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A Mathematical Model for a Contracting Interstellar Cloud

MERI LISI - SILVIA TOTARO

Abstract. – *In this paper, we study a one-dimensional mathematical model for a contracting interstellar cloud, with a star inside. Existence and uniqueness of a positive solution are proved by means of the fixed point theorem. A time discretization procedure is given and the case of an expanding interstellar cloud is also considered.*

1. – Introduction

The interstellar (and intergalactic) medium, far from being a vacuum, can be regarded as a "chemical laboratory" responsible of the birth of stars and galaxies. Developments in astrophysics during the last few decades have kept the subject in the forefront of general scientific and popular interest.

The intergalactic medium is mainly filled with hydrogen (90%), about 10% of the atoms are helium, and a further 0.1% of atoms are carbon, nitrogen and oxygen. Other elements are even less abundant. Mixed with gas are "dust" grains of silicon and carbonates, [4].

Matter is concentrated in big clouds (nebulae or intestellar clouds), whose dimensions are of the order of ten lights years, i.e., between 10^{-1} and 10 parsec (one parsec is about $3 \cdot 10^{13}$ kilometers). Note that the diameter of the solar system is of the order of 10^{-4} parsec. The numerical density of the particles inside an interstellar cloud ranges from 10^3 to 10^6 particles per cubic centimetre (Earth atmosphere density, at sea level, is approximately 10^{19} particles per cubic centimetre, whereas in the intergalactic vacuum one can find 10^0 particles/cm^3). This means that a nebula is rarefied, but not so much as the intergalactic vacuum.

Interstellar cloud can be classified in dark nebulae, emission nebulae and reflection nebulae, [8]. *Dark nebulae* are clouds that become opaque because of their internal dust grains. They obscure, or absorb, the light coming from stars behind them. This is the reason why dark nebulae are sometimes called *absorption nebulae*. The form of such dark clouds is very irregular: they have no clearly defined boundaries and sometimes take on convoluted serpentine shapes. The largest dark nebulae are visible to the naked eye: a famous example is given by the *Horsehead Nebula* (B33, [1]).

Emission nebulae are clouds that emit light from one or more stars which are inside the cloud itself. They are usually the sites of recent and ongoing star formation. An example of this kind of nebulae is given by the *Orion Nebula* (M42, [7]).

Finally, *reflection nebulae* are clouds of dust which are simply reflecting the light of a nearby star or stars. Also reflection nebulae are usually sites of star formation: an example is given by *Pleiades* (M45, [7]).

Consider now an interstellar cloud, occupying a convex region of the interstellar medium. The boundaries of the nebula are not fixed, but they move slowly in time. In most cases, when a source (like for example a star) is present inside the nebula, the movement of the boundaries are such that the cloud collapses into the source itself. Of course, the time needed to conclude the whole "contraction" is very long (a million of years), because of the enormous dimension involved and the extremely slow speed of the boundaries, which is of the order of 30 km/s, i.e., 10^{-12} pc/s. Note that to have an appreciable "variation" of 10^{-3} pc, about 31 years are needed, [13].

This process can be simplified going to consider the one-dimensional case, where the nebula can be represented by a slab. Consider a fixed reference system, defining a suitable x axis and assume the nebula to have a plane simmetry with respect to a 0 point. Moreover, imagine the nebula at time $t = 0$ to be bounded by the two surfaces $x = -b(0)$ and $x = b(0)$, i.e., assume the initial dimension of the cloud to be $2b(0)$. Note that, because of the geometrical properties of a nebula, $2b(0) < 20$ parsec. As time goes on, consider the movement of these boundaries to be such that the dimension of the cloud become smaller and smaller. The case under investigation is that when the boundaries $x = -b(t), x = b(t)$ ($2b(t)$ indicates the dimension of the cloud at time t) tend to the "critical position" $x = -b_{min}, x = b_{min}$, i.e., when pressure and temperature conditions tend to so high values to cause nuclear reactions. The consequence of these facts is that the interstellar cloud collapses into the photon source, i.e., it becomes a star, that is going to shine for millions of years. Figure 1 shows a sketch-plan of the situation.

Note that what we assumed implies that $b(t)$ is a bounded function such that:

$$b(t) \leq b(0) < 20 \text{ parsec.} \tag{1}$$

If the transport phenomenon is assumed to be one-dimensional, i.e., the photon number density N depends on the space variable x, on the "angle" variable μ (μ represents the cosine of the angle between the speed direction and the horizontal x axis) and on time t, the photon transport equation in the interstellar space reads as follows, [9]:

$$\begin{aligned} \frac{\partial}{\partial t} N(x,\mu,t) = &-c\mu \frac{\partial}{\partial x} N(x,\mu,t) - c\Sigma(x,t) N(x,\mu,t) \\ &+ \frac{1}{2} c\Sigma_s(x,t) \int_{-1}^{+1} N(x,\mu',t) d\mu' + cq(x,t), \end{aligned} \tag{2}$$

with $x \in (-b(0), b(0))$, $\mu \in (-1, +1)$ and $t \in (0, \bar{t})$, where $\bar{t}$ is a suitable positive constant that will be chosen later.

In Eq. (2), Σ and Σ_s are the total and the scattering cross sections respectively (the scattering phenomenon is assumed to be isotropic), q is the source term and c is the light speed.

Moreover, Eq. (2) is supplemented with the following assumptions on N, q, Σ and Σ_s:

$$(3) \quad N(x, \mu, 0) = N_0(x, \mu), \qquad x \in (-b(0), b(0)), \quad \mu \in [-1, +1],$$

$$(4) \quad N_0(x, \mu) = 0, \qquad x \notin (-b(0), b(0)), \quad \mu \in [-1, +1],$$

$$(5) \quad N(-b(0), \mu, t) = 0, \qquad \mu > 0, t \in [0, \bar{t}\,],$$

$$(6) \quad N(b(0), \mu, t) = 0, \qquad \mu < 0, t \in [0, \bar{t}\,],$$

$$(7) \quad \Sigma(x, t) = \Sigma_s(x, t) = 0, \qquad x \notin (-b(t), b(t)), t \in [0, \bar{t}\,],$$

$$(8) \quad q(x, t) = 0, \qquad x \notin (-b_{min}, b_{min}), t \in [0, \bar{t}\,].$$

Note that relations (3)-(4) are the initial conditions for the photon number density N, assumptions (5)-(6) represents non-reentry boundary conditions and assumption (7) indicates the fact that the particle density outside the nebula (i.e., in the intergalactic vacuum) may be considered equal to zero. Finally, since the source term is assumed to be present inside the region bounded by $x = -b_{min}$ and $x = b_{min}$, relation (8) holds.

In order to study the model, make a change of variables, by considering $x - r\mu$ instead of x and $t - \dfrac{r}{c}$ instead of t, with r a nonnegative real number (see Fig. 1). Moreover, dividing by c and multiplying by $\exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \dfrac{r'}{c}\right) dr'\right]$, Eq. (2) becomes:

$$(9) \quad -\frac{\partial}{\partial r}\left\{N\left(x - r\mu, \mu, t - \frac{r}{c}\right)\exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right)dr'\right]\right\}$$
$$= \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right)dr'\right]$$
$$\cdot\left\{\frac{1}{2}\Sigma_s\left(x - r\mu, t - \frac{r}{c}\right)\int_{-1}^{1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right)d\mu' + q\left(x - r\mu, t - \frac{r}{c}\right)\right\}.$$

In what follows, the model will be studied by considering the cases $\mu > 0$ and $\mu < 0$ separately and existence and uniqueness of the solution will be proved. In particular, ct represents the space crossed in t seconds by a given particle along

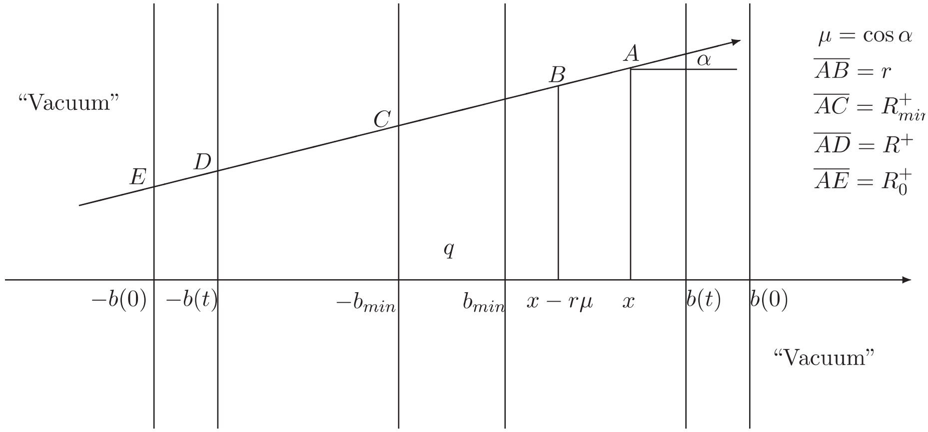

Fig. 1.

the velocity direction. Since the maximum dimension of the nebula is $2b(0)$, that can be estimated in about 20 parsec (see (1)), the maximum time, t_{max}, necessary to cross the whole nebula can be given by:

$$t_{max} = \frac{20 \text{ parsec}}{c}, \tag{10}$$

i.e., about 10 days.

The following notation will be used: considering a generic particle moving with direction μ, if it occupies position x (for example, $x \in (-b_{min}, b_{min})$), then:

$$-b(t) = x - R^+(t)\mu, \text{ for } \mu \in (0,1), x \in (-b(t), b(0)) \text{ or } \mu \in (-1,0), x \in (-b(0), -b(t)),$$

$$b(t) = x - R^-(t)\mu, \text{ for } \mu \in (0,1), x \in (b(t), b(0)) \text{ or } \mu \in (-1,0), x \in (-b(0), b(t)),$$

where, with a compact notation, $R^\pm(t) = R^\pm(x,\mu,t)$ are suitable nonnegative real numbers, for any fixed t. In a similar way:

$$-b(0) = x - R_0^+\mu, \ \mu \in (0,1), \qquad b(0) = x - R_0^-\mu, \ \mu \in (-1,0), \tag{11}$$

$$-b_{min} = x - R^+_{min}\mu, \text{ for } \mu \in (0,1), \quad x \in (-b_{min}, b(0)) \text{ or } \mu \in (-1,0), x \in (-b(0), -b_{min}), \tag{12}$$

$$b_{min} = x - R^-_{min}\mu, \text{ for } \mu \in (0,1), \quad x \in (b_{min}, b(0)) \text{ or } \mu \in (-1,0), x \in (-b(0), b_{min}), \tag{13}$$

where, with a compact notation, $R_0^\pm = R_0^\pm(x,\mu), R^\pm_{min} = R^\pm_{min}(x,\mu)$ are suitable nonnegative real numbers (see Fig. 1).

Now, integrating Eq. (9) with respect to r, between 0 and ct, the mathematical equation describing the photon number density for a contracting intestellar cloud becomes:

$$(14)\quad N(x,\mu,t) = N_0(x-ct\mu,\mu)\exp\left[-\int_0^{ct}\Sigma\left(x-r'\mu,t-\frac{r'}{c}\right)dr'\right]$$
$$+\int_0^{R_0^\star}\left\{\exp\left[-\int_0^{r}\Sigma\left(x-r'\mu,t-\frac{r'}{c}\right)dr'\right]q\left(x-r\mu,t-\frac{r}{c}\right)\right\}dr$$
$$+\int_0^{R_0^\star}\left\{\exp\left[-\int_0^{r}\Sigma\left(x-r'\mu,t-\frac{r'}{c}\right)dr'\right]\frac{1}{2}\Sigma_s\left(x-r\mu,t-\frac{r}{c}\right)\int_{-1}^{+1}N\left(x-r\mu,\mu',t-\frac{r}{c}\right)d\mu'\right\}dr,$$

where $x\in[-b(0),b(0)]$, $\mu\in[-1,+1]$, $t\in[0,\bar{t}\,]$ and

$$(15)\qquad R_0^\star = R_0^\star(x,\mu,t) = \begin{cases}\min\{ct,R_0^+\}, & \mu>0,\\ \min\{ct,R_0^-\}, & \mu<0,\end{cases}$$

with $R_0^\pm$ defined by relations (11). Note that $R_0^\star$ has been defined because, in the case $\mu>0$, for the photon density $N(x,\mu,t)$ one has to consider only the contribution due to the values of r between 0 and R_0^+ if $ct\geq R_0^+$, i.e., if the space crossed by a generic photon in a time t is such to "bring" it outside the nebula, whereas, the integral has to be made for r between 0 and ct, if $ct<R_0^+$. In a similar way, for $\mu<0$, if $ct\geq R_0^-$, the integration with respect to r has to be done between 0 and R_0^-, whereas, if $ct<R_0^-$, between 0 and ct. Let us remember that $ct<ct_{max}$, see (10).

2. – Analysis of the model.

Now, in order to analyze Eq. (14), the case $\mu>0$ is first studied.

For x belonging to $[-b(0),-b(t)]$, since $x-r\mu\leq x\leq -b(t)\leq -b_{min}$, from conditions (7)-(8) it results $q(x-r\mu,t)=\Sigma_s(x-r\mu,t)=\Sigma(x-r\mu,t)=0$. Moreover, in the case $ct\geq R_0^+$, $x-ct\mu\leq -b(0)$ and thus, by using assumptions (4)-(5), one can deduce that $N_0(x-ct\mu,\mu)=0$. Hence, for $x\in[-b(0),-b(t)]$ and $\mu>0$, Eq. (14) reduces to:

$$N(x,\mu,t) = \begin{cases} N_0(x-ct\mu,\mu), & 0\leq t<\dfrac{R_0^+}{c}\\[2ex] 0, & \dfrac{R_0^+}{c}\leq t\leq\bar{t}\end{cases}.$$

For x belonging to $(-b(t),b(0)]$, first consider the case $ct\geq R_0^+$. For the same reasons as before, it results $N_0(x-ct\mu,\mu)=0$. Moreover, in Eq. (14), $R_0^\star=R_0^+$ (see definition (15)). From assumptions (7)-(8) and definitions (11), for $x\in(-b(t),b(0)]$ and $t\geq R_0^+/c$, Eq. (14) reduces to:

$$N(x,\mu,t) = \overbrace{\int_0^{\frac{x+b_{min}}{\mu}} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] q\left(x - r\mu, t - \frac{r}{c}\right) \right\} dr}^{A}$$

$$\underbrace{+\int_0^{\frac{x+b(t)}{\mu}} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] \frac{1}{2} \Sigma_s\left(x - r\mu, t - \frac{r}{c}\right) \int_{-1}^{+1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right) d\mu' \right\} dr,}_{B}$$

where it is assumed that A is zero if $\frac{x + b_{min}}{\mu} < 0$ and B is zero if $\frac{x + b(t)}{\mu} < 0$.

For x belonging to $(-b(t), b(0)]$, considering now the case $ct < R_0^+$, we have that $R_0^\star = ct$ (see (15)). Hence, the results can be generalized such that, for $x \in (-b(t), b(0)]$, Eq. (14) reduces to:

$$(16) \quad N(x,\mu,t) = N_0(x - ct\mu, \mu) \exp\left[-\int_0^{ct} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right]$$

$$+ \int_0^{\delta^+_{min}} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] q\left(x - r\mu, t - \frac{r}{c}\right) \right\} dr$$

$$+\int_0^{\delta^+} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] \frac{1}{2} \Sigma_s\left(x - r\mu, t - \frac{r}{c}\right) \int_{-1}^{+1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right) d\mu' \right\} dr,$$

with

$$(17) \quad \delta^+_{min} = \delta^+_{min}(x,\mu,t) = \min\left\{ ct, \frac{x + b_{min}}{\mu} \right\}, \quad \delta^+ = \delta^+(x,\mu,t) = \min\left\{ ct, \frac{x + b(t)}{\mu} \right\}$$

and where the first part on the right handside of Eq. (16) is equal to 0 for $0 \le t < R_0^+/c$. Note that δ^+ is such that $b(t) < b(0) < 10$ parsec, $t < t_{max}$.

The case $\mu < 0$ can be studied in a similar way. In particular, since $R_0^\star = R_0^-$ for $ct \ge R_0^-$ and $R_0^\star = ct$ for $ct < R_0^-$ (see (15)), if $x \in [-b(0), b(t))$ Eq. (14) reduces to:

$$(18) \quad N(x,\mu,t) = N_0(x - ct\mu, \mu) \exp\left[-\int_0^{ct} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right]$$

$$\overbrace{+ \int_0^{\delta^-_{min}} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] q\left(x - r\mu, t - \frac{r}{c}\right) \right\} dr}^{C}$$

$$\underbrace{+\int_0^{\delta^-} \left\{ \exp\left[-\int_0^r \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr' \right] \frac{1}{2} \Sigma_s\left(x - r\mu, t - \frac{r}{c}\right) \int_{-1}^{+1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right) d\mu' \right\} dr,}_{D}$$

with:

$$\delta^-_{min} = \delta^-_{min}(x,\mu,t) = \min\left\{ct, \frac{x-b_{min}}{\mu}\right\}, \quad \delta^- = \delta^-(x,\mu,t) = \min\left\{ct, \frac{x-b(t)}{\mu}\right\} \tag{19}$$

and where the first part on the right handside of Eq. (18) is equal to 0 for $0 \le t < R_0^-/c$. Note that δ^- is such that $b(t) < b(0) < 10$ parsec, $t < t_{max}$. Moreover, note that in Eq. (18), it is assumed that C is zero if $\frac{x+b_{min}}{\mu} < 0$ and D is zero if $\frac{x+b(t)}{\mu} < 0$.

Finally, for $x \in [b(t), b(0)]$, Eq. (14) reduces to:

$$N(x,\mu,t) = \begin{cases} N_0(x - ct\mu, \mu), & 0 \le t < \dfrac{R_0^-}{c} \\ 0, & \dfrac{R_0^-}{c} \le t \le \bar{t} \end{cases},$$

because $b_{min} \le b(t) \le x \le x - r\mu$ and from conditions (7)-(8) it follows that $q(x - r\mu, t) = \Sigma_s(x - r\mu, t) = 0$. Moreover, since $b(0) \le x - ct\mu$ for $ct \ge R_0^-$, then $N_0(x - ct\mu, \mu) = 0$ (see assumptions (4)-(6)).

Looking at definitions (17) and (19) for $\delta^\pm$, it is useful to analyze the cases:

$$ct = \frac{x+b(t)}{\mu}, \quad \text{for } \mu > 0; \qquad ct = \frac{x-b(t)}{\mu}, \quad \text{for } \mu < 0.$$

Consider the definition of δ^+, for $\mu > 0$. By means of a graphyc representation of the functions $y = ct$ and $y = \frac{x+b(t)}{\mu}$ (with a fixed x), it is easy to see that a $t_+^\star$ exists such that

$$ct_+^\star = \frac{x + b(t_+^\star)}{\mu}. \tag{20}$$

If $t > t_+^\star$ then $ct > [x + b(t)]/\mu$ and $\delta^+ = [x + b(t)]/\mu$. The case $\mu < 0$ is similar: in particular, a $t_-^\star$ exists such that:

$$ct_-^\star = \frac{x - b(t_-^\star)}{\mu}. \tag{21}$$

Thus, if $t > t^\star$, then $ct > [x + b(t)]/\mu$ if $\mu > 0$, $ct > [x - b(t)]/\mu$ if $\mu < 0$ and $\delta^\pm = [x \pm b(t)]/\mu$, where, from relations (20)-(21):

$$t^\star = \begin{cases} t_+^\star, & \mu > 0 \\ t_-^\star, & \mu < 0 \end{cases}. \tag{22}$$

Hence, the trasport equation for the photon number density $N(x,\mu,t)$ for a contracting interstellar cloud, initially occupying the convex region $[-b(0), b(0)]$, can be summarized as follows.

For $x \in [-b(0), -b(t)], \mu > 0, \frac{R_0^+}{c} \le t \le \bar{t}$ and $x \in [b(t), b(0)], \mu < 0, \frac{R_0^-}{c} \le t \le \bar{t}$:

$$N(x,\mu,t) = 0. \tag{23}$$

For $x \in [-b(0), -b(t)], \mu > 0, 0 \le t < \frac{R_0^+}{c}$ and $x \in [b(t), b(0)], \mu < 0, 0 \le t < \frac{R_0^-}{c}$:

$$N(x,\mu,t) = N_0(x - ct\mu, \mu) \exp\left[-\int_0^{ct} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr'\right].$$

For $x \in (-b(t), b(0)], \mu > 0$ and $x \in [-b(0), b(t)), \mu < 0$:

$$\begin{aligned}
N(x,\mu,t) &= N_0(x - ct\mu, \mu) \exp\left[-\int_0^{ct} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr'\right] \\
&+ \int_0^{\delta_{min}} \left\{ \exp\left[-\int_0^{r} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr'\right] q\left(x - r\mu, t - \frac{r}{c}\right) \right\} dr \\
&+ \int_0^{\delta} \left\{ \exp\left[-\int_0^{r} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right) dr'\right] \frac{1}{2} \Sigma_s\left(x - r\mu, t - \frac{r}{c}\right) \int_{-1}^{+1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right) d\mu' \right\} dr,
\end{aligned} \tag{24}$$

where the first part on the right handside of Eq. (24) is equal to 0 for $0 \le t < R_0^+/c, \mu > 0$ and for $0 \le t < R_0^-/c, \mu < 0$. Moreover, in Eq. (24), we put:

$$\delta_{min} = \delta_{min}(x,\mu,t) = \begin{cases} \delta_{min}^+, & \mu > 0 \\ \delta_{min}^-, & \mu < 0 \end{cases}; \qquad \delta = \delta(x,\mu,t) = \begin{cases} \delta^+, & \mu > 0 \\ \delta^-, & \mu < 0 \end{cases}, \tag{25}$$

with $\delta_{min}^{\pm}, \delta^{\pm}$ defined by relations (17), (19).

Note that, since relation (23) holds, the no-reentry boundary conditions (5)-(6) “move” from $x = -b(0), x = b(0)$, to the contracting boundaries $x = -b(t), x = b(t)$, respectively.

To study existence and uniqueness of the solution $N(x,\mu,t)$ of Eq. (24), the following Banach space is introduced, because of the physical meaning of its standard norm:

$$X = L^1([-b(0), b(0)] \times [-1, +1] \times [0, \bar{t}\,]),$$

with $\bar{t} > t^\star$, ($t^\star$ is given by (22)), and where the norm

$$\|f\|_X = \int_0^{\bar{t}} dt \int_{-b(0)}^{b(0)} dx \int_{-1}^{+1} d\mu, \quad \forall x \in X \tag{26}$$

represents the integral between 0 and $\bar{t}$ of the photon number inside the cloud at time t.

Since the number of particles inside the nebula is constant in time, because of the mass conservation law, it follows:

$$\int_{-b(t)}^{b(t)} \Sigma(x,t)\,dx = \int_{-b(0)}^{b(0)} \Sigma(x,0)\,dx;$$

and analogously

$$\int_{-b(t)}^{b(t)} \Sigma_s(x,t)\,dx = \int_{-b(0)}^{b(0)} \Sigma_s(x,0)\,dx.$$

Assuming that the cloud is homogeneous at any $t \in [0,\bar{t}]$, one has that the cross sections do not depend on x (i.e., $\Sigma(x,t) = \Sigma(t)$, $\Sigma_s(x,t) = \Sigma_s(t)$; $\Sigma(x,0) = \Sigma(0) = \Sigma_0 =$ a constant, $\Sigma_s(x,0) = \Sigma_s(0) = \Sigma_{s0} =$ a constant) and the above relations yield

$$\Sigma(t)b(t) = \Sigma_0 b(0), \;\; \Sigma_s(t)b(t) = \Sigma_{s0} b(0). \tag{27}$$

This means that the interactions of photons with the particles of the nebula do not alterate the number and the "quality" of the particles themselves (for instance, no particle is destroyed or modified, if its diffusion or absorption coefficient has been changed).

Define now the operator $Q : X \to X$, such that:

$$Qf = \varphi + Hf, \qquad \forall f \in X,$$

with

$$\varphi(x,\mu,t) = N_0(x - ct\mu, \mu) \exp\left[-\Sigma_0 b(0) \int_0^{ct} \frac{1}{b\left(t - \frac{r'}{c}\right)} dr'\right]$$
$$+ \int_0^{\delta_{min}} \left\{ \exp\left[-\Sigma_0 b(0) \int_0^{r} \frac{1}{b\left(t - \frac{r'}{c}\right)} dr'\right] q\left(x - r\mu, t - \frac{r}{c}\right) \right\} dr,$$

and where $H : X \to X$ is a linear, bounded operator such that

$$(Hf)(x,\mu,t) = \int_0^{\delta} \left\{ \exp\left[-\Sigma_0 b(0) \int_0^{r} \frac{1}{b\left(t - \frac{r'}{c}\right)} dr'\right] \frac{\Sigma_{s0} b(0)}{2b\left(t - \frac{r}{c}\right)} \int_{-1}^{+1} f\left(x - r\mu, \mu', t - \frac{r}{c}\right) d\mu' \right\} dr,$$

where relations (27) have been used. Note that φ is a known function, because all the parameters appearing in it are known.

In order to prove existence and uniqueness of the solution of Eq. (24), it is sufficient to prove that the operator Q is a contraction on the space X, i.e., that the operator H is such that $\|H\| < 1$.

Since

$$\frac{\partial}{\partial r}\left\{\exp\left[-\Sigma_0 b(0)\int_0^r \frac{1}{b(t-\frac{r'}{c})}dr'\right]\right\} = \exp\left[-\Sigma_0 b(0)\int_0^r \frac{1}{b(t-\frac{r'}{c})}dr'\right]\left[-\frac{\Sigma_0 b(0)}{b(t-\frac{r'}{c})}\right],$$

from definition (26), we have:

$$\|Hf\|_X \leq 2\,\frac{\Sigma_{s0}b(0)}{b_{min}}\int_{-b(0)}^{b(0)} dy \int_0^{\bar{t}} d\vartheta \int_{-1}^{+1} |f(y,\mu',\vartheta)|d\mu' \Bigg\} dr,$$

where $y = x - r\mu$, $\vartheta = t - \frac{r}{c}$ and the fact that $\int_{-1}^{1} d\mu = 2$ is used. Hence:

$$\|Hf\|_X \leq \frac{\Sigma_{s0}}{\Sigma_0}\frac{b(0)}{b_{min}}\|f\|_X.$$

Moreover, since $\Sigma_{s0} < 10^{-3}\Sigma_0$ (see [3], [5], [6]), and $10^{-1} < b_{min} < b(t) < b(0) < 10$ parsec, $b(0)/b_{min} < 10^2$, we proved that $\|H\| < 1$. By using the fixed point theorem, existence and uniqueness of an a.e. positive solution of Eq. (24) are proved. The solution can be found by means of a successive approximations method.

3. – Time-discretization.

In this section a time-discretization procedure is provided and an inverse problem for the computation of the interstellar cloud "dimension" $b(t)$ is proposed. Note that $b(t)$ is a bounded function such that $b(t) < b(0) < 10$ parsec, for any t.

Consider Eq. (24) with $\vartheta = t - \frac{r}{c}$ and $\vartheta' = t - \frac{r'}{c}$, and choose a generic time instant t_j, such that $t_j = j\tau$, with $j \in \mathbb{N}$ and $n\tau = \bar{t}$. In this way, the interval $[0, \bar{t}\,]$ is divided into n subintervals of lenght τ. The time step τ must obviously be such that $[b(t+\tau) - b(t)]/b(t) \ll 1$. Since $b(t+\tau) - b(t) \cong \tau\dot{b}(t)$, τ must be chosen so that $\tau|\dot{b}(t)/b(t)| \ll 1$, i.e., $\tau \ll b(t)/|\dot{b}(t)|, \tau \ll \min\{b(t)/|\dot{b}(t)|, t \in [t_0, t_j]\}$. Hence, Eq. (24) at "time" $t = t_j$ reads as follows:

$$N(x,\mu,t_j) = N_0(x - ct_j\mu,\mu)\exp\left[-c\Sigma_0 b(0)\int_0^{t_j}\frac{1}{b(\vartheta')}d\vartheta'\right]$$
$$+ c\int_{t_j-\frac{\delta_{min}}{c}}^{t_j}\left\{\exp\left[-c\Sigma_0 b(0)\int_{\vartheta}^{t_j}\frac{1}{b(\vartheta')}d\vartheta'\right]q(x - c(t_j-\vartheta)\mu,\vartheta)\right\}d\vartheta$$
$$+ c\int_{t_j-\frac{\delta(t_j)}{c}}^{t_j}\left\{\exp\left[-c\Sigma_0 b(0)\int_{\vartheta}^{t_j}\frac{1}{b(\vartheta')}d\vartheta'\right]\frac{1}{2}\Sigma_{s0}\frac{b(0)}{b(\vartheta)}\int_{-1}^{+1}N(x - c(t_j-\vartheta)\mu,\mu',\vartheta)d\mu'\right\}d\vartheta.$$

By discretizing the time variable in the various integrals, the above equation becomes:

$$(28)\quad N(x,\mu,t_j) = N_0(x-ct_j\mu,\mu)\exp\left[-c\Sigma_0 b(0)\sum_{i=0}^{j-1}\frac{\tau}{b(t_i)}\right]$$
$$+c\sum_{h=j^\star_{min}}^{j}\tau\exp\left[-c\Sigma_0 b(0)\sum_{i=h-1}^{j-1}\frac{\tau}{b(t_i)}\right]q(x-c(t_{j-1}-t_{h-1})\mu,t_{h-1})$$
$$+c\sum_{h=j^\star}^{j}\tau\exp\left[-c\Sigma_0 b(0)\sum_{i=h-1}^{j-1}\frac{\tau}{b(t_i)}\right]\frac{1}{2}\Sigma_{s0}\frac{b(0)}{b(t_h)}\int_{-1}^{+1}N(x-c(t_{j-1}-t_{h-1})\mu,\mu',t_{h-1})d\mu',$$

where $j^\star_{min}$ is the index that gives the time $t^\star_{j_{min}}$ closer to $\left(t_j-\frac{\delta_{min}}{c}\right)$, whereas $j^\star$ is that index that gives the time $t^\star_j$ closer to $\left(t_j-\frac{\delta(t_j)}{c}\right)$. This discretization procedure permits to find $N_j = N(x,\mu,t_j)$ in terms of previous known "steps" $N_0, N_1, ..., N_{j-1}$ and this may be used in connection with a numerical procedure to solve (24).

Assume to know a measure of N at time $t=t_j$, at the point $\hat{x}$, along the direction $\hat{\mu}$, i.e., $\hat{N}_j=\hat{N}_j(\hat{x},\hat{\mu},t_j)$: this is a so called "far field measure", that can be made from Earth by using suitable astrophysics instruments.

Then, putting $N_j=\hat{N}_j$, Eq. (28) becomes:

$$(29)\quad \hat{N}(x,\mu,t_j) = N_0(x-ct_j\mu,\mu)\exp\left[-c\Sigma_0 b(0)\sum_{i=0}^{j-1}\frac{\tau}{b(t_i)}\right]$$
$$+c\sum_{h=j^\star_{min}}^{j}\tau\exp\left[-c\Sigma_0 b(0)\sum_{i=h-1}^{j-1}\frac{\tau}{b(t_i)}\right]q(x-c(t_{j-1}-t_{h-1})\mu,t_{h-1})$$
$$+c\sum_{h=j^\star}^{j}\tau\exp\left[-c\Sigma_0 b(0)\sum_{i=h-1}^{j-1}\frac{\tau}{b(t_i)}\right]\frac{1}{2}\Sigma_{s0}\frac{b(0)}{b(t_h)}\int_{-1}^{+1}\hat{N}(x-c(t_{j-1}-t_{h-1})\mu,\mu',t_{h-1})d\mu',$$

which permits to find $b(t_j)$, that is the unknown. By inserting again $b(t_j)$ into Eq. (28), this allows to obtain $N(x,\mu,t_j)$ at time $t=t_j$, for any point x and direction μ. Now, the process can be iterate going to find $N(x,\mu,t_{j+1})$ at time $t=t_{j+1}$, for any x and μ. Thanks to far field measures, this "machinery" permits to understand the movement of the boundaries of a cloud in dependence of discretized times ("inverse problem"). A "continuous" form of $b(t)$ may then be obtained by some interpolation method.

Note that whereas the literature on time independent inverse problems in photon transport (in particle transport) is rather abundant (see the references listed in [3]), on the other hand, only a few papers deals with time dependent inverse problems (see for istance [10], [11], [12]).

Finally, observe that a "small" error on the computation of $b(t)$, obtained by means of the time-discretization procedure proposed in this section, causes a "small" error on $N(x,\mu,t)$. In fact, evaluating $\Delta N_j = |N(x,\mu,t_j) - \bar{N}(x,\mu,t_j)|$, with $N(x,\mu,t_j)$ the photon number density at time $t = t_j$ corresponding to $b(t_j)$ and $\bar{N}(x,\mu,t)$ that corresponding to $\bar{b}(t_j)$ (see (28)), one has that ΔN_j goes to zero as $|b(t_j) - \bar{b}(t_j)|$ goes to zero.

4. – Concluding remarks.

A similar analysis can be done when a mathematical model for an expanding interstellar cloud is considered. In this case, as time goes on, the movement of the boundaries $x = -b(0), x = b(0)$ is such that the cloud dimension becomes bigger and bigger, till its boundaries tend to a critical position $x = -b_{max}, x = b_{max}$.

Note that, in any case, $b_{max} < 10$ parsec. Figure 2 shows a sketch-plan of the situation.

Hence, Eq. (2) represents the transport equation in the interstellar cloud, with $x \in (-b_{max}, b_{max})$, $\mu \in (-1, +1)$ and $t \in (0, \bar{t}\,)$ (it is possible to prove that $\bar{t} > t^\star$, with $t^\star$ given by (22)), and it is supplemented by conditions (3), (4), (7) and the following ones:

$$N(-b_{max},\mu,t) = 0, \qquad \mu > 0, t \in [0,\bar{t}\,], \tag{30}$$

$$N(b_{max},\mu,t) = 0, \qquad \mu < 0, t \in [0,\bar{t}\,], \tag{31}$$

$$q(x,t) = 0, \qquad x \notin (-b(0), b(0)), t \in [0,\bar{t}\,].$$

In a way similar to that followed for the contracting interstellar cloud model, the following results can be obtained: for $x \in [-b_{max}, -b(t)], \mu > 0$ and $x \in [b(t), b_{max}], \mu < 0$,

$$N(x,\mu,t) = 0. \tag{32}$$

For $x \in (-b(t), b_{max}], \mu > 0$ and $x \in [-b_{max}, b(t)), \mu < 0$:

$$N(x,\mu,t) = N_0(x - ct\mu, \mu)\exp\left[-\int_0^{ct} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right)dr'\right] \tag{33}$$
$$+\int_0^{\delta_0}\left\{\exp\left[-\int_0^{r} \Sigma\left(x - r'\mu, t - \frac{r'}{c}\right)dr'\right] q\left(x - r\mu, t - \frac{r}{c}\right)\right\}dr$$
$$+\int_0^{\delta}\left\{\exp\left[-\int_0^{r}\Sigma\left(x - r'\mu, t - \frac{r'}{c}\right)dr'\right]\frac{1}{2}\Sigma_s\left(x - r\mu, t - \frac{r}{c}\right)\int_{-1}^{+1} N\left(x - r\mu, \mu', t - \frac{r}{c}\right)d\mu'\right\}dr,$$

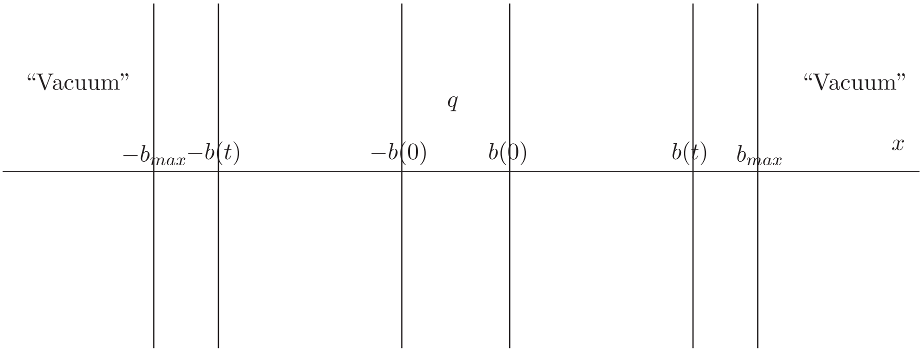

Fig. 2.

where δ is given by definition (25) and:

$$\delta_0 = \delta_0(x,\mu,t) = \begin{cases} \delta_0^+ = \delta_0^+(x,\mu) = \min\left\{ct, \dfrac{x+b(0)}{\mu}\right\}, & \mu > 0 \\ \delta_0^- = \delta_0^-(x,\mu) = \min\left\{ct, \dfrac{x-b(0)}{\mu}\right\}, & \mu < 0 \end{cases}.$$

Note that, since relation (32) holds, the non-reentry boundary conditions (30)-(31) "move" from $x = -b_{max}, x = b_{max}$ to the expanding boundaries $x = -b(t)$, $x = b(t)$, respectively.

To study existence and uniqueness of the solution of Eq. (33), a procedure similar to that used for the case of a contracting nebula can be followed, considering the Banach space $Y = L^1([-b_{max}, b_{max}] \times [-1,+1] \times [0,\bar{t}\,])$, with $\bar{t} > t^\star$ (see definition (22)). Moreover, a time discretization process similar to that made in the case of a contracting intestellar cloud can be performed. Finally, analogous considerations on the error ΔN_j can be made.

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