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Abstract. – *Let*

$$S(x) = \log(1+x) + \int_0^1 \left[1 - \left(\frac{1+t}{2}\right)^x\right] \frac{dt}{\log t} \quad \text{and} \quad F(x) = \log 2 - S(x) \quad (0 < x \in \boldsymbol{R}).$$

We prove that F is completely monotonic on $(0, \infty)$. This complements a result of Miller and Moskowitz (2006), *who proved that F is positive and strictly decreasing on $(0, \infty)$. The sequence $\{S(k)\}$ $(k = 1, 2, \ldots)$ plays a role in information theory.*

1. – Introduction.

A detailed investigation of covert communication channels led Miller and Moskowitz [3] and Moskowitz et al. [4] to the following remarkable sequence:

$$S(k) = \frac{1}{2^k} \sum_{v=0}^{k} \left[\frac{1}{2}\binom{k+1}{v}\log\binom{k+1}{v} - \binom{k}{v}\log\binom{k}{v}\right] \quad (k = 1, 2, \ldots). \tag{1.1}$$

In [3] the authors prove that

$$S(k) < S(k+1) \quad (k = 1, 2, \ldots) \quad \text{and} \quad \lim_{k\to\infty} S(k) = \log 2.$$

A key role in the proof of this monotonicity property plays the elegant integral representation

$$S(k) = \log(1+k) + \int_0^1 \left[1 - \left(\frac{1+t}{2}\right)^k\right] \frac{dt}{\log t}, \tag{1.2}$$

which leads to a continuous counterpart of the sum given in (1.1). Miller and Moskowitz replace the natural number k by the positive real variable x and prove that $(d/dx)S(x) > 0$, that is, S is a strictly increasing function on $(0, \infty)$.

In view of these results it is natural to study the monotonicity properties of higher derivatives of S. In the next section we prove that $(-1)^{n+1}(d^n/dx^n)S(x) > 0$ for $x > 0$ and $n = 1, 2, \ldots$. In particular we show that S is strictly concave on $(0, \infty)$.

We recall that a function $f:(a,\infty)\to \boldsymbol{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(1.3)\qquad (-1)^n f^{(n)}(x)\geq 0 \quad \text{for} \quad x>a \quad \text{and} \quad n=0,1,2,\ldots.$$

Dubourdieu [2] pointed out that if f is non-constant and completely monotonic, then the strict inequality holds in (1.3).

Since these functions have relevant applications in various fields, like, for instance, probability theory, potential theory, numerical and asymptotic analysis, they have been investigated thoroughly by many mathematicians. A collection of the most important properties of completely monotonic functions can be found in [5, Chapter IV]. In several recently published papers it is shown that certain functions, which are defined in terms of the gamma function and its relatives are completely monotonic. We refer to [1] and the references therein.

Let

$$F(x)=\log 2 - S(x) \quad (x>0),$$

where S is given in (1.2). The results of Miller and Moskowitz lead to

$$(1.4)\qquad (-1)^n F^{(n)}(x) > 0 \quad \text{for} \quad x>0 \quad \text{and} \quad n=0,1.$$

We prove that F is completely monotonic on $(0,\infty)$, that is, (1.4) holds for all nonnegative integers n.

2. – Main result.

The following monotonicity theorem complements the results of Miller and Moskowitz.

THEOREM. – *The function* $F(x)=\log 2 - S(x)$ *is completely monotonic on* $(0,\infty)$.

PROOF. – In order to show that (1.4) is valid for $n\geq 1$, we make use of the formulas

$$\frac{d^m}{dx^m}\log(1+x)=\frac{(-1)^{m-1}(m-1)!}{(1+x)^m} \quad (x>0;\ m=1,2,\ldots)$$

and

$$\frac{d^m}{dx^m}c^x = c^x(\log c)^m \quad (c>0;\ x>0;\ m=0,1,2,\ldots).$$

Then we get for $x > 0$ and $n \geq 1$:

$$(2.1)\quad (-1)^n F^{(n)}(x) = (-1)^{n+1} S^{(n)}(x) = \frac{(n-1)!}{(1+x)^n} + \int_0^1 \Big(\frac{1+t}{2}\Big)^x \Big(-\log\frac{1+t}{2}\Big)^n \frac{dt}{\log t}.$$

We have for $z > 0$:

$$\frac{(n-1)!}{z^n} = \int_0^\infty e^{-zt} t^{n-1} dt.$$

The substitution $t = -\log(1/2 + u/2)$ leads to

$$(2.2)\quad \frac{(n-1)!}{(1+x)^n} = \frac{1}{2}\int_{-1}^1 \Big(\frac{1+u}{2}\Big)^x \Big(-\log\frac{1+u}{2}\Big)^{n-1} du$$

$$= \frac{1}{2}\int_0^1 \Big[\Big(\frac{1+t}{2}\Big)^x \Big(-\log\frac{1+t}{2}\Big)^{n-1} + \Big(\frac{1-t}{2}\Big)^x \Big(-\log\frac{1-t}{2}\Big)^{n-1}\Big] dt.$$

From (2.1) and (2.2) we obtain

$$(-1)^n F^{(n)}(x) = \int_0^1 \big(A_n(x,t) + B_n(x,t)\big) dt$$

with

$$A_n(x,t) = \frac{1}{2}\Big(\frac{1+t}{2}\Big)^x \Big(-\log\frac{1+t}{2}\Big)^{n-1}\Big[1 - \frac{\log(1/2+t/2)}{\log\sqrt{t}}\Big]$$

and

$$B_n(x,t) = \frac{1}{2}\Big(\frac{1-t}{2}\Big)^x \Big(-\log\frac{1-t}{2}\Big)^{n-1}.$$

Since

$$A_n(x,t) > 0 \quad \text{and} \quad B_n(x,t) > 0 \quad \text{for} \quad x > 0, \quad 0<t<1, \quad n = 1,2,...,$$

we conclude that $(-1)^n F^{(n)}(x) > 0$. □

REMARK. – A sequence $\{\mu_k\}$ $(k = 1,2,...)$ is called completely monotonic, if

$$(-1)^n \Delta^n \mu_k \geq 0 \quad \text{for} \quad k = 1,2,... \quad \text{and} \quad n = 0,1,2,...,$$

where

$$\Delta^0 \mu_k = \mu_k, \quad \Delta^n \mu_k = \Delta^{n-1}\mu_{k+1} - \Delta^{n-1}\mu_k \quad (n \geq 1).$$

A connection between completely monotonic functions and completely monotonic sequences is given by the following result (see [5, Theorem 11d]):

PROPOSITION. – *If* f *is completely monotonic on* $[a, \infty)$ *and if* δ *is any positive number, then the sequence* $\{f(a + \delta k)\}$ $(k = 0, 1, 2, \ldots)$ *is completely monotonic.*

As an immediate consequence of the Theorem and the Proposition we obtain that the sequence $\{\log 2 - S(k)\}$ $(k = 1, 2, \ldots)$ is completely monotonic.

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