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A Montel Type Result for Subharmonic Functions

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Abstract. – *This article is devoted to sequences $(u_n)_n$ of subharmonic functions in $\mathbb{R}^N$, with finite order, whose means $J_{u_n}(r)$ (over spheres centered at the origin, with radius r) satisfy such a condition as: $\forall r > 0\ \exists A_r > 0$ such that $J_{u_n}(r) \leq A_r\ \forall n \in \mathbb{N}$. The paper investigates under which conditions one may extract a pointwise or uniformly convergent subsequence.*

1. – Introduction.

For a sequence $(f_n)_{n\in\mathbb{N}}$ of holomorphic functions in $\mathbb{C}$ which is uniformly bounded on each compact, Montel's Theorem asserts that there exists a subsequence which converges uniformly on any compact (see [11], pp. 54–56). Since the $\ln|f_n|$ are subharmonic functions in $\mathbb{R}^2$, the question arises whether such a result still holds for a sequence of subharmonic functions $(u_n)_{n\in\mathbb{N}}$ which is uniformly majorized on each compact. It is already available that a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ can be extracted, which converges in the distribution sense (see [8], p. 47 for the detailed statement). But this kind of convergence does not give any information on the pointwise behaviour. Without any growth condition on the u_n, the articles [1] and [5] obtain a convergent subsequence, but the convergence is not uniform: it is mean convergence over spheres (see also Remark 7 in Section 3). The present paper is devoted to the case of subharmonic functions u_n in $\mathbb{R}^N$ ($N \in \mathbb{N}$, $N \geq 3$) with a finite order of growth, under the assumption that the means $J_{u_n}(r)$ of u_n over spheres with radius r and center O are majorized independently of n, whereas the analogous condition in [1] and [5] dealt with the means of $|u_n|$ instead of u_n. We obtain the existence of a subharmonic function u and a subsequence $(u_{n_p})_{p\in\mathbb{N}}$ converging towards u uniformly on any compact set $\Xi \subset \mathbb{R}^N$ which remains distant from the supports of the Riesz measures associated to u and u_{n_p} (more precisely: there exists $\theta > 0$ such that the θ–neighborhood Ξ_θ of Ξ does not intersect any of these supports). Besides that

$$\limsup_{p\to+\infty} u_{n_p}(x) \leq u(x) \qquad \forall x \in \mathbb{R}^N$$

with equality quasi–everywhere (outside a set of outer capacity zero). We refer to Theorem 3 in Section 3 and Theorem 4 in Section 5 for more precisions on the exact assumptions. Such a result of quasi–everywhere convergence was already known for subharmonic functions of the kind

$$x \mapsto \int_{\mathbb{R}^N} \frac{-dv_p(\xi)}{|x-\xi|^{N-2}}$$

(Riesz potential of the measure v_p, see [6], p. 58). But this result required that

$$\int_{|\xi|>1} \frac{dv_p(\xi)}{|\xi|^{N-2}} < +\infty$$

(see [6], p. 190), whereas the Riesz measures μ_n associated to the functions u_n under study in this article only satisfy

$$\int_{|\xi|>1} \frac{d\mu_n(\xi)}{|\xi|^{N-1+q}} < +\infty$$

(with $q \in \mathbb{N}$ related to the order of growth of u_n, see Sections 3 and 5 for more explanations).

The paper is organized as follows: the case of subharmonic functions with arbitrary finite order is postponed to Section 5, whereas Sections 3 and 4 start with the study of the case where the order is ≤ 1. Throughout the paper, all measures are non-negative measures on the σ-algebra of Borel subsets of $\mathbb{R}^N$ and they assume finite values on all compact sets.

2. – Sequence of measures.

THEOREM 1. – ([7] p. 351, [2] pp. 11 and p. 16). – *Let E be a compact in $\mathbb{R}^N$ and $(\mu_n)_{n\in\mathbb{N}}$ a sequence of measures on E. If there exists $A > 0$ such that $\mu_n(E) \leq A$ $\forall n \in \mathbb{N}$, then a subsequence $(\mu_{n_p})_{p\in\mathbb{N}}$ can be extracted which converges to a measure μ in the following sense:*

$$\lim_{p\to+\infty} \int_E \varphi\, d\mu_{n_p} = \int_E \varphi\, d\mu \tag{1}$$

for any continuous function φ on E.

The sequence $(\mu_{n_p})_{p\in\mathbb{N}}$ is said to be vaguely convergent (see [6], pages 7 and 3) or convergent in the w*-topology (see [4], p. 231). If φ was only upper semi-continuous on E, it could only be asserted that

$$\limsup_{p\to+\infty} \int_E \varphi\, d\mu_{n_p} \leq \int_E \varphi\, d\mu.$$

Similarly, for a lower semi-continuous φ:

$$\int_E \varphi\, d\mu \le \liminf_{p\to+\infty} \int_E \varphi\, d\mu_{n_p}$$

(see [3], pp. 205–209 for the proof of these results). For instance, when $N \ge 3$:

$$\int_E \frac{1}{|x-\xi|^{N-2}}\, d\mu(x) \le \liminf_{p\to+\infty} \int_E \frac{1}{|x-\xi|^{N-2}}\, d\mu_{n_p}(x) \qquad \forall \xi \in \mathbb{R}^N \tag{2}$$

with $|\,.\,|$ the Euclidean norm in $\mathbb{R}^N$.

LEMMA 1. – *Given E, μ_{n_p}, μ as in Theorem 1 and a compact $\Xi \subset \mathbb{R}^N$ distant from the supports of measures μ_{n_p} and μ, then*

$$\lim_{p\to+\infty} \int_E \frac{d\mu_{n_p}(x)}{|x-\xi|^{N-2}} = \int_E \frac{d\mu(x)}{|x-\xi|^{N-2}} \qquad \textit{uniformly on } \Xi.$$

REMARK 1. – The ponctual convergence was already given by [2], p. 18.

PROOF OF LEMMA 1. – Let Ξ_θ be the θ–neighborhood of Ξ, relatively to $|\,.\,|$, with $\theta > 0$ chosen small enough so that Ξ_θ does not intersect the supports of the measures μ_{n_p} and μ. We thus have to prove that

$$\lim_{p\to+\infty} \int_E \varphi_\xi\, d\mu_{n_p} = \int_E \varphi_\xi\, d\mu \qquad \text{uniformly on } \Xi,$$

with the functions $\varphi_\xi : x \mapsto [1/\max\{|x-\xi|, \theta\}]^{N-2}$. In the demonstration of Theorem 1 (see [3], p. 207), one of the main steps is the proof of $I(\varphi) \le J(\varphi)$ where $I(\varphi) := \sup_\Delta s(\Delta, \varphi)$ and $J(\varphi) := \inf_\Delta S(\Delta, \varphi)$, the supremum and infimum involving any finite collection $\Delta = \{C_1, C_2, \ldots, C_t\}$ of disjoint hypercubes whose union contains E (and a fortiori the supports of μ_{n_p} and μ), with

$$s(\Delta, \varphi) = \sum_{j=1}^{t} \mu(C_j) \inf_{C_j} \varphi \qquad \text{and} \qquad S(\Delta, \varphi) = \sum_{j=1}^{t} \mu(C_j) \sup_{C_j} \varphi.$$

Here, for any $\xi \in \Xi$, we have:

$$J(\varphi_\xi) - I(\varphi_\xi) \le S(\Delta, \varphi_\xi) - s(\Delta, \varphi_\xi) = \sum_{j=1}^{t} \mu(C_j)\,[\varphi_\xi(x^{(j)}) - \varphi_\xi(y^{(j)})]$$

for some $x^{(j)}$ and $y^{(j)}$ in the closure $\overline{C_j}$ of C_j. We can assume that all $\overline{C_j}$ (at least those such that $\mu(C_j) > 0$) have small enough diameters and do not intersect Ξ_θ.

For all x and $y \in \mathbb{R}^N \setminus \Xi_\theta$, the following holds:

$$\varphi_\xi(x) - \varphi_\xi(y) = \frac{1}{|x-\xi|^{N-2}} - \frac{1}{|y-\xi|^{N-2}} = f(0) - f(1)$$

with f defined by $f(t) = |x - \xi + t(y-x)|^{-N+2}$ $\forall t \in \mathbb{R}$. Now

$$f'(t) = \frac{-N+2}{|x-\xi+t(y-x)|^N} \sum_{i=1}^{N} [x_i - \xi_i + t(y_i - x_i)]\,(y_i - x_i)$$

and Cauchy-Schwarz formula yields

$$|f'(t)| \leq \frac{N-2}{|(1-t)x + ty - \xi|^N}\,(\,|x-\xi|.|y-x| + |y-x|^2\,) \qquad \forall t \in [0,1].$$

When x and y both belong to the convex $\overline{C_j}$, then $(1-t)x + ty$ is also located there, hence $|(1-t)x + ty - \xi| \geq \theta$. There exists $M > 0$ (depending only on the compacts E and Ξ) such that $|x-\xi| < M$ and $|y-x| < M$ for all $\xi \in \Xi$, x and $y \in E$. Finally

$$\varphi_\xi(x^{(j)}) - \varphi_\xi(y^{(j)}) \leq \frac{N-2}{\theta^N}\, 2M\, |y^{(j)} - x^{(j)}|.$$

For any $\varepsilon > 0$, there exists a collection $\Delta = \{C_1, C_2, ..., C_t\}$ such that the diameter of every C_j does not exceed ε, hence

$$J(\varphi_\xi) - I(\varphi_\xi) \leq \frac{2M(N-2)}{\theta^N}\,\varepsilon A \qquad \forall \xi \in \Xi.$$

Including this argument in the demonstration [3] (p. 206-208) will point out that the convergence (1) is actually uniform when we work with the equicontinuous family $\{\varphi_\xi\}_{\xi\in\Xi}$.

Notation. – We consider the ball $\overline{B}(O,r) = \{x \in \mathbb{R}^N : |x| \leq r\}$ and the sphere $S(O,r) = \{x \in \mathbb{R}^N : |x| = r\}$ $\forall r \geq 0$

Remark 2. – With $\varphi \equiv 1$ in Theorem 1, we observe that

$$\lim_{p\to+\infty} \mu_{n_p}(E) = \mu(E) \leq A.$$

The vague convergence on E of $(\mu_{n_p})_{p\in\mathbb{N}}$ towards μ does not necessarily imply its vague convergence on a compact $K \subset E$ towards the same limit μ. Otherwise, one would have $\lim\limits_{p\to+\infty} \mu_{n_p}(K) = \mu(K)$. A counterexample is provided by:

Example 1. – For any $n \in \mathbb{N}^*$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the continuous function, affine on $\left[1 - \frac{1}{n}, 1\right]$ and on $\left[1, 1 + \frac{1}{n}\right]$, defined by: $f_n(1) = n$ and $f_n(t) = 0$ if

$t \leq 1 - \dfrac{1}{n}$ or if $t \geq 1 + \dfrac{1}{n}$. Let μ_n be the measure defined by:

$$\int_{\mathbb{R}^N} \varphi(x)\, d\mu_n(x) = \int_0^2 \varphi(t, 0, ..., 0) f_n(t)\, dt$$

for any measurable function $\varphi : \mathbb{R}^N \to \mathbb{R}^+$. The sequence $(\mu_n)_{n\in\mathbb{N}^*}$ converges vaguely on $E = \overline{B}(O, 2)$ to the measure $\mu = \delta$ the Dirac mass concentrated at the point $(1, 0, ..., 0) \in \mathbb{R}^N$, because

$$\lim_{n\to+\infty} \int_0^2 \varphi(t, 0, ..., 0) f_n(t)\, dt = \varphi(1, 0, ..., 0)$$

for continuous functions $\varphi : E \to \mathbb{R}$. Now let $K = \overline{B}(O, 1)$. Then $\mu_n(K) = \int_0^1 f_n(t)\, dt = \dfrac{1}{2}$ hence $\lim\limits_{n\to+\infty} \mu_n(K) < \mu(K) = 1$.

REMARK 3. – If M is a non-empty measurable set contained in E, such that $\mu(\partial M) = 0$ (where ∂M denotes the boundary of M), then

$$\lim_{p\to+\infty} \mu_{n_p}(M) = \mu(M)$$

(see [6], p. 9 for a proof). When $\mu(\partial M) = 0$, the set M is said to be regular relatively to μ (see [2], pp. 9-10). Denoting by μ'_{n_p} and μ' the restrictions to M of μ_{n_p} and μ respectively, then $(\mu'_{n_p})_{p\in\mathbb{N}}$ converges vaguely to μ' (see [6], p. 10), but this result does not hold any longer without the assumption $\mu(\partial M) = 0$.

EXAMPLE 2. – Let $(f_n)_{n\in\mathbb{N}^*}$, $(\mu_n)_{n\in\mathbb{N}^*}$, μ and E be defined as in Example 1. Let $M = \overline{B}(O, 1)$ and μ'_n be the restriction to M of μ_n, in other words:

$$\int_M \varphi\, d\mu'_n = \int_E \psi\, d\mu_n$$

for any measurable function $\varphi : M \to \mathbb{R}^+$, with ψ defined by

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in M \\ 0 & \text{if } x \in E \setminus M. \end{cases}$$

Thus $\int_M \varphi\, d\mu'_n = \int_0^1 \varphi(t, 0, ..., 0) f_n(t)\, dt$. Here $\mu' = \delta$, but $(\mu'_n)_{n\in\mathbb{N}^*}$ converges vaguely on M to the measure $\dfrac{1}{2}\,\delta$ because

$$\lim_{n\to+\infty} \int_0^1 \varphi(t, 0, ..., 0) f_n(t)\, dt = \frac{1}{2}\, \varphi(1, 0, ..., 0)$$

for any continuous function $\varphi : M \to \mathbb{R}$.

A more general version of Theorem 1 is available:

THEOREM 2. – ([6], pp. 11-12 and p. 7). – *Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of measures on* $\mathbb{R}^N$. *Suppose that for any compact* $E \subset \mathbb{R}^N$ *a constant* $A_E > 0$ *exists such that* $\mu_n(E) \le A_E$ $\forall n \in \mathbb{N}$. *There is then a subsequence* $(\mu_{n_p})_{p\in\mathbb{N}}$ *which converges vaguely to a measure* μ *on* $\mathbb{R}^N$, *in other words:*

$$\lim_{p\to+\infty} \int_{\mathbb{R}^N} \varphi \, d\mu_{n_p} = \int_{\mathbb{R}^N} \varphi \, d\mu$$

for any continuous function φ *on* $\mathbb{R}^N$ *with compact support.*

REMARK 4. – This implies (1) on any compact set E such that $\mu(\partial E) = 0$, according to [6], p. 10. This allows to apply (2) and Lemma 1 on such sets E.

LEMMA 2. – *Given* μ_n, μ_{n_p}, μ *defined as in Theorem 2, let* $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ *be the repartition function associated to the measure* μ, *defined by* $\rho(t) = \mu\big(\overline{B}(O,t)\big)$ $\forall t \ge 0$. *Similarly, for any* $n \in \mathbb{N}$, *let* ρ_n *denote the repartition function associated to* μ_n. *Then* $\rho(t) = \lim_{p\to+\infty} \rho_{n_p}(t)$ *at any continuity point of* ρ *(hence for any* t *in the set* $[0,+\infty[$ *deprived of an at most countable subset). If there exists a right–continuous function* $\beta : [0,+\infty[\,\to [0,+\infty[$ *such that* $\rho_n(t) \le \beta(t)$ $\forall n \in \mathbb{N}$ $\forall t \ge 0$, *then* $\rho(t) \le \beta(t)$ $\forall t \ge 0$.

REMARK 5. – The term "repartition function" is explained as follows: let μ' be the measure defined on $[0,+\infty[$ by $\mu'(I) = \mu(\{x \in \mathbb{R}^N \,:\, |x| \in I\})$ for any Borel set $I \subset [0,+\infty[$. In the case where $\mu'([0,+\infty[) = \mu(\mathbb{R}^N) = 1$, the function ρ coincides with the repartition function of μ' in the classical probabilistic meaning.

PROOF OF LEMMA 2. – Since $\rho(s) \ge \rho(t)$ $\forall s \ge t \ge 0$, the set of all discontinuity points of ρ on $]0,+\infty[$ is at most countable. At any point $t_0 > 0$ the following holds:

$$\mu(S(O,t_0)) = \rho(t_0) - \lim_{t\to t_0\,,\,t<t_0} \rho(t).$$

Let $t > 0$ be a point where the function ρ is continuous. Hence $\mu(\partial M) = 0$ with $M = \overline{B}(O,t)$ and $\lim_{p\to+\infty} \mu_{n_p}(M) = \mu(M)$ according to [6], p. 9.

The estimation $\rho(t) \le \beta(t)$ is obvious for points t where ρ is continuous. Let t_0 be a point of discontinuity for ρ. As ρ is right-continuous at any point of $[0,+\infty[$, one has:

$$\rho(t_0) = \lim_{t\to t_0\,,\,t>t_0} \rho(t) = \lim_{k\to+\infty} \rho(t_k)$$

where $(t_k)_{k\in\mathbb{N}^*}$ is a sequence of continuity points for ρ with $t_k > t_0$ and

$$\lim_{k\to+\infty} t_k = t_0$$

(such a sequence exists since the set of discontinuity points is at most countable). The conclusion follows from $\lim_{k\to+\infty} \beta(t_k) = \beta(t_0)$.

3. – Subharmonic functions of order less than one.

A subharmonic function u in $\mathbb{R}^N$ of finite order $\lambda \geq 0$ satisfies such an estimation:

$$\forall\gamma > \lambda \qquad \exists A > 0 \qquad u(x) \leq A + |x|^\gamma \qquad \forall x \in \mathbb{R}^N.$$

Assuming that u is moreover harmonic in some neighborhood of the origin with $u(O) = 0$, the repartition function ρ associated to the Riesz measure μ of u satisfies (see [9]):

$$\forall\gamma > \lambda \qquad \exists C > 0 \qquad \rho(r) \leq C\, r^{N-2+\gamma} \qquad \forall r \geq 0.$$

When $\lambda < 1$, this leads to $\int_0^{+\infty} \frac{\rho(r)}{r^N}\, dr < +\infty$ which provides the following representation (see [8], pp. 67-69) where $K_0(x,\xi) = |\xi|^{2-N} - |x-\xi|^{2-N}$:

$$u(x) = \int_{\mathbb{R}^N} K_0(x,\xi)\, d\mu(\xi) \qquad \forall x \in \mathbb{R}^N \tag{3}$$

This representation remains valid for $\lambda = 1$ (see [3], pp. 155-156) provided that moreover u is of convergence class, that is: $\int_1^{+\infty} \frac{M_u(r)}{r^{\lambda+1}}\, dr < +\infty$ with $M_u(r) = \max_{|x|=r} u(x)$ (see [3], p. 143) or equivalently (see [10])

$$\int_1^{+\infty} \frac{J_u(r)}{r^{\lambda+1}}\, dr < +\infty \qquad \text{where } J_u(r) = (N-2)\int_0^r \frac{\rho(t)}{t^{N-1}}\, dt.$$

Jensen-Privalov formula (see [8], p. 44), in other words: Nevanlinna's first fundamental theorem (see [3], p. 127), provides another expression:

$$J_u(r) = \frac{1}{\sigma_N}\int_{S_N} u(rx)\, d\sigma_x$$

with $d\sigma$ the area element on the unit sphere S_N and $\sigma_N = \int_{S_N} d\sigma = \frac{2\,\pi^{N/2}}{\Gamma(N/2)}$ (see [3], p. 29).

LEMMA 3. – *Given u subharmonic in $\mathbb{R}^N$, harmonic in $B(O,\varepsilon)$ (for some $\varepsilon > 0$) with $u(O) = 0$, let ρ denote the repartition function of its Riesz measure. Then*

$$2^{N-3}\,\rho(r/2)\,\frac{1}{r^{N-2}} \leq J_u(r) \qquad \forall r > 0.$$

PROOF. – This minoration follows from $J_u(r) \geq (N-2)\,\rho(r/2)\int\limits_{r/2}^{r}\frac{dt}{t^{N-1}}$, together with $2^{N-2}-1 \geq 2^{N-3}$.

DEFINITION 1. – *Given* $\varepsilon > 0$, *let* $\mathcal{S}_\varepsilon$ *denote the set of all subharmonic functions* u *in* $\mathbb{R}^N$, *harmonic in* $B(O,\varepsilon)$ *with* $u(O)=0$, *of order* <1 *or at most of order* 1 *convergence class.*

For $u \in \mathcal{S}_\varepsilon$, note that $\rho \equiv 0$ on $[0,\varepsilon[$ and that $\lim\limits_{r\to+\infty}\frac{\rho(r)}{r^{N-1}} = 0$ (see [10]).

THEOREM 3. – *Given* $\varepsilon > 0$ *and two increasing sequences of positive numbers* $(M_k)_{k\in\mathbb{N}}$ *and* $(R_k)_{k\in\mathbb{N}}$ *satisfying* $R_0 > 2\varepsilon$ *and* $\lim\limits_{k\to+\infty} R_k = +\infty$ *together with*

$$\sum_{k\geq 1}\frac{M_k}{R_k}\left[\left(\frac{R_k}{R_{k-1}}\right)^{N-1} - 1\right] < +\infty,$$

let $(u_n)_{n\in\mathbb{N}}$ *be a sequence of elements in* $\mathcal{S}_\varepsilon$ *such that*

$$J_{u_n}(R_k) \leq M_k \qquad \forall n\in\mathbb{N} \qquad \forall k\in\mathbb{N}. \tag{4}$$

a) Then there exist $u\in\mathcal{S}_\varepsilon$ *and a subsequence* $(u_{n_p})_{p\in\mathbb{N}}$ *converging towards* u *uniformly on any compact set* $\Xi \subset \mathbb{R}^N$ *distant from the supports of the Riesz measures associated to* u *and* u_{n_p}. *Besides that*

$$\limsup_{p\to+\infty} u_{n_p}(x) \leq u(x) \qquad \forall x\in\mathbb{R}^N.$$

b) There exists a set $Q\subset\mathbb{R}^N$, *with outer capacity zero, such that*

$$\limsup_{p\to+\infty} u_{n_p}(x) = u(x) \qquad \forall x\in\mathbb{R}^N\setminus Q.$$

LEMMA 4. – *Given* $\varepsilon>0$, $u\in\mathcal{S}_\varepsilon$, μ *the Riesz measure of* u *and* ρ *its repartition function, let* $w(u,x,R) = \int\limits_{|\xi|>R} K_0(x,\xi)\,d\mu(\xi)$ $\forall x\in\mathbb{R}^N$ $\forall R\geq 0$. *If* $|x|\leq R/2$, *then*

$$|w(u,x,R)| \leq 2^{N-1}\,|x|\int\limits_R^{+\infty}\frac{d\rho(t)}{t^{N-1}} \leq 2^{N-1}\,|x|\,(N-1)\int\limits_R^{+\infty}\frac{\rho(t)}{t^N}\,dt.$$

PROOF. – See [3] p. 139 and [10].

REMARK 6. – When $R\geq 2|x|$, the representation (3) becomes:

$$u(x) = \int\limits_{|\xi|\leq R}\frac{d\mu(\xi)}{|\xi|^{N-2}} - \int\limits_{|\xi|\leq R}\frac{d\mu(\xi)}{|x-\xi|^{N-2}} + w(u,x,R)$$

with $u(x) = -\infty$ if and only if $\int\limits_{|\xi|\leq R} \dfrac{d\mu(\xi)}{|x-\xi|^{N-2}} = +\infty$ (both other terms on the right–hand side being finite).

REMARK 7. – Keeping Jensen–Privalov formula in mind, it turns out that (4) is fulfilled for instance when $u_n(x) \leq M_k \;\; \forall x \in \overline{B}(O, R_k) \;\; \forall n \in \mathbb{N} \;\; \forall k \in \mathbb{N}$. It also points out that the mean convergent subsequence from the works [1] and [5] was obtained in a different situation than here: the articles [1] and [5] assume that

$$\forall r > 0 \qquad \exists A_r > 0 \qquad \int\limits_{S_N} |u_n(rx)|\, d\sigma_x \leq A_r \qquad \forall n \in \mathbb{N}$$

which implies (4) but the converse is not valid.

PROOF OF THEOREM 3a). –The Riesz measures μ_n ($n \in \mathbb{N}$) associated to u_n respectively satisfy for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$:

$$\mu_n\big(\overline{B}(O, R_k/2)\big) = \rho_n(R_k/2) \leq \frac{1}{2^{N-3}}\, M_k\, {R_k}^{N-2} \tag{5}$$

with ρ_n the repartition function of μ_n. Thanks to Theorem 2, a subsequence $(\mu_{n_p})_{p\in\mathbb{N}}$ can be extracted, which converges vaguely to a measure μ whose repartition function, denoted by ρ, satisfies: $\rho \leq \beta$ on $[0, \infty[$ according to Lemma 2 applied to the increasing piecewise-constant function β defined by:

$$\beta(t) = \frac{1}{2^{N-3}}\, M_k\, {R_k}^{N-2} := L_k \qquad \forall t \in [R_{k-1}/2\,,\, R_k/2[\qquad \forall k \geq 1$$

$$\beta(t) = \frac{1}{2^{N-3}}\, M_0\, {R_0}^{N-2} \qquad \forall t \in [\varepsilon, R_0/2[$$

$$\beta \equiv 0 \text{ on } [0, \varepsilon[.$$

Now

$$I_k := \int\limits_{\frac{1}{2}R_{k-1}}^{\frac{1}{2}R_k} \frac{\beta(r)}{r^N}\, dr \leq L_k \int\limits_{\frac{1}{2}R_{k-1}}^{\frac{1}{2}R_k} r^{-N}\, dr = \frac{L_k}{N-1}\left[\left(\frac{2}{R_{k-1}}\right)^{N-1} - \left(\frac{2}{R_k}\right)^{N-1}\right].$$

Thus

$$\sum_{k=1}^{+\infty} I_k \leq \frac{2^{N-1}}{N-1} \sum_{k=1}^{+\infty} \frac{M_k\, {R_k}^{N-2}}{2^{N-3}}\left[\left(\frac{1}{R_{k-1}}\right)^{N-1} - \left(\frac{1}{R_k}\right)^{N-1}\right]$$
$$\leq \frac{4}{N-1} \sum_{k=1}^{+\infty} \frac{M_k}{R_k}\left[\left(\frac{R_k}{R_{k-1}}\right)^{N-1} - 1\right].$$

This shows that

$$\int\limits_0^{+\infty} \frac{\beta(r)}{r^N}\, dr < +\infty \qquad \text{thus} \quad \int\limits_0^{+\infty} \frac{\rho(r)}{r^N}\, dr < +\infty.$$

Hence the function u defined by (3) is subharmonic in $\mathbb{R}^N$ with Riesz measure μ according to [8], pp. 67-68. The growth order σ of the function $r \mapsto r^{2-N}\rho(r)$ coincides with the convergence exponent

$$\inf\left\{ c : \int_0^{+\infty} \frac{\rho(r)}{r^{N-1+c}}\, dr < +\infty \right\} \qquad \text{(see [8], p. 66)}$$

whence $\sigma \leq 1$. Thus

$$\forall \gamma > \sigma \qquad \exists A > 0 \qquad \rho(r) \leq A + r^{N-2+\gamma} \qquad \forall r \geq 0.$$

For such a fixed γ, we compute

$$J_u(r) = (N-2)\int_\varepsilon^r \frac{\rho(t)}{t^{N-1}}\, dt \leq \frac{A}{\varepsilon^{N-2}} + \frac{N-2}{\gamma}\, r^\gamma \qquad \forall r \geq \varepsilon.$$

The order λ of u being given by:

$$\lambda = \limsup_{r\to+\infty} \frac{\log M_u(r)}{\log r} = \limsup_{r\to+\infty} \frac{\log J_u(r)}{\log r}$$

(as a consequence of Poisson formula: see [10]), we obtain $\lambda \leq 1$. When $\lambda = 1$, it remains to check

$$\int_0^{+\infty} \frac{J_u(r)}{r^2}\, dr < +\infty$$

in order to conclude that $u \in \mathcal{S}_\varepsilon$. Fubini theorem leads to:

$$\int_0^{+\infty} \frac{1}{r^2}\left(\int_0^r \frac{\rho(t)}{t^{N-1}}\, dt\right) dr = \int_0^{+\infty} \frac{\rho(t)}{t^{N-1}} \underbrace{\left(\int_t^{+\infty} \frac{dr}{r^2}\right)}_{=1/t} dt = \int_0^{+\infty} \frac{\rho(t)}{t^N}\, dt.$$

For all $R > 0$ and $x \in \mathbb{R}^N$ such that $|x| \leq R/2$, it follows from Lemma 4 that

$$|w(u,x,R)| \leq 2^{N-1}\, |x|\, (N-1) \int_R^{+\infty} \frac{\beta(r)}{r^N}\, dr.$$

Since $\rho_{n_p} \leq \beta$ ($\forall p \in \mathbb{N}$), the same bound holds for $|w(u_{n_p}, x, R)|$ independantly of p. For all $x \in \mathbb{R}^N$, $R \geq 2\,|x|$ and $p \in \mathbb{N}$, the integral representations of u and u_{n_p} (see (3) and Remark 6) lead to:

$$|u(x) - u_{n_p}(x)| \leq |w(u,x,R)| + \left| \int_{|\xi|\leq R} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} - \int_{|\xi|\leq R} \frac{d\mu(\xi)}{|\xi|^{N-2}} \right|$$

$$+|w(u_{n_p},x,R)| + \left| \int_{|\xi|\leq R} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} - \int_{|\xi|\leq R} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \right|,$$

provided that $u(x) \neq -\infty$ and $u_{n_p}(x) \neq -\infty$ (this is fulfilled for instance for $x \in \Xi$). Let $D > 0$ such that $\Xi \subset \overline{B}(O, D)$. Given $\eta > 0$, let $T > 2D$ be chosen such that

$$2^N D\,(N-1) \int_T^{+\infty} \frac{\beta(r)}{r^N}\,dr \leq \eta$$

and such that T is a point of continuity for ρ. Remark 4 then applies with $E = \overline{B}(O,T)$ (since $\mu(\partial E) = 0$), hence it follows from (1) with $\varphi(\xi) = [\max(\varepsilon, |\xi|)]^{2-N}$ that

$$\lim_{p\to+\infty} \int_{|\xi|\leq T} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} = \int_{|\xi|\leq T} \frac{d\mu(\xi)}{|\xi|^{N-2}}$$

and moreover, according to Lemma 1, there exists $P_\eta \in \mathbb{N}$ for which

$$\left| \int_{|\xi|\leq T} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} - \int_{|\xi|\leq T} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \right| \leq \eta \qquad \forall x \in \Xi \qquad \forall p \geq P_\eta$$

and

$$\left| \int_{|\xi|\leq T} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} - \int_{|\xi|\leq T} \frac{d\mu(\xi)}{|\xi|^{N-2}} \right| \leq \eta \qquad \forall p \geq P_\eta.$$

For any $x \in \Xi$, we have $|w(u,x,T)| + |w(u_{n_p},x,T)| \leq \eta$ ($\forall p \in \mathbb{N}$) since $T \geq 2|x|$. Finally $|u(x) - u_{n_p}(x)| \leq 3\eta\ \forall x \in \Xi\ \forall p \geq P_\eta$, hence the uniform convergence on Ξ.

Besides that, the following holds for all $x \in \mathbb{R}^N$ and $R \geq 2\,|x|$:

$$u_{n_p}(x) - \int_{|\xi|\leq R} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \leq \left| \int_{|\xi|\leq R} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} - \int_{|\xi|\leq R} \frac{d\mu(\xi)}{|\xi|^{N-2}} \right| + |w(u,x,R)|$$

$$+|w(u_{n_p},x,R)| + u(x) - \int_{|\xi|\leq R} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}}.$$

Given $x \in \mathbb{R}^N$ fixed and $\eta > 0$, let $T \geq 2\,|x|$ be chosen large enough so that $|w(u,x,T)| + |w(u_{n_p},x,T)| \leq \eta\ \forall p \in \mathbb{N}$ and satisfying moreover $\mu(\partial E) = 0$ with $E = \overline{B}(O,T)$, in order to use (1) and (2). Thus:

$$\limsup_{p\to+\infty} u_{n_p}(x) - \int_{|\xi|\leq T} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \leq \eta + u(x) - \liminf_{p\to+\infty} \int_{|\xi|\leq T} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}}.$$

When $\int_{|\xi|\leq T} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} < +\infty$, we obtain: $\limsup_{p\to+\infty} u_{n_p}(x) \leq \eta + u(x)\ \forall \eta > 0$.

When $\int\limits_{|\xi| \le T} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} = +\infty$ (in other words $u(x) = -\infty$), then (2) leads to

$$\limsup_{p \to +\infty} \int\limits_{|\xi| \le T} \frac{-d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} = -\infty$$

and the result follows from Remark 6:

$$u_{n_p}(x) = \int\limits_{|\xi| \le T} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} - \int\limits_{|\xi| \le T} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} + w(u_{n_p}, x, T)$$

the last term being bounded independantly of p.

The proof of Theorem 3*b)* is postponed to Section 4.

4. – Null capacity sets.

The notion of capacity $Cap(K)$ is first defined for compact sets K in $\mathbb{R}^N$ (see [6] pp. 58 and 131-133 with $a = 2$). This gives rise to the notions of inner capacity $\underline{Cap}(E)$ and then outer capacity $\overline{Cap}(E)$ for arbitrary sets $E \subset \mathbb{R}^N$ (see [6] p. 143 with $a = 2$). Such a set is said to be capacitable if $\underline{Cap}(E) = \overline{Cap}(E)$, in which case this common value defines the capacity $Cap(E)$. For arbitrary sets $(E_n)_{n \in \mathbb{N}}$, the following holds:

$$\overline{Cap}\left(\bigcup_{n=0}^{+\infty} E_n\right) \le \sum_{n=0}^{+\infty} \overline{Cap}(E_n) \tag{6}$$

whereas

$$\underline{Cap}\left(\bigcup_{n=0}^{+\infty} E_n\right) \le \sum_{n=0}^{+\infty} \underline{Cap}(E_n)$$

requires that the E_n are Borel sets (see [6] p. 144 with $a = 2$).

LEMMA 5. – ([6] p. 190 with $a = 2$). – *Let v_p ($p \in \mathbb{N}$) and v be measures such that $(v_p)_{p \in \mathbb{N}}$ converges vaguely towards v and that*

$$\lim_{r \to +\infty} \int\limits_{|\xi| > r} \frac{dv_p(\xi)}{|\xi|^{N-2}} = 0 \qquad \textit{uniformly with respect to } p.$$

Then there exists $Q \subset \mathbb{R}^N$, with $\overline{Cap}(Q) = 0$ such that

$$\int\limits_{\mathbb{R}^N} \frac{dv(\xi)}{|x-\xi|^{N-2}} = \liminf_{p \to +\infty} \int\limits_{\mathbb{R}^N} \frac{dv_p(\xi)}{|x-\xi|^{N-2}} \qquad \forall x \in \mathbb{R}^N \setminus Q.$$

PROOF OF THEOREM 3*b*). – For the statement $\int_0^{+\infty} \frac{\beta(t)}{t^N}\, dt < +\infty$, we refer to the proof of Theorem 3*a*). For any $m \in \mathbb{N}^*$, there exists $T_m > 0$ such that

$$2^N (N-1) \int_{T_m}^{+\infty} \frac{\beta(t)}{t^N}\, dt \leq \frac{1}{m^2}$$

it may moreover be assumed that T_m is a continuity point for ρ and that $\lim_{m\to+\infty} T_m = +\infty$. Given $x \in \mathbb{R}$, Lemma 4 provides for all $m \in \mathbb{N}^*$ satisfying $m \geq |x|$ and $T_m \geq 2\,|x|$:

$$|w(u, x, T_m)| + |w(u_{n_p}, x, T_m)| \leq \frac{1}{m} \qquad \forall p \in \mathbb{N}.$$

With $m \in \mathbb{N}^*$ fixed, let $B_m = \overline{B}(O, T_m)$ and μ'_{n_p} (resp. μ') the restriction to B_m of μ_{n_p} (resp. μ). Since $\mu(\partial B_m) = 0$, it follows from [6] (p. 10) that $(\mu'_{n_p})_{p\in\mathbb{N}}$ converges vaguely towards μ'. Now

$$\int_{|\xi|>r} \frac{d\mu'_{n_p}(\xi)}{|\xi|^{N-2}} = 0 \qquad \forall r > T_m \qquad \forall p \in \mathbb{N}$$

thus Lemma 5 applies: there exists $Q_m \subset \mathbb{R}^N$ with $\overline{Cap}(Q_m) = 0$ such that

$$\liminf_{p\to+\infty} \int_{|\xi|\leq T_m} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} = \int_{|\xi|\leq T_m} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \qquad \forall x \in \mathbb{R}^N \setminus Q_m.$$

Moreover, according to (1)

$$\lim_{p\to+\infty} \int_{|\xi|\leq T_m} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} = \int_{|\xi|\leq T_m} \frac{d\mu(\xi)}{|\xi|^{N-2}}.$$

Let $Q = \bigcup_{m=1}^{+\infty} Q_m$. It follows from (6) that $\overline{Cap}(Q) = 0$. Now, given $x \in \mathbb{R}^N \setminus Q$ and $m \in \mathbb{N}^*$ with $m \geq |x|$ and $T_m \geq 2\,|x|$, the representation (3):

$$u_{n_p}(x) - \int_{|\xi|\leq T_m} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} = \int_{|\xi|\leq T_m} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-2}} - \int_{|\xi|\leq T_m} \frac{d\mu(\xi)}{|\xi|^{N-2}} - w(u, x, T_m)$$
$$+ w(u_{n_p}, x, T_m) + u(x) - \int_{|\xi|\leq T_m} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}}.$$

leads to:

$$-\frac{1}{m} + u(x) \leq \limsup_{p\to+\infty} u_{n_p}(x) \leq \frac{1}{m} + u(x)$$

(valid for all sufficiently large m) if $\int\limits_{|\xi|\le T_m} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} < +\infty$. The case where $\int\limits_{|\xi|\le T_m} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} = +\infty$ has already been studied at the end of Section 3.

REMARK 8. – It was not allowed to apply Lemma 5 to the measures μ_{n_p}. Of course

$$\int\limits_{|\xi|\ge r} \frac{d\mu_{n_p}(\xi)}{|\xi|^{N-1}} = \int_r^{+\infty} \frac{d\rho_{n_p}(t)}{t^{N-1}} \le (N-1)\int_r^{+\infty} \frac{\rho_{n_p}(t)}{t^N}\,dt \le (N-1)\int_r^{+\infty} \frac{\beta(t)}{t^N}\,dt$$

tends towards 0 (as $r \to +\infty$) uniformly with respect to p, but with $|\xi|^{N-1}$ replaced by $|\xi|^{N-2}$, this result does not necessarily hold any longer, since $\frac{1}{|\xi|^{N-1}} \le \frac{1}{|\xi|^{N-2}}$ when $|\xi| \ge 1$.

REMARK 9. – A similar conclusion as in Lemma 5 is provided by [6] p. 195, under the assumption that the measures ν_p have uniformly bounded energies, the energy of ν_p being given by

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}}\,d\nu_p(x)\,d\nu_p(y).$$

But neither this result can apply to the measures μ_{n_p} of Theorem 3, because their energy is not necessarily bounded. For instance, the measure $\mu = \delta_a$ (the Dirac mass at some fixed point $a \in \mathbb{R}^N$, $a \ne O$) has the following repartition function

$$\rho : t \mapsto \begin{cases} 0 & \text{if } 0 \le t < |a| \\ 1 & \text{if } t \ge |a| \end{cases}$$

hence $\int_0^{+\infty} \frac{\rho(t)}{t^N}\,dt < +\infty$ so that the function u defined by (3) is subharmonic in $\mathbb{R}^N$ with Riesz measure δ_a according to [8] pp. 67-68. Explicitely

$$u(x) = \frac{1}{|a|^{N-2}} - \frac{1}{|x-a|^{N-2}} \le \frac{1}{|a|^{N-2}}.$$

Thus the growth order of u is 0, hence $u \in \mathcal{S}_\varepsilon$ (provided that $|a| > \varepsilon$). However the energy of δ_a is not bounded since

$$\int\limits_{\mathbb{R}^N} \frac{d\delta_a(x)}{|x-y|^{N-2}} = \frac{1}{|a-y|^{N-2}} \qquad \text{and} \qquad \int\limits_{\mathbb{R}^N} \frac{d\delta_a(y)}{|a-y|^{N-2}} = +\infty$$

5. – Subharmonic functions of arbitrary finite order.

DEFINITION 2. – *Given* $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *let* $\mathcal{S}_{\varepsilon,q}$ *denote the set of all subharmonic functions* u *in* $\mathbb{R}^N$, *harmonic in* $B(O,\varepsilon)$ *with* $u(O) = 0$, *of order* $< q+1$ *or at most of order* $(q+1)$ *convergence class.*

For any $u \in \mathcal{S}_{\varepsilon,q}$, Fubini Theorem leads to

$$\int_0^{+\infty} \frac{J_u(r)}{r^{q+2}}\,dr = \frac{N-2}{q+1}\int_0^{+\infty} \frac{\rho(t)}{t^{N+q}}\,dt \qquad \text{thus} \quad \int_0^{+\infty} \frac{\rho(t)}{t^{N+q}}\,dt < +\infty.$$

This provides an integral representation of u similar to (3) but with $K_0(x,\xi)$ replaced by

$$K_q(x,\xi) = -|x-\xi|^{2-N} + A_q(x,\xi) \qquad \text{with } A_q(x,\xi) = \sum_{m=0}^{q} a_m(x,\xi)$$

and $a_m(x,\xi)$ the homogeneous polynomial of degree m (with respect to x_1, $x_2, \ldots, x_N$) in the Taylor expansion of $x \mapsto |x-\xi|^{2-N}$ (see [3] p. 137 or [8] p. 66). More precisely: there exists a harmonic polynomial $H_u \in \mathbb{R}[x_1, x_2, \ldots, x_N]$ of degree at most q, such that

$$u(x) = H_u(x) + \int_{\mathbb{R}^N} K_q(x,\xi)\,d\mu(\xi) \qquad \forall x \in \mathbb{R}^N \tag{7}$$

(see [3] pp. 141-146 or [8] pp. 67–69).

NOTATION. – For any $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{N}^N$, let $s(\nu) = \nu_1 + \nu_2 + \ldots + \nu_N$.

LEMMA 6. – *Given* $\varepsilon > 0$, $q \in \mathbb{N}$ *and measures* τ_p *and* τ *such that* $(\tau_p)_{p\in\mathbb{N}}$ *converges vaguely towards* τ *and that* $\tau_p\big(B(O,\varepsilon)\big) = 0$ $\forall p \in \mathbb{N}$, *let* $R > 0$ *satisfying* $\tau\big(S(O,R)\big) = 0$. *Then*

$$\lim_{p\to+\infty} \int_{|\xi|\le R} A_q(x,\xi)\,d\tau_p(\xi) = \int_{|\xi|\le R} A_q(x,\xi)\,d\tau(\xi)$$

uniformly on any compact of $\mathbb{R}^N$.

PROOF. – Let Φ_ξ be defined by $\Phi_\xi(x) = |x-\xi|^{2-N} = \left(\sum_{j=1}^{N}(x_j-\xi_j)^2\right)^{\frac{2-N}{2}}$. Hence:

$$A_q(x,\xi) = \sum_{\nu\,:\,s(\nu)\le q} a_\nu(\xi)\, x_1^{\nu_1} x_2^{\nu_2} \ldots x_N^{\nu_N}$$

with

$$a_\nu(\xi) = \frac{\partial^{s(\nu)} \Phi_\xi}{\partial x_1^{\nu_1} \ldots \partial x_N^{\nu_N}} (O) = \frac{Q_\nu(\xi)}{|\xi|^{N-2+2s(\nu)}}$$

where $Q_\nu \in \mathbb{R}[\xi_1, \xi_2, \ldots \xi_N]$. Now it follows from Remark 4 and (1) applied with $\varphi(\xi) = Q_\nu(\xi) [\max(\varepsilon, |\xi|)]^{2-N-2s(\nu)}$ that:

$$\lim_{p \to +\infty} \int_{|\xi| \leq R} \frac{Q_\nu(\xi)}{|\xi|^{N-2+2s(\nu)}} \, d\tau_p(\xi) = \int_{|\xi| \leq R} \frac{Q_\nu(\xi)}{|\xi|^{N-2+2s(\nu)}} \, d\tau(\xi)$$

for all $\nu \in \mathbb{N}^N$ such that $s(\nu) \leq q$. Whence

$$\left| \int_{|\xi| \leq R} A_q(x, \xi) \, d\tau_p(\xi) - \int_{|\xi| \leq R} A_q(x, \xi) \, d\tau(\xi) \right|$$

$$\leq \sum_{\nu \,:\, s(\nu) \leq q} |x_1^{\nu_1} \ldots x_N^{\nu_N}| \cdot \left| \int_{|\xi| \leq R} a_\nu(\xi) \, d\tau_p(\xi) - \int_{|\xi| \leq R} a_\nu(\xi) \, d\tau(\xi) \right| .$$

Given $D \geq 1$, we have $|x_1^{\nu_1} x_2^{\nu_2} \ldots x_N^{\nu_N}| \leq D^{\nu_1} D^{\nu_2} \ldots D^{\nu_N} = D^{s(\nu)} \leq D^q$ for all $x \in \overline{B}(O, D)$. Thus the convergence is uniform on $\overline{B}(O, D)$.

LEMMA 7. – *Given $\varepsilon > 0$, $q \in \mathbb{N}$, $u \in \mathcal{S}_{\varepsilon,q}$, μ the Riesz measure of u and ρ its repartition function, let*

$$w_q(u, x, R) = \int_{|\xi| > R} K_q(x, \xi) \, d\mu(\xi) \qquad \textit{and}$$

$$w'_q(u, x, R) = \int_{|\xi| \leq R} K_q(x, \xi) \, d\mu(\xi) \qquad \forall x \in \mathbb{R}^N \qquad \forall R \geq 0.$$

If $|x| \leq R/2$, then

$$|w_q(u, x, R)| \leq 4^{N+q} \, |x|^{q+1} \int_R^{+\infty} \frac{d\rho(t)}{t^{N-1+q}} \leq 4^{N+q} \, |x|^{q+1} \, (N - 1 + q) \int_R^{+\infty} \frac{\rho(t)}{t^{N+q}} \, dt$$

If $|x| \leq \varepsilon/2$, then

$$|w'_q(u, x, R)| \leq \frac{1}{\varepsilon^{N-2}} \left[2^{N-2} + \sum_{m=0}^{q} \frac{b_m}{2^m} \right] \rho(R) \qquad \textit{with } b_m = \frac{(m + N - 3)!}{m! \, (N - 1)!} .$$

PROOF. – The estimation $|K_q(x, \xi)| \leq 4^{N+q} \dfrac{|x|^{q+1}}{|\xi|^{N-1+q}}$ (if $|x| \leq |\xi|/2$) is available in [3] p. 139. Observing that

$$|w'_q(u, x, R)| \leq \int_{\varepsilon \leq |\xi| \leq R} \left[\frac{1}{|x - \xi|^{N-2}} + \sum_{m=0}^{q} |a_m(x, \xi)| \right] d\mu(\xi),$$

the second result follows from $|a_m(x,\xi)| \le b_m \dfrac{|x|^m}{|\xi|^{N+m-2}}$ (see [3] p. 137), since $|x-\xi| \ge |\xi| - |x| \ge \varepsilon - \varepsilon/2 = \varepsilon/2$ and $|a_m(x,\xi)| \le b_m \dfrac{(\varepsilon/2)^m}{\varepsilon^{N+m-2}} = \dfrac{b_m}{2^m\,\varepsilon^{N-2}}$.

LEMMA 8. – *Given $q \in \mathbb{N}$, let $(F_n)_{n\in\mathbb{N}}$ be a sequence of polynomials in $\mathbb{R}[x_1, x_2, ..., x_N]$ with degrees at most q, which converges pointwise on $\overline{B}(O,q)$ towards some function f. Then f is a polynomial of degree $\le q$ and $(F_n)_{n\in\mathbb{N}}$ converges towards f uniformly on every compact of $\mathbb{R}^N$. If the F_n are harmonic, then so is f.*

REMARK 10. – This result remains valid if the pointwise convergence holds on $\overline{B}(O,r)$ with some radius $r > 0$ independant of q (Lemma 8 merely applies to polynomials G_n defined by $G_n(x) = F_n(r^{-1}qx)\ \forall x \in \mathbb{R}^N$).

PROOF OF LEMMA 8. – For any $k \in \mathbb{N}$, let $P_k \in \mathbb{R}[t]$ be defined by

$$P_k(t) = \frac{1}{k!}\, t(t-1)(t-2)\dots(t-k+1)$$

with $P_0 \equiv 1$. Thus $P_k(l) = C_l^k\ \forall l \in \mathbb{N}$. These P_k provide a basis for the space of polynomials with degree $\le q$ in $\mathbb{R}[x_1, x_2, ..., x_N]$. With respect to this basis, let $a_{n,\nu}$ denote the coefficients of F_n, with $a_{n,\nu} = 0$ for any $\nu \in \mathbb{N}^N$ such that $s(\nu) > q$, in other words:

$$F_n(x) = \sum_{\nu\in\mathbb{N}^N} a_{n,\nu}\, P_{\nu_1}(x_1)\, P_{\nu_2}(x_2)\dots P_{\nu_N}(x_N) \qquad \forall x \in \mathbb{R}^N.$$

For any $\lambda \in \mathbb{N}^N$, we have $F_n(\lambda) = \sum a_{n,\nu}\, C_{\lambda_1}^{\nu_1}\, C_{\lambda_2}^{\nu_2} \dots C_{\lambda_N}^{\nu_N}$, this sum being restricted to the $\nu \in \mathbb{N}^N$ such that $\nu_j \le \lambda_j\ \forall j \in \{1, 2, ..., N\}$ together with $s(\nu) \le q$. When $s(\nu) = s(\lambda)$ and $\nu \ne \lambda$, the corresponding term in the sum vanishes since there exists i such that $\nu_i > \lambda_i$. If $s(\lambda) \le q$, then $\lambda \in \overline{B}(O,q)$ since

$$\sum_{j=1}^{N} \lambda_j^2 \le \left(\sum_{j=1}^{N} \lambda_j\right)^2 \le q^2.$$

If $\lambda = O$, then $F_n(\lambda) = a_{n,O}$ hence $\lim\limits_{n\to+\infty} a_{n,O} = f(O) := c_O$.
If $s(\lambda) = 1$, then $F_n(\lambda) = a_{n,O} + a_{n,\lambda}$, thus $\lim\limits_{n\to+\infty} a_{n,\lambda} = f(\lambda) - c_O := c_\lambda$.
More generally, for $2 \le s(\lambda) \le q$, we have

$$a_{n,\lambda} = F_n(\lambda) - \sum_{\nu\,:\, s(\nu) < s(\lambda)} a_{n,\nu}\, C_{\lambda_1}^{\nu_1}\, C_{\lambda_2}^{\nu_2} \dots C_{\lambda_N}^{\nu_N}.$$

Having defined recurrently the coefficients c_ν for $s(\nu) < s(\lambda)$, we obtain

$$\lim_{n\to+\infty} a_{n,\lambda} = f(\lambda) - \sum_{\nu\,:\, s(\nu) < s(\lambda)} c_\nu\, C_{\lambda_1}^{\nu_1}\, C_{\lambda_2}^{\nu_2} \dots C_{\lambda_N}^{\nu_N}$$

and this value is denoted by c_λ. Let F be the polynomial defined by

$$F(x) = \sum_{\nu \in \mathbb{N}^N} c_\nu \, P_{\nu_1}(x_1) \, P_{\nu_2}(x_2) \dots P_{\nu_N}(x_N) \qquad \forall x \in \mathbb{R}^N$$

with $c_\nu = 0$ if $s(\nu) > q$. Given K a compact in $\mathbb{R}^N$, we will next show that $\|F_n - F\| := \sup_{x \in K} |F_n(x) - F(x)|$ tends towards 0 as $n \to +\infty$ (on $K = \overline{B}(O, q)$, it will provide $f = F$). Now

$$|F_n(x) - F(x)| \leq \sum_{\nu \in \mathbb{N}^N} |a_{n,\nu} - c_\nu| \, . \, |P_{\nu_1}(x_1)| \, . \, |P_{\nu_2}(x_2)| \dots |P_{\nu_N}(x_N)|.$$

Let $R \geq 1$ be such that $K \subset \overline{B}(O, R)$. Thus

$$|P_{\nu_j}(x_j)| \leq R(R+1) \dots (R + \nu_j - 1) \leq (R + \nu_j)^{\nu_j} \leq (R + q)^{\nu_j} \qquad \forall x \in K.$$

Finally $||F_n - F|| \leq (R+q)^q \sum_\nu |a_{n,\nu} - c_\nu|$. This last sum containing only a fixed number of terms ($\leq q^N$), each of which tending towards 0 as $n \to +\infty$, the result follows.

Now $P_k'(t) = P_k(t) \sum_{l=0}^{k-1} \dfrac{1}{t-l}$ and $P_k''(t) = P_k(t) \, S_k(t) \; \forall t \in \mathbb{R}$, where

$$S_k(t) = \left(\sum_{l=0}^{k-1} \frac{1}{t-l} \right)^2 - \sum_{l=0}^{k-1} \frac{1}{(t-l)^2}$$

with $S_k \equiv 0$ for $k = 0$ and $k = 1$. Thus

$$\frac{\partial^2 F_n}{\partial x_i^2}(x) = \sum_{\nu \in \mathbb{N}^N} a_{n,\nu} \left(\prod_{j=1}^N P_{\nu_j}(x_j) \right) S_{\nu_i}(x_i) \qquad \forall x \in \mathbb{R}^N,$$

hence

$$\Delta F_n(x) = \sum_{\nu \in \mathbb{N}^N} a_{n,\nu} \left(\prod_{j=1}^N P_{\nu_j}(x_j) \right) \sum_{i=1}^N S_{\nu_i}(x_i).$$

As $n \to +\infty$, this quantity tends towards

$$\sum_{\nu \in \mathbb{N}^N} c_\nu \left(\prod_{j=1}^N P_{\nu_j}(x_j) \right) \sum_{i=1}^N S_{\nu_i}(x_i) = \Delta F(x)$$

at any point $x \in \mathbb{R}^N$. Whence $\Delta F_n \equiv 0 \; \forall n \in \mathbb{N}$ implies $\Delta F \equiv 0$.

THEOREM 4. – *Given $\varepsilon > 0$, $q \in \mathbb{N}$ and two increasing sequences of positive numbers $(M_k)_{k \in \mathbb{N}}$ and $(R_k)_{k \in \mathbb{N}}$ satisfying $R_0 > 2\varepsilon$, $\lim_{k \to +\infty} R_k = +\infty$ and*

$$\sum_{k \geq 1} \frac{M_k}{R_k^{\,1+q}} \left[\left(\frac{R_k}{R_{k-1}} \right)^{N+q-1} - 1 \right] < +\infty,$$

let $(u_n)_{n\in\mathbb{N}}$ *a sequence of elements in* $\mathcal{S}_{\varepsilon,q}$ *satisfying* (4) *and* $u_n(x) \le M_0$ $\forall x \in \overline{B}(O,\varepsilon/2)$ $\forall n \in \mathbb{N}$. *Then the same conclusions hold as in Theorem* 3 *(with here* $u \in \mathcal{S}_{\varepsilon,q}$*).*

LEMMA 9. – ([8] p. 47). – *Let* v_n *(*$n \in \mathbb{N}$*) be subharmonic functions in a domain* G *of* $\mathbb{R}^N$. *Suppose that:*

(i) *the sequence* $(v_n)_{n\in\mathbb{N}}$ *is uniformly bounded above on every compact subset of* G

(ii) *there is a compact subset* $K \subset G$ *such that* $(v_n)_{n\in\mathbb{N}}$ *does not uniformly converge on* K *to* $-\infty$ *as* $n \to +\infty$.

Then we can choose a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ *which converges in the distribution sense (in* $\mathcal{D}'(G)$ *in other words) to some function* v *subharmonic in* G.

LEMMA 10. – ([8] p. 48). – *Let* $(v_k)_{k\in\mathbb{N}}$ *be an uniformly bounded above sequence of functions subharmonic in* G *which converges in the distribution sense towards some function* v. *If we assume in addition that the* v_k *are harmonic, then they converge uniformly to* v *on every compact subset of the domain* G.

PROOF OF THEOREM 4. – Each function u_n has an integral representation of the kind (7) involving a harmonic polynomial H_{u_n} of degree $\le q$, written H_n for sake of brevity. Now

$$H_n(x) \le u_n(x) + |w'_q(u_n,x,R)| + |w_q(u_n,x,R)| \qquad \forall x \in \mathbb{R}^N \qquad \forall R > 0.$$

We obtain for all $x \in \overline{B}(O,\varepsilon/2)$ the following majorants which are independant of n:

$$u_n(x) \le M_0$$

$$|w'_q(u_n,x,R_0/2)| \le \frac{1}{\varepsilon^{N-2}}\left[2^{N-2} + \sum_{m=0}^{q}\frac{b_m}{2^m}\right]\frac{1}{2^{N-3}}\,M_0\,{R_0}^{N-2}$$

$$|w_q(u_n,x,R_0/2)| \le 4^{N+q}\left(\frac{\varepsilon}{2}\right)^{q+1}(N-1+q)\int_0^{+\infty}\frac{\beta(t)}{t^{N+q}}\,dt$$

according to Lemma 7 and (5). With β and L_k defined as in the proof of Theorem 3 a), we now have:

$$\int_{\frac12 R_{k-1}}^{\frac12 R_k}\frac{\beta(r)}{r^{N+q}}\,dr \le \frac{2^{N+q-1}}{N+q-1}\,\frac{L_k}{(R_k)^{N+q-1}}\left[\left(\frac{R_k}{R_{k-1}}\right)^{N+q-1}-1\right],$$

hence

$$\int_{\frac{1}{2}R_0}^{+\infty} \frac{\beta(r)}{r^{N+q}}\,dr \leq \frac{2^{q+2}}{N+q-1} \sum_{k\geq 1} \frac{M_k}{{R_k}^{1+q}} \left[\left(\frac{R_k}{R_{k-1}}\right)^{N+q-1} -1\right] < +\infty.$$

Finally: the polynomials H_n are uniformly bounded above on $\overline{B}(O, \varepsilon/2)$. When $q \in \mathbb{N}^*$, the polynomials F_n defined by

$$F_n(x) = H_n\left(\frac{\varepsilon x}{4q}\right)$$

are uniformly bounded above on $\overline{B}(O, 2q)$. They are harmonic too:

$$\frac{\partial F_n}{\partial x_j}(x) = \frac{\varepsilon}{4q}\frac{\partial H_n}{\partial x_j}\left(\frac{\varepsilon x}{4q}\right) \qquad \text{hence} \qquad \Delta F_n = \left(\frac{\varepsilon}{4q}\right)^2 \Delta H_n \equiv 0.$$

Moreover $F_n(O) = H_n(O) = 0$ and $\deg F_n \leq q\ \forall n \in \mathbb{N}$. We first apply Lemma 9 with $v_n = F_n$, $G = B(O, 2q)$, $K = \{O\}$ and next, Lemma 10. We thus obtain a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ converging uniformly on $\overline{B}(O, q)$ towards some function f which turns out to be a harmonic polynomial of degree $\leq q$ thanks to Lemma 8. So is H, defined by

$$H(x) = f\left(\frac{4qx}{\varepsilon}\right).$$

Conclusion: up to the extraction of a subsequence, $(H_n)_{n\in\mathbb{N}}$ converges to H uniformly on every compact of $\mathbb{R}^N$.

Through Lemma 3 and Theorem 2, a subsequence $(\mu_{n_p})_{p\in\mathbb{N}}$ is extracted, which converges vaguely to a measure μ whose repartition function ρ satisfies $\rho \leq \beta$ thus

$$\int_{\frac{1}{2}R_0}^{+\infty} \frac{\rho(t)}{t^{N+q}}\,dt \leq \int_{\frac{1}{2}R_0}^{+\infty} \frac{\beta(t)}{t^{N+q}}\,dt < +\infty.$$

Now the function U defined by

$$U(x) = \int_{\mathbb{R}^N} K_q(x, \xi)\,d\mu(\xi)$$

is subharmonic in $\mathbb{R}^N$ with Riesz measure μ according to [8] pp. 67-68. As in the proof of Theorem 3*a)*, we deduce that the order λ of U satisfies $\lambda \leq q+1$. If $\lambda = q+1$, then moreover

$$\int_0^{+\infty} \frac{J_U(r)}{r^{1+\lambda}}\,dr = \frac{N-2}{q+1} \int_0^{+\infty} \frac{\rho(t)}{t^{N+q}}\,dt < +\infty$$

thus $U \in \mathcal{S}_{\varepsilon,q}$. Now $u = H + U \in \mathcal{S}_{\varepsilon,q}$. For all $p \in \mathbb{N}$, $x \in \mathbb{R}^N$ and $R \geq 2|x|$ we have

$$u_{n_p}(x) - \int\limits_{|\xi|\leq R} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} = w_q(u_{n_p}, x, R) - w_q(u, x, R) - H(x) + H_{n_p}(x)$$

$$+ u(x) - \int\limits_{|\xi|\leq R} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} + \int\limits_{|\xi|\leq R} A_q(x,\xi)\, d\mu_{n_p}(\xi) - \int\limits_{|\xi|\leq R} A_q(x,\xi)\, d\mu(\xi).$$

• *Proof of the uniform convergence of* $(u_{n_p})_{p\in\mathbb{N}}$ *towards* u *on* Ξ.

Let $D > 0$ such that $\Xi \subset \overline{B}(O, D)$. Given $\eta > 0$, we fix $T \geq 2D$ in order to be a continuity point for ρ and such that

$$4^{N+q+\frac{1}{2}} D^{q+1} (N+q-1) \int\limits_T^{+\infty} \frac{\beta(t)}{t^{N+q}}\, dt \leq \eta \tag{8}$$

thus $|w_q(u,x,T)| + |w_q(u_{n_p},x,T)| \leq \eta \ \forall p \in \mathbb{N}\ \forall x \in \Xi$. There exists $P_\eta \in \mathbb{N}$ such that the following three estimations hold for all $p \geq P_\eta$ and $x \in \Xi$:

$$|H(x) - H_{n_p}(x)| \leq \eta\,, \qquad \left| \int\limits_{|\xi|\leq T} \frac{d\mu_{n_p}(\xi)}{|x-\xi|^{N-2}} - \int\limits_{|\xi|\leq T} \frac{d\mu(\xi)}{|x-\xi|^{N-2}} \right| \leq \eta$$

and

$$\left| \int\limits_{|\xi|\leq T} A_q(x,\xi)\, d\mu_{n_p}(\xi) - \int\limits_{|\xi|\leq T} A_q(x,\xi)\, d\mu(\xi) \right| \leq \eta.$$

Conclusion: $|u(x) - u_{n_p}(x)| \leq 4\eta \ \forall p \geq P_\eta \ \forall x \in \Xi$.

• *Proof of* $\limsup\limits_{p\to+\infty} u_{n_p}(x) \leq u(x)$ *for all* $x \in \mathbb{R}^N$.

Given $x \in \mathbb{R}^N$ and $\eta > 0$, we choose $T \geq 2|x|$ such that T is a continuity point for ρ and satisfies (8) with D replaced by $|x|$. Thus,

$$w_q(u_{n_p}, x, R) - w_q(u, x, R) \leq \eta \ \forall p \in \mathbb{N}.$$

We next proceed as at the end of Section 3.

• *Proof of* $\limsup\limits_{p\to+\infty} u_{n_p}(x) = u(x)$ *for all* x *outside a set of outer capacity zero.*

A sequence $(T_m)_{m\in\mathbb{N}^*}$ of positive numbers tending towards $+\infty$ is built in such a way that each T_m is a continuity point for ρ and that

$$4^{N+q+\frac{1}{2}} (N+q-1) \int\limits_{T_m}^{+\infty} \frac{\beta(t)}{t^{N+q}}\, dt \leq \frac{1}{m^{q+2}}.$$

Thus $|w_q(u, x, T_m)| + |w_q(u_{n_p}, x, T_m)| \le \dfrac{1}{m}$ $\forall p \in \mathbb{N}$, $\forall x \in \mathbb{R}^N$ and $m \in \mathbb{N}^*$ such that $|x| \le m$ and $T_m \ge 2|x|$. The proof now ends as in Section 4.

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