$$

For every chamber $c_0 \in \mathcal{C}$ and for every pair (γ_1,γ_2) we set

$$\mathcal{S}_{c_0}(\gamma_1,\gamma_2) = \{f \in \mathcal{S}(\gamma_1,\gamma_2) : f \text{ constant on the } r_{c_0}\text{-fibers}\}.$$

As for a building of type $\widetilde{A}_2$ we prove the following lemma.

LEMMA 5.2.2. – *Let* (a, β) *be a non-singular pair and* $\gamma_i = \gamma_i(a, \beta), i = 1, 2$. *Consider* $f = \mathcal{P}^{x_0}_{a,\beta}\nu$, *for* $\nu \in H'(\Omega)$. *Then* $f \in \mathcal{S}_{c_0}(\gamma_1, \gamma_2)$ *if and only if* ν *is quasi-isotropic with respect to* c_0.

PROOF. – See proof of Lemma 4.6 of [12], with the obvious change of notation. □

As a consequence of the previous lemma, we get the following result.

PROPOSITION 5.2.3. – *Let* (a, β) *be a non-singular pair and* $\gamma_i = \gamma_i(a, \beta), i = 1, 2$. *Then, for every* $f \in \mathcal{S}_{c_0}(\gamma_1, \gamma_2)$, *there exists a unique measure* $\nu \in H'(\Omega)$, *quasi-isotropic with respect to* c_0, *such that* $f = \mathcal{P}^{x_0}_{a,\beta}\nu$.

PROOF. – The retraction with respect to c_0 induces a linear injection of $\mathcal{S}_{c_0}(\gamma_1, \gamma_2)$, into the space $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, which has dimension 12; hence $dim\ \mathcal{S}_{c_0}(\gamma_1, \gamma_2) \leq 12$. On the other hand the quasi-isotropic measures with respect to c_0 form a linear subspace $H'_{c_0}(\Omega)$ of $H'(\Omega)$, having dimension 12, as any $\nu \in H'_{c_0}(\Omega)$ is determined by its values on the twelve sets $\Omega_{c_0,j}$. Thus the injectivity of the Poisson transform $\mathcal{P}^{x_0}_{a}$ from $H'_{c_0}(\Omega)$ to $\mathcal{S}_{c_0}(\gamma_1, \gamma_2)$ implies $dim\ \mathcal{S}_{c_0}(\gamma_1, \gamma_2) = 12$ and hence the surjectivity of the map. □

REMARK 5.2.4. – For every (γ_1, γ_2), the eigenspace $\mathcal{S}(\gamma_1, \gamma_2))$ splits as

$$\mathcal{S}(\gamma_1, \gamma_2)) = \mathcal{S}_{c_0}(\gamma_1, \gamma_2)) + Ker(r_{c_0}),$$

where r_{c_0} denotes the map $f \to \widetilde{f}_{c_0}$.

Also the technical result of Lemma 4.8 of [12] holds in the present case, with the obvious change of notation.

LEMMA 5.2.5. – *Let* (a, β) *be a non-singular pair and* $\gamma_i = \gamma_i(a, \beta), i = 1, 2$. *If* f *belongs to* $\mathcal{S}(\gamma_1, \gamma_2)$, *and* $\widetilde{f}_{c_0} = 0$, *then*

$$\sum_{c \in r_{c_0}^{-1}(C)} \widetilde{f}_c = 0, \tag{12}$$

for every chamber $C \subset Q_{k_{c_0}}$.

PROOF. – For ease of notation, suppose $r_c(c) = C$, $\forall c \in r_{c_0}^{-1}(C)$. Since $\widetilde{f}_c$ belongs to $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, for every $c \in r_{c_0}^{-1}(C)$, it suffices to prove that the sum in (12) is zero on the vertices of a fundamental region $\mathcal{R}_0$. We choose $\mathcal{R}_0$ in the subsector of Q based at C and denote its vertices $Y_1, \ldots, Y_{12}$ as usual. Then it is easy to observe that

$$\widetilde{f}_{c_0}(Y_j) = \sum_{c \in r_{c_0}^{-1}(C)} \widetilde{f}_c(Y_j), \quad \forall j = 1, \ldots, 12.$$

This allows us to conclude. □

Using these preliminary results, we may obtain, by the same argument as in [12], the following theorem, which gives the required characterization of the eigenfunctions of the Laplacians.

THEOREM 5.2.6. – *Let* $\gamma_1, \gamma_2 \in \mathbb{C}$ *and let* (α, β) *be a non-singular pair such that* $\gamma_i = \gamma_i(\alpha, \beta),\ i = 1, 2$. *For every* $f \in \mathcal{S}(\gamma_1, \gamma_2)$ *there exists a unique finitely additive measure* $\nu \in H'(\Omega)$ *such that*

$$f = \mathcal{P}^e_{\alpha,\beta}\nu.$$

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