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An Elliptic Problem with a Lower Order Term Having Singular Behaviour

Daniela Giachetti - François Murat

Dedicated to the memory of Guido Stampacchia

Abstract. – *We prove the existence of distributional solutions to an elliptic problem with a lower order term which depends on the solution u in a singular way and on its gradient Du with quadratic growth. The prototype of the problem under consideration is*

$$\begin{cases} -\Delta u + \lambda u = \pm \dfrac{|Du|^2}{|u|^k} + f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$*,* $k > 0$*,* $f(x) \in L^\infty(\Omega)$*,* $f(x) \geq 0$ *(and so* $u \geq 0$*). If* $0 < k < 1$*, we prove the existence of a solution for both the "+" and the "−" signs, while if* $k \geq 1$*, we prove the existence of a solution for the "+" sign only.*

1. – Introduction.

Second order quasilinear elliptic problems involving a first order term $b(x, u, Du)$ depending on the solution u and on its gradient Du with a quadratic growth with respect to Du have been studied by many authors. Let us just quote [6], [7], [8], [9], [10], [11], [12], [13] and [17] and references therein.

The first order term $b(x, u, Du)$ appears in a natural way when one considers the Euler equations of functionals of the type

$$\frac{1}{2}\int_\Omega a(x,u)|Du|^2 - \int_\Omega fu\,, \tag{1.1}$$

which are perturbations of the classical energy functional

$$\frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega fu\,,$$

since the Euler equation of (1.1) reads as

$$-\operatorname{div}\,(a(x,u)Du) + \frac{1}{2}\frac{\partial a}{\partial u}(x,u)|Du|^2 = f \quad \text{in } \Omega.$$

More in general, the literature deals with the case where $b(x,s,\xi)$ is a Carathéodory function, which implies, in particular, that $b(x,s,\xi)$ is continuous in the s variable.

In contrast, in the present paper, we want to study the case where $b(x,s,\xi)$ is singular in $s=0$. In particular, since we are interested in homogeneous Dirichlet problems in a bounded open set Ω of $\mathbb{R}^N$, the function $b(x,u,Du)$ is singular at each point of the boundary of Ω.

As far as we know, only few results have been obtained in this case, see [1], [2], [3], [4], [5], [14], [15], and [16]. The papers [14] and [15] are concerned with the parabolic case. In [4], the authors consider the equation

$$(1.2)\qquad \begin{cases} -\Delta u + g(u)|Du|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $g \geq 0$, $\limsup_{s\to 0} sg(s) < +\infty$ and a datum f which is supposed to satisfy

$$f \in L^\infty(\Omega), \quad \inf\{f(x) : x \in \omega\} > 0 \qquad \forall\, \omega \Subset \Omega.$$

In [1], a variation on the hypothesis $g \geq 0$ is considered. In [2], existence and nonexistence results are given for the previous problem. In [5], if $0 \leq f \in L^m(\Omega)$, existence results of strictly positive solutions are proved for

$$(1.3)\qquad \begin{cases} -a\Delta u + \dfrac{1}{u^k}|Du|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for $a > 0$ and $m \geq \left(\frac{2^*}{k}\right)'$ if $0 < k < 1$ and for $a > 2$ and $m \geq \frac{2N}{N+2}$ if $k = 1$ (see also [16]). In [3], the case of variational inequalities with obstacles associated to (1.3) and $k = 1$ is considered.

The problem that we consider in the present paper is actually

$$(1.4)\qquad \begin{cases} -\operatorname{div}(a(x,u,Du)) + \lambda u = b(x,u,Du) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the principal part $-\operatorname{div}(a(x,u,Du))$ of the equation is a Leray-Lions operator acting on $H_0^1(\Omega)$ and where $\lambda > 0$. The datum f is supposed to satisfy

$$f \in L^\infty(\Omega), \quad f(x) \geq 0,$$

so that, in view the assumptions made on the term $b(x,u,Du)$, it results that $u \geq 0$. As for the nonlinear term $b(x,u,Du)$, we will assume that

$$|b(x,s,\xi)| \leq \frac{C_2}{|s|^k}|\xi|^2;$$

if $0 < k < 1$, this will be the sole assumption on b, and in particular no sign condition will be imposed on b. We make a more restrictive assumption if $k \geq 1$, assuming in this case that for some $C_1 > 0$

$$\frac{C_1}{|s|^k}|\xi|^2 \leq b(x,s,\xi) \leq \frac{C_2}{|s|^k}|\xi|^2. \tag{1.5}$$

In both cases, we will prove the existence of a solution of problem (1.4), namely of a function u which satisfies

$$\begin{cases} u \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega), \quad u \geq 0, \quad \dfrac{|Du|^2}{u^k}\chi_{u>0} \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,Du)D\Phi + \lambda\int_\Omega u\Phi = \int_\Omega b(x,u,Du)\chi_{u>0}\Phi + \int_\Omega f\Phi, \quad \forall\, \Phi \in C_c^\infty(\Omega). \end{cases}$$

Moreover, when $0 < k < 1$, the function u satisfies $u \in H^1_0(\Omega)$ (and not only $u \in H^1_{loc}(\Omega)$), and therefore satisfies the boundary condition $u = 0$ in the usual weak sense, while in the case $k \geq 1$, the function u satisfies $\psi(u) \in H^1_0(\Omega)$, where

$$\psi(s) = \int_0^s e^{\gamma(\sigma)}d\sigma, \quad \gamma(s) \sim -\frac{1}{s^{k-1}} \quad \text{when } k > 1,$$

which also expresses the homogeneous Dirichlet boundary condition, but not in the usual weak sense.

The fact that nonlinear functions of u appear in the formulation of the problem is not really surprising. It is indeed well known that, in this kind of problems, functions which are related to the behaviour of the nonlinearity $b(x,s,\xi)$ in the s variable play an essential role. In particular test functions of the type $e^{\gamma(u)}\varphi(u)$ (with φ a convenient function) are often used to get a priori estimates.

Let us emphasize that there is an important difference between the case $0 < k < 1$ and the case $k \geq 1$. In particular, the stronger hypothesis (1.5) made in the case $k \geq 1$ is probably a crucial and not only a technical hypothesis. Indeed it has recently been proved in [2] that there is no solution $u \in H^1_0(\Omega)$ of (1.2) in the case where $g(u) = \dfrac{1}{u^k}$ and $k \geq 2$, see Remark 2.5 below.

To conclude this Introduction, let us note that, as far as we know, the case where the datum f (and therefore the solution u) takes both positive and negative values is an open problem. Also, as far as we know, the problem with f positive but where the singularity in u takes place in a point $m > 0$ (and is therefore of the type $\dfrac{1}{|s-m|^k}$ and no more of the type $\dfrac{1}{|s|^k}$) is an open problem, which seems to exhibit difficulties similar to the previous one.

The plan of the present paper as follows: In Section 2 we give the precise hypotheses and statements of our results. In Section 3 we define the approximating problems. In Section 4 we prove Theorem 2.2 (case $0 < k < 1$), while in Section 5 we prove Theorem 2.3 (case $k \geq 1$). Let us explicitly note that the first two steps are the same in the proofs of Theorems 2.2 and 2.3.

2. – Hypotheses and results.

In the present paper we consider the problem

$$\begin{cases} -\mathrm{div}\,(a(x,u,Du)) + \lambda u = b(x,u,Du) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, where

$$\lambda > 0, \tag{2.2}$$

$$f(x) \in L^\infty(\Omega), \quad f(x) \geq 0, \tag{2.3}$$

where the function

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a Carathéodory function which satisfies for some $\alpha > 0$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^2, \tag{2.4}$$

$$|a(x,s,\xi)| \leq \nu|\xi|, \tag{2.5}$$

$$(a(x,s,\xi) - a(x,s,\eta))(\xi - \eta) > 0, \quad \forall\, \xi \neq \eta, \tag{2.6}$$

$$\text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

and where the function

$$b(x,s,\xi) : \Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function on $\Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N$, i.e. a function which is, for every $(s,\xi) \in (\mathbb{R} - \{0\}) \times \mathbb{R}^N$, a measurable function $x \in \Omega \to b(x,s,\xi) \in \mathbb{R}$, and which is, for almost every $x \in \Omega$, a continuous function $(s,\xi) \in (\mathbb{R} - \{0\}) \times \mathbb{R}^N \to$ $\to b(x,s,\xi) \in \mathbb{R}$ (see Remark 2.6 below concerning the definition of $b(x,s,\xi)$ when $s = 0$).

As far as the behaviour of $b(x,s,\xi)$ near $s=0$ (for x and ξ fixed) is concerned, it is worth to distinguish two cases, the case $0<k<1$ and the case $k\geq 1$, that present different features and that will be treated separately.

We will suppose either that

$$\begin{cases} 0<k<1, \\ |b(x,s,\xi)| \leq \dfrac{C_2}{|s|^k}|\xi|^2, \quad \text{a.e. } x\in\Omega, \quad \forall s\in\mathbb{R}, \quad s\neq 0, \quad \forall\xi\in\mathbb{R}^N, \end{cases} \tag{2.7}$$

or that for some $C_1>0$,

$$\begin{cases} k\geq 1, \\ \dfrac{C_1}{|s|^k}|\xi|^2 \leq b(x,s,\xi) \leq \dfrac{C_2}{|s|^k}|\xi|^2, \ \text{a.e. } x\in\Omega, \ \forall s\in\mathbb{R}, \quad s\neq 0, \ \forall\xi\in\mathbb{R}^N. \end{cases} \tag{2.8}$$

Note that (2.8) is much more restrictive than (2.7), since (2.8) is a growth condition for $b(x,s,\xi)$ both from above and from below, while (2.7) is only a growth condition from above. In particular, when (2.7) holds true, $b(x,s,\xi)$ is not assumed to have a specified sign, while $b(x,s,\xi)$ has in particular to be strictly positive (for $\xi\neq 0$) when (2.8) holds true.

Let $M>0$ and $\beta:(0,M]\to\mathbb{R}$ be defined by

$$M=\frac{\|f\|_\infty}{\lambda}, \quad \beta(s)=\frac{1}{s^k}\,. \tag{2.9}$$

In the case where $0<k<1$, the function β belongs to $L^1(0,M)$, while in the case where $k\geq 1$, the function β is not integrable in 0.

Let us introduce the following function $\gamma(s)$, defined for $s\in(0,M]$, which is a primitive function of the function $\dfrac{C_2}{a}\beta(s)$, defined by

$$\gamma(s)=\begin{cases} \dfrac{C_2}{a}\displaystyle\int_0^s \frac{1}{\sigma^k}\,\mathrm{d}\sigma = \frac{C_2}{a}\frac{s^{1-k}}{1-k}, & \text{if } \ 0<k<1, \\ \dfrac{C_2}{a}\displaystyle\int_M^s \frac{1}{\sigma}\,\mathrm{d}\sigma = \frac{C_2}{a}\log\Big(\frac{s}{M}\Big), & \text{if } \ k=1, \\ \dfrac{C_2}{a}\displaystyle\int_M^s \frac{1}{\sigma^k}\,\mathrm{d}\sigma = \frac{C_2}{a}\frac{1}{k-1}\Big(\frac{1}{M^{k-1}}-\frac{1}{s^{k-1}}\Big), & \text{if } \ k>1\,. \end{cases} \tag{2.10}$$

Let us finally define, for $s\in[0,M]$, the function ψ by

$$\psi(s)=\int_0^s e^{\gamma(\sigma)}\,\mathrm{d}\sigma, \tag{2.11}$$

and, for $m > 0$ and $s \in \mathbb{R}$, the function S_m by

$$S_m(s) = \begin{cases} m & \text{if } s \leq m, \\ s & \text{if } s \geq m\,. \end{cases} \tag{2.12}$$

REMARK 2.1. – Let us point out that, in the case where $0 < k < 1$ (i.e. when (2.7) holds), the function $\gamma(s)$ is an increasing, non negative bounded function on $[0, M]$, while in the case where $k \geq 1$ (i.e. when (2.8) holds), the function $\gamma(s)$ is an increasing, non positive function on $(0, M]$ with $\lim_{s \to 0^+} \gamma(s) = -\infty$.

In both cases $e^{\gamma(s)}$ is a bounded function on $[0, M]$ and, therefore, the function $\psi(s)$ is well defined by (2.11).

Our results are the following.

THEOREM 2.2. – *Suppose that* (2.2)-(2.7) *hold true. Then there exists at least a function u such that*

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u \geq 0, \tag{2.13}$$

$$\psi(u) \in H_0^1(\Omega), \quad \frac{|Du|^2}{u^k}\chi_{u>0} \in L^1(\Omega), \tag{2.14}$$

$$\int_\Omega a(x,u,Du)D\Phi + \lambda\int_\Omega u\Phi = \int_\Omega b(x,u,Du)\chi_{u>0}\Phi + \int_\Omega f\Phi, \quad \forall\, \Phi \in C_c^\infty(\Omega). \tag{2.15}$$

THEOREM 2.3. – *Suppose that* (2.2)-(2.6) *and* (2.8) *hold true. Then there exists at least a function u such that*

$$u \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \quad u \geq 0, \tag{2.16}$$

$$S_m(u) \in H^1(\Omega), \quad \forall\, m > 0, \quad \psi(u) \in H_0^1(\Omega), \quad \frac{|Du|^2}{u^k}\chi_{u>0} \in L^1_{\text{loc}}(\Omega), \tag{2.17}$$

$$\int_\Omega a(x,u,Du)D\Phi + \lambda\int_\Omega u\Phi = \int_\Omega b(x,u,Du)\chi_{u>0}\Phi + \int_\Omega f\Phi, \quad \forall\, \Phi \in C_c^\infty(\Omega). \tag{2.18}$$

REMARK 2.4. – In view of assumptions (2.7) and (2.8), the fact that $\dfrac{|Du|^2}{u^k}\chi_{u>0}$ belongs to $L^1_{\text{loc}}(\Omega)$ implies that $b(x,u,Du)\chi_{u>0}$ belongs to $L^1_{\text{loc}}(\Omega)$, which gives a meaning to the first terms of the right-hand sides of equations (2.15) and (2.18).

REMARK 2.5. – Let us observe that in Theorem 2.2, where (2.7) is assumed to hold true, we can consider a general term $b(x, u, Du)$ without any sign condition, while hypothesis (2.8), assumed in Theorem 2.3, obliges $b(x, s, \xi)$ to be strictly positive for $\xi \neq 0$.

It is likely that the more restrictive condition (2.8) is necessary in order to have the existence of a solution of (2.18) if $k \geq 2$. Indeed, it has been proved in [2] that for $\lambda > 0$, f strictly positive on every compactly embedded subset of Ω and $k \geq 2$, there is no solution of the problem

$$\begin{cases} u \in H_0^1(\Omega), \quad u \geq 0, \\ -\Delta u + \dfrac{1}{u^k}|Du|^2 = f(x) \quad \text{in } \mathcal{D}'(\Omega). \end{cases}$$

REMARK 2.6. – The first terms of the right-hand sides of equations (2.15) and (2.18) involve the function $b(x, u, Du)\chi_{u>0}$, which can also be written as $\tilde{b}(x, u, Du)$, where $\tilde{b}(x, s, \xi)$ is the function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ (and no more on $\Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N$) by

$$\tilde{b}(x, s, \xi) = \begin{cases} b(x, s, \xi) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0\,. \end{cases}$$

Note that $\tilde{b}$ is not a Carathéodory function since it is not continuous at the point $s = 0$. Nevertheless, for u and v measurable functions with values in $\mathbb{R}$ and $\mathbb{R}^N$, the function $x \in \Omega \to \tilde{b}(x, u(x), v(x))$ is measurable (1) under the assumption made on $b(x, s, \xi)$ that b is a Carathéodory function on $\Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N$.

We could therefore have replaced the first terms of the right-hand sides of (2.15) and (2.18) by $\int_\Omega \tilde{b}(x, u, Du)\Phi$, but we chose not to do so, in order to emphasize the fact that $\tilde{b}$ is not a Carathéodory function.

On the other hand, in the case where hypothesis (2.8) holds, it is natural to consider the function $\tilde{\tilde{b}}(x, s, \xi)$ defined by

$$\tilde{\tilde{b}}(x, s, \xi) = \begin{cases} b(x, s, \xi) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0,\ \xi = 0, \\ +\infty & \text{if } s = 0,\ \xi \neq 0, \end{cases}$$

(1) Consider indeed a sequence (u_n, v_n) of step functions which converge almost everywhere on Ω to (u, v) and which satisfy $u_n(x) \neq 0$ for every x and every n (it is always possible to build such a sequence u_n from a given sequence of step functions $\hat{u}_n$ by defining the functions u_n by $u_n(x) = \hat{u}_n(x)$ if $\hat{u}_n(x) \neq 0$, $u_n(x) = \dfrac{1}{n}$ if $\hat{u}_n(x) = 0$). Then $\tilde{b}(x, u_n(x), v_n(x))$ is a measurable function and $\tilde{b}(x, u_n(x), v_n(x))\chi_{u>0}$ converges a.e. to $\tilde{b}(x, u, v)$, which is therefore a measurable function.

since the function $\tilde{\tilde{b}}$ is continuous for every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^N$, except in the point $s = 0$ and $\xi = 0$. With this definition, we have

$$\tilde{\tilde{b}}(x, s, \xi) = \tilde{b}(x, s, \xi) + (+\infty)\chi_{\{s=0\}\cap\{\xi\neq 0\}}\,.$$

But for $u \in H^1_{\mathrm{loc}}(\Omega)$, we have

$$\tilde{\tilde{b}}(x, u, Du) = \tilde{b}(x, u, Du) \quad \text{a.e. in } \Omega,$$

since, when $u \in H^1_{\mathrm{loc}}(\Omega)$, one has $Du = 0$ almost everywhere on the set where $u = 0$.

Therefore, since Theorem 2.3 asserts that $u \in H^1_{\mathrm{loc}}(\Omega)$, we could also have replaced the integral $\int_\Omega b(x, u, Du)\chi_{u>0}\Phi$ by the integral $\int_\Omega \tilde{\tilde{b}}(x, u, Du)\Phi$ in the first term of the right-hand side of (2.18).

3. – Approximating problems.

In order to prove Theorem 2.2 and 2.3, we introduce in this Section a sequence of approximating problems.

For $n \in \mathbb{N}$, we consider the problems

$$\begin{cases} u_n \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}\,(a(x, u_n, Du_n)) + \lambda u_n = b_n(x, u_n, Du_n) + f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{3.1}$$

where, for almost every $x \times \Omega$, for every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^N$, the function $b_n : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by

$$b_n(x, s, \xi) = \begin{cases} b(x, s, \xi) & \text{if } s > \dfrac{1}{n}, \\ b(x, \dfrac{1}{n}, \xi) & \text{if } \quad s \leq \dfrac{1}{n}. \end{cases}$$

We also define, for $s \in \mathbb{R}$,

$$\beta_n(s) = \begin{cases} \beta(s) & \text{if } s \geq \dfrac{1}{n}, \\ \beta\left(\dfrac{1}{n}\right) & \text{if } \quad s \leq \dfrac{1}{n}, \end{cases}$$

$$\gamma_n(s) = \begin{cases} \dfrac{C_2}{a}\displaystyle\int_0^s \beta_n(\sigma)\mathrm{d}\sigma & \text{if } 0 < k < 1, \\ \dfrac{C_2}{a}\displaystyle\int_M^s \beta_n(\sigma)\mathrm{d}\sigma & \text{if } k \geq 1, \end{cases} \tag{3.2}$$

$$\psi_n(s) = \int_0^s e^{\gamma_n(\sigma)}\, \mathrm{d}\sigma\,. \tag{3.3}$$

In contrast with the function b, which is defined on $\Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N$, the function b_n is a Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and we have

$$|b_n(x, s, \xi)| \le C_2 \beta_n(s) |\xi|^2, \text{ a.e. } x \in \Omega, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.4}$$

which implies in particular that

$$|b_n(x, s, \xi)| \le C_2 \beta\left(\frac{1}{n}\right) |\xi|^2, \quad \text{a.e. } x \in \Omega,\ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N\,. \tag{3.5}$$

For n fixed, the function b_n is now a classical Carathéodory function with quadratic growth with respect to ξ, and since $f \in L^\infty(\Omega)$, it is well known that problem (3.1) has at least one solution u_n (see e.g. [7]).

Moreover, since $f \ge 0$, this solution satisfies $u_n \ge 0$ a.e. in Ω. Indeed, denoting for every $s \in \mathbb{R}$

$$(s)_+ = \max\{s, 0\}, \quad (s)_- = \max\{-s, 0\},$$

the result $u_n \ge 0$ is easily obtained, in the case where $b(x, s, \xi) \ge 0$ (which implies $b_n(x, s, \xi) \ge 0$), by using $-(u_n)_-$ as test function in (3.1); in the general case, the result $u_n \ge 0$ is proved by using in (3.1) the test function $-(u_n)_- e^{-k_n u_n}$, with $a k_n \ge C_2 \beta\left(\frac{1}{n}\right)$, since

$$k_n\, a(x, u_n, Du_n) Du_n e^{-k_n u_n} (u_n)_- + b_n(x, u_n Du_n) e^{-k_n u_n} (u_n)_- \ge 0.$$

This proves the existence of a function u_n such that

$$\begin{cases} u_n \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u_n \ge 0, \\ -\operatorname{div}(a(x, u_n, Du_n)) + \lambda u_n = b_n(x, u_n, Du_n) + f(x) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.6}$$

4. – Proof of Theorem 2.2.

We begin by proving Theorem 2.2 and divide the proof in several steps.

Let us explicitly note that the first two steps of the proof of Theorem 2.3 will be identical to the first two steps of the present proof.

STEP 1. – Uniform estimate of $(u_n)_{n \in \mathbb{N}}$ in $L^\infty(\Omega)$.

Let us use as test function in (3.1) the function $v_n = e^{\gamma_n(u_n)}(u_n - M)_+$ which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ since u_n belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, where M is defined by (2.9). Using (2.4), (3.2) and (3.4) we get

$$a\int_\Omega |D(u_n - M)_+|^2 e^{\gamma_n(u_n)}$$

$$+ C_2\int_\Omega |Du_n|^2 \beta_n(u_n) e^{\gamma_n(u_n)}(u_n - M)_+ + \int_\Omega \lambda u_n e^{\gamma_n(u_n)}(u_n - M)_+$$

$$\le C_2\int_\Omega \beta_n(u_n)|Du_n|^2 e^{\gamma_n(u_n)}(u_n - M)_+ + \int_\Omega f e^{\gamma_n(u_n)}(u_n - M)_+ .$$

We simplify the two terms which are equal, forget the first (non negative) term in the left-hand side and add to both sides $-\lambda M \int_\Omega e^{\gamma_n(u_n)}(u_n - M)_+$, getting

$$\lambda\int_\Omega (u_n - M)_+^2 e^{\gamma_n(u_n)} \le \int_\Omega (f - \lambda M) e^{\gamma_n(u_n)}(u_n - M)_+ \le 0,$$

which implies $u_n \le M$. Therefore we have

$$0 \le u_n \le M \quad \forall n \in \mathbb{N}\,. \tag{4.1}$$

STEP 2. – Uniform estimate of $(\psi(u_n))_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$.

Let us take as test function in (3.1) the function $v_n = \psi(u_n)e^{\gamma_n(u_n)}$ which belongs to $H_0^1(\Omega)$. Using (2.4), (3.3) and (3.4), we get

$$a\int_\Omega |Du_n|^2 e^{2\gamma_n(u_n)} + C_2\int_\Omega |Du_n|^2 e^{\gamma_n(u_n)}\beta_n(u_n)\psi_n(u_n) + \lambda\int_\Omega u_n\psi_n(u_n)e^{\gamma_n(u_n)}$$

$$\le C_2\int_\Omega \beta_n(u_n)|Du_n|^2 e^{\gamma_n(u_n)}\psi_n(u_n) + \int_\Omega f e^{\gamma_n(u_n)}\psi(u_n)\,.$$

Recalling that, by Step 1, $fe^{\gamma_n(u_n)}\psi_n(u_n)$ is uniformly bounded in n, we get

$$a\int_\Omega |D\psi_n(u_n)|^2 = a\int_\Omega |Du_n|^2 e^{2\gamma_n(u_n)} \le \text{cst} \quad \forall\, n \in \mathbb{N}\,. \tag{4.2}$$

Note that, in these first two steps, only the growth condition of hypothesis (2.7) (and never the growth condition from below of hypothesis (2.8)) is used.

STEP 3. – Uniform estimate of $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$, of $(a(x, u_n, Du_n))_{n\in\mathbb{N}}$ in $L^2(\Omega)^N$ and of $(b_n(x, u_n, Du_n))_{n\in\mathbb{N}}$ in $L^1(\Omega)$.

In this step we use for the first time the fact that $k < 1$.

Since here $0 < k < 1$, the functions γ and γ_n defined by (2.10) and (3.2) are non negative. Therefore $\gamma_n(u_n) \geq 0$ and (4.2) implies that

$$\int_\Omega |Du_n|^2 \leq \text{cst} \quad \forall n \in \mathbb{N}, \tag{4.3}$$

and from (2.5) and (4.3) it follows that

$$\int_\Omega |a(x, u_n, Du_n)|^2 \leq \nu^2 \int_\Omega |Du_n|^2 \leq \text{cst} \quad \forall n \in \mathbb{N}\,. \tag{4.4}$$

We then use as test function in (3.1) the function $v_n = e^{\gamma_n(u_n)} - 1$ which belongs to $H_0^1(\Omega)$. Since $\gamma_n(u_n) \geq 0$, we have $v_n \geq 0$, and using (3.4) we get

$$\begin{aligned} &\frac{C_2}{a}\int_\Omega a(x, u_n, Du_n)Du_n\, \beta_n(u_n)\, e^{\gamma_n(u_n)} + \lambda \int_\Omega u_n(e^{\gamma_n(u_n)} - 1) \\ &\leq C_2 \int_\Omega \beta_n(u_n)|Du_n|^2(e^{\gamma_n(u_n)} - 1) + \int_\Omega f(e^{\gamma_n(u_n)} - 1)\,. \end{aligned}$$

Using (2.4), simplifying the two terms which are equal, then using the fact that the last term of the left-hand side is non negative, and the fact that $e^{\gamma_n(u_n)}$ is uniformly bounded in n, we get

$$C_2 \int_\Omega \beta_n(u_n)|Du_n|^2 \leq \text{cst} \quad \forall n \in \mathbb{N}, \tag{4.5}$$

which by (3.4) gives

$$\int_\Omega |b_n(x, u_n, Du_n)| \leq \text{cst} \quad \forall n \in \mathbb{N}. \tag{4.6}$$

STEP 4. – Weak convergence of $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ and strong convergence of $(DS_m(u_n))_{n\in\mathbb{N}}$ in $(L^2(\Omega))^N$ for any fixed $m > 0$.

By estimate (4.3), we deduce that, up to a subsequence, there exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in}\, H_0^1(\Omega)\,, \qquad u_n \to u \quad \text{a.e. in}\, \Omega\,.$$

Therefore, for every $m > 0$,

$$S_m(u_n) \rightharpoonup S_m(u) \text{ weakly in } H_0^1(\Omega), \quad S_m(u_n) \to S_m(u) \text{ a.e. in } \Omega$$

where S_m is defined by (2.12).

We want now to prove that

$$\lim_{n\to+\infty} \int_\Omega |D(S_m(u_n) - S_m(u))|^2 = 0 \quad \forall\, m > 0. \tag{4.7}$$

We first use as test function in (3.1) the function

$$v_n = \Phi((S_m(u_n) - S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))},$$

where $\Phi(s) = e^{\mu s} - 1$, for some $\mu \geq 2\beta(m)$. Using (3.2) and (3.4), we get

$$\int_\Omega a(x,u_n,Du_n)D(S_m(u_n)-S_m(u))_+\Phi'((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$+\frac{C_2}{a}\int_\Omega a(x,u_n,Du_n)Du_n\Phi((S_m(u_n)-S_m(u))_+)\beta_n(u_n)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$-\frac{C_2}{a}\int_\Omega a(x,u_n,Du_n)DS_m(u_n)\Phi((S_m(u_n)-S_m(u))_+)\beta_n(S_m(u_n))e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$+\lambda\int_\Omega u_n\Phi((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$\leq C_2\int_\Omega \beta_n(u_n)|Du_n|^2\Phi((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$+\int_\Omega f\Phi((S_m(u_n)-S_m(u))_+e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}.$$

Using (2.4), then simplifying the two terms which are equal and forgetting the last term of the left-hand side, which is non negative, we get

$$\int_\Omega a(x,u_n,Du_n)D(S_m(u_n)-S_m(u))_+\Phi'((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$-\frac{C_2}{a}\int_\Omega a(x,u_n,Du_n)D(S_m(u_n))\Phi((S_m(u_n)-S_m(u))_+)\beta_n(S_m(u_n))e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$

$$\leq\int_\Omega f\Phi((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}.$$

Due to (4.1) and to the almost everywhere convergence of $(u_n)_{n\in\mathbb{N}}$ to u, we have

$$\int_\Omega f\Phi((S_m(u_n)-S_m(u))_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))} = \omega(n),$$

where here and in the sequel $\omega(n)$ is a sequence of real numbers for which

$\lim_{n\to+\infty} \omega(n) = 0$. Therefore

$$\int_\Omega a(\,)D(\,)_+\Phi'((\,)_+)e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$
$$(4.8)\qquad -\frac{C_2}{a}\int_\Omega a(\,)DS_m(u_n)\Phi((\,)_+)\beta_n(S_m(u_n))e^{\gamma_n(u_n)-\gamma_n(S_m(u_n))}$$
$$= I + II = \omega(n)\,.$$

Let us split the term I as

$$(4.9)\qquad I = \int_{u_n\le m} + \int_{u_n\ge m} = I^1 + I^2\,.$$

Now

$$I^1 = 0$$

$$I^2 = \int_{u_n\ge m} [a(x,u_n,Du_n) - a(x,u_n,DS_m(u))]D(u_n - S_m(u))_+\Phi'((u_n - S_m(u))_+)$$
$$+ \int_{u_n\ge m} a(x,u_n,DS_m(u))D(u_n - S_m(u))_+\Phi'((u_n - S_m(u))_+)\,.$$

The last term tends to zero as n tends to $+\infty$ by the fact that

$$DS_m(u_n) \rightharpoonup DS_m(u)$$

weakly in $(L^2(\Omega))^N$ while

$$a(x,u_n,DS_m(u))\Phi'((u_n - S_m(u))_+)\chi_{u_n\ge m}\chi_{u>m} \to 0 \text{ strongly in } (L^2(\Omega))^N\,.$$

Therefore, by (4.8), we have

$$(4.10)\qquad \int_{u_n\ge m}[a(x,u_n,Du_n) - a(x,u_n,DS_m(u))]D(u_n - S_m(u))_+\Phi'((u_n - S_m(u))_+) + II$$
$$= H + II = \omega(n)$$

where we have defined

$$(4.11)\qquad H = \int_{u_n\ge m}[a(x,u_n,Du_n) - a(x,u_n,DS_m(u))]D(u_n - S_m(u))_+\Phi'((u_n - S_m(u))_+).$$

We now estimate term II in (4.10)

$$II = -\frac{C_2}{a}\int\limits_{u_n\ge m} a(x,u_n,Du_n)Du_n\Phi((u_n-S_m(u))_+)\beta_n(u_n)$$

$$= -\frac{C_2}{a}\int\limits_{u_n\ge m} [a(x,u_n,Du_n)-a(x,u_n,DS_m(u))]$$

$$D(u_n-S_m(u))\Phi((u_n-S_m(u))_+)\beta_n(u_n)$$

(4.12)

$$-\frac{C_2}{a}\int\limits_{u_n\ge m} a(x,u_n,DS_m(u))D(u_n-S_m(u))\Phi((u_n-S_m(u))_+)\beta_n(u_n)$$

$$= -\frac{C_2}{a}\int\limits_{u_n\ge m} a(x,u_n,Du_n)DS_m(u)\Phi((u_n-S_m(u))_+)\beta_n(u_n)$$

$$= I^1 + II^2 + II^3 .$$

We have

$$II^1 \ge -\tilde{c}(m)\int\limits_{u_n\ge m} [\ ]D(u_n-S_m(u))\Phi((u_n-S_m(u))_+) \ge -\frac{1}{2}H \tag{4.13}$$

where H is defined in (4.11) and

$$\tilde{c}(m) = \max_{s\in[m,k]} \beta_n(s) .$$

Here we used the fact that we can choose λ sufficiently large that

$$\tilde{c}(m)\Phi(s) \le \frac{1}{2}\Phi'(s) .$$

Moreover it is easy to check that

$$II^2 = \omega(n) \tag{4.14}$$

$$|II^3| \le c(m)\left(\int\limits_{\Omega} \Phi^2((S_m(u_n)-S_m(u))_+)|DS_m(u)|^2\right)^{\frac{1}{2}} = \omega(n) \tag{4.15}$$

(here we used estimate (4.4)).

By (4.10)-(4.15), by the fact that $\Phi'(s)\ge 1$ for all $s\ge 0$ and by [8], we have

$$\lim_{n\to+\infty}\int\limits_{\Omega} |D(S_m(u_n)-S_m(u))_+|^2 = 0 \quad \forall\, m>0 .$$

In a similar way, using as test function in (3.1)

$$v = \Phi(-(S_m(u_n) - S_m(u))_-)e^{\gamma_n(S_m(u_n))-\gamma_n(u_n)}$$

we get

$$\lim_{n\to+\infty} \int_\Omega |D(S_m(u_n) - S_m(u))_-|^2 = 0 \quad \forall\, m > 0$$

which concludes the proof of Step 4.

STEP 5. – In this step we prove that for every $C \Subset \Omega$, we have

$$\lim_{m\to 0} \int_{C\cap\{u_n\le m\}} |b_n(x, u_n, Du_n)| = 0 \quad \text{uniformly in } n\,. \tag{4.16}$$

Let us take as test function in (3.1)

$$v_n = -(e^{\gamma_n(m)-\gamma_n(u_n)} - 1)_+\varphi^2 \qquad \varphi \in C_c^\infty(\Omega)\,.$$

By (2.4)

$$\begin{aligned} &-2\int_\Omega a(x,u_n,Du_n)D\varphi\,(e^{\gamma_n(m)-\gamma_n(u_n)} - 1)_+\varphi \\ &+C_2\int_{\{u_n\le m\}} |Du_n|^2\varphi^2 e^{\gamma_n(m)-\gamma_n(u_n)}\beta_n(u_n) - \lambda\int_{\{u_n\le m\}} u_n(e^{\gamma_n(m)-\gamma_n(u_n)} - 1)\varphi^2 \\ &\le \int_{\{u_n\le m\}} |b_n(x,u_n,Du_n)|(e^{\gamma_n(m)-\gamma_n(u_n)} - 1)\varphi^2 \end{aligned}$$

where we used the fact that $f(x) \ge 0$ and the test function is non positive.

Using (2.7), cancelling similar terms, we have, by Holder inequality, observing that in this case $\gamma_n(m) \le \gamma(m)$:

$$\begin{aligned} \int_{\{u_n\le m\}} |b_n(x,u_n,Du_n)|\varphi^2 &\le \lambda\int_{\{u_n\le m\}} u_n e^{\gamma(m)}\varphi^2 \\ &+ c\left(\int_{\{u_n\le m\}} |a(x,u_n,Du_n)|^2\varphi^2\right)^{\frac12}\left(\int_\Omega |D\varphi|^2\varphi^2\right)^{\frac12} = A + B\,. \end{aligned} \tag{4.17}$$

We have

$$A \le \text{cst}\ m\,; \tag{4.18}$$

we are going to prove that we have also

$$B \le \text{cst } m\,. \tag{4.19}$$

Indeed, to see the last inequality, it is sufficient to take $-(u_n - m)_- \varphi^2$, $\varphi \in C_c^\infty(\Omega)$, as test function in (3.1):

$$\begin{aligned}
&\int_{\{u_n \le m\}} a(x,u_n,Du_n)Du_n\varphi^2 - 2\int_{\{u_n \le m\}} a(x,u_n,Du_n)D\varphi\,(u_n - m)_-\varphi \\
&\le \lambda \int_{u_n \le m} u_n(m-u_n)\varphi^2 + \int_{\{u_n \le m\}} |b(x,u_n,Du_n)|(m-u_n)\varphi^2 \\
&\quad + \int_{\{u_n \le m\}} f(u_n - m)\varphi^2\,.
\end{aligned} \tag{4.20}$$

The last integral is non positive, while the second one at the right-hand side is bounded as

$$\int_{\{u_n \le m\}} |b(\,)|(m-u_n)\varphi^2 \le \text{cst } m\,,$$

taking into account (4.5).

The same holds true for the first integral in (4.20) by Step 1. Let us finally estimate:

$$\begin{aligned}
&\int_{\{u_n \le m\}} |a(x,u_n,Du_n)|\,|\varphi|\,|D\varphi|(m-u_n) \\
&\qquad \le m\left(\int_\Omega |D\varphi|^2\right)^{\frac{1}{2}}\left(\int_\Omega |a(x,u_n,Du_n)|^2\varphi^2\right)^{\frac{1}{2}} \\
&\qquad \le \text{cst } m
\end{aligned}$$

by (4.4).

Therefore, by (4.20), $\int_{\{u_n \le m\}} |Du_n|^2\varphi^2 \le \text{cst } m$ for all $n \in \mathbb{N}$, which implies, by (2.5)

$$\int_{\{u_n \le m\}} |a(x,u_n,Du_n)|^2\varphi^2 \le \text{cst } m \quad \forall\, n \in \mathbb{N} \tag{4.21}$$

and, consequently, also (4.19).

Therefore (4.17) and (4.19) imply (4.16), taking $\varphi \equiv 1$ on the compact set C.

STEP 6. – Equintegrability of the sequence $(b_n(x,u_n,Du_n))_{n\in\mathbb{N}}$ on $C \Subset \Omega$.

In this step we are going to prove that for any fixed $C \Subset \Omega$

$$\forall\, \varepsilon > 0, \quad \exists\, \delta_\varepsilon > 0 : \forall E \subseteq C, \quad |E| < \delta_\varepsilon \quad \sup_n \int_E |b_n(x,u_n,Du_n)| < \varepsilon \,. \tag{4.22}$$

Indeed

$$\int_E |b_n(x,u_n,Du_n)| = \int_{E\cap\{u_n \le m\}} |b_n(x,u_n,Du_n)|$$

$$+ \int_{E\cap\{u_n \ge m\}} |b_n(x,u_n,Du_n)| = I_1 + I_2 \,.$$

By (4.16) in Step 5, we have $\lim\limits_{m\to 0} I_1 = 0$ uniformly in n, so that

$$\forall\, \varepsilon > 0 \quad \exists m_0 : \forall\, m \le m_0 \quad I_1 < \frac{\varepsilon}{2} \quad \forall\, n \in \mathbb{N} \,.$$

Therefore, for such an m, due to the strong convergence of $|DS_m(u_n)|$ in $L^2(\Omega)$ (see Step 4), we can choose $|E|$ so small that

$$I_2 \le C_2 \beta(m) \int_E |DS_m(u_n)|^2 < \frac{\varepsilon}{2} \quad \forall\, n \in \mathbb{N} \,.$$

STEP 7. – Passage to the limit.

By (4.22), taking into account the a.e. convergence of $(Du_n)_{n\in\mathbb{N}}$ and of $(u_n)_{n\in\mathbb{N}}$ (up to a subsequence), we have, for any compact set $C \Subset \Omega$

$$b_n(x,u_n,Du_n) \to b(x,u,Du) \quad \text{strongly in}\, L^1(C \cap \{u > 0\}) \,.$$

It remains to prove that

$$b_n(x,u_n,Du_n) \to 0 \quad \text{strongly in}\, L^1(C \cap \{u = 0\}) \,.$$

To this aim, for any $\varepsilon > 0$, we have

$$\int_{C\cap\{u=0\}} |b_n(x,u_n,Du_n)|$$

$$= \int_{C^\varepsilon\cap\{u=0\}} |b_n(x,u_n,Du_n)| + \int_{(C-C^\varepsilon)\cap\{u=0\}} |b_n(x,u_n,Du_n)| = J_1 + J_2,$$

where C^ε is a subset of C (such a subset exists by Egoroff's theorem) such that $|C^\varepsilon| < \delta_\varepsilon$ and in $C - C^\varepsilon$ the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly. Here δ_ε is the number defined in (4.22), and, in fact, by this condition it follows

$$J_1 < \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N},$$

while, for m sufficiently small, $m = m(\varepsilon)$, by (4.16), we get

$$J_2 \leq \int\limits_{C\cap\{u_n \leq m\}} |b_n(x, u_n, Du_n)| < \frac{\varepsilon}{2} \quad \forall n \geq n_0(m(\varepsilon)) = n_0(\varepsilon)\,.$$

So we have proved that

$$\lim_{n\to+\infty} b_n(x, u_n, Du_n) = 0 \text{ strongly in } L^1(C \cap \{u = 0\}),$$

and we can pass to the limit in the term $\int\limits_\Omega b_n(x, u_n, Du_n)\Phi$ in the weak formulation of (3.1).

As far as the term $\int\limits_\Omega a_n(x, u_n, Du_n)D\varphi$ is concerned, we can use (4.21) and Step 4 and repeat exactly the same arguments used for the term $b_n(x, u_n, Du_n)$, obtaining

$$a_n(x, u_n, Du_n) \to a(x, u, Du) \quad \text{strongly in } L^1(C \cap \{u > 0\}),$$

$$a_n(x, u_n, Du_n) \to 0 \quad \text{strongly in } L^1(C \cap \{u = 0\})\,.$$

Therefore we proved the existence of a distributional solution u for (2.1), in the sense that (2.15) holds true.

5. – Proof of Theorem 2.3.

As observed at the beginning of Section 4, the first two steps of the proof of Theorem 2.2 remain valid here, since these steps only use the fact that $|b(x, s, \xi)| \leq \frac{C_2}{|s^k|}|\xi|^2$. Let us emphasize that the main difference in the proof below consists in the fact that we cannot achieve global estimates on the main terms, but only local ones.

STEP 3. – Uniform L^1_{loc}-estimate on $(b_n(x, u_n, Du_n))_{n\in\mathbb{N}}$.

Let $\eta(x) \in C_c^\infty(\Omega)$.

We use $v = (e^{\gamma_n(u_n)} - 1)\eta^2 \in H_0^1(\Omega)$ as test function in (3.1). Let us explicitly point out that we need to use a cut-off function $\eta^2(x)$, since, in this case, by the definition (2.10) of $\gamma_n(s)$, $e^{\gamma_n(u_n)} - 1$ does not vanish on $\partial\Omega$. Note also that $v \leq 0$.

Since $\gamma_n(u_n) \le 0$, we have $\int_\Omega f(e^{\gamma_n(u_n)} - 1)\eta^2 \le 0$, and

$$
\begin{aligned}
&C_2 \int_\Omega \eta^2 |Du_n|^2 e^{\gamma_n(u_n)} \beta_n(u_n) \\
&\le \nu \int_\Omega (1 - e^{\gamma_n(u_n)}) 2|\eta|\,|D\eta|\,|Du_n| + \lambda \int_\Omega u_n(1 - e^{\gamma_n(u_n)})\eta^2 \\
&\quad + \int_\Omega b_n(x, u_n, Du_n)(e^{\gamma_n(u_n)} - 1)\eta^2 + \int_\Omega f(e^{\gamma_n(u_n)} - 1)\eta^2 \\
&\le 2\nu \int_\Omega |\eta|\,|D\eta|\,|Du_n| + \lambda \int_\Omega u_n \eta^2 \\
&\quad + C_2 \int_\Omega \beta_n(u_n)|Du_n|^2 e^{\gamma_n(u_n)} \eta^2 - \int_\Omega b_n(x, u_n, Du_n)\eta^2 \,.
\end{aligned}
\tag{5.23}
$$

We absorb the term $C_2 \int_\Omega \beta_n(u_n)|Du_n|^2 e^{\gamma_n(u_n)} \eta^2$ by the term in the left-hand side of (5.23).

Moreover, for any $\varepsilon > 0$, there exists $C(\varepsilon)$ such that

$$
\begin{aligned}
2\nu \int_\Omega |\eta|\,|D\eta|\,|Du_n| &\le C(\varepsilon) \int_\Omega |D\eta|^2 \\
&\quad + \varepsilon \int_\Omega \eta^2 |Du_n|^2 \le C(\varepsilon) \int_\Omega |D\eta|^2 + \varepsilon M^k \int_\Omega \eta^2 \beta_n(u_n)|Du_n|^2 \,.
\end{aligned}
$$

We have also, by Step 1,

$$
\lambda \int_\Omega u_n \eta^2 \le \mathrm{cst} \quad \forall\, n \in \mathbb{N} \,.
$$

Using the last two estimates in (5.23), recalling (2.8) and taking $\varepsilon = \dfrac{C_1}{2M^k}$, we get

$$
\int_\Omega \beta_n(u_n)|Du_n|^2 \eta^2(x) \le \mathrm{cst} \quad \forall\, n \in \mathbb{N}
\tag{5.24}
$$

which, by (2.8) and the fact that $\beta_n(u_n) \ge C > 0$, gives also, for $\eta \in C_c^\infty(\Omega)$

$$
\int_\Omega b_n(x, u_n, Du_n)\eta^2(x) \le \mathrm{cst} \quad \forall\, n \in \mathbb{N}
\tag{5.25}
$$

$$
\int_\Omega |Du_n|^2 \eta^2(x) \le \mathrm{cst} \quad \forall\, n \in \mathbb{N} \,.
\tag{5.26}
$$

STEP 4. – Weak convergence of $(u_n)_{n\in\mathbb{N}}$ in $H^1_{\mathrm{loc}}(\Omega)$ and strong convergence of $(DS_m(u_n))_{n\in\mathbb{N}}$ in $(L^2_{\mathrm{loc}}(\Omega))^N$ for any fixed $m > 0$.

By the previous steps we deduce that there exists $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } H^1_{\mathrm{loc}}(\Omega)$$
$$u_n \to u \quad \text{a.e. in } \Omega\,.$$

Moreover, with minor modifications, we can prove, as in Step 4 of Theorem 2.2, that

$$\lim_{n\to+\infty} \int_\Omega |D(S_m(u_n) - S_m(u))|^2 \eta^2 = 0 \quad \forall\, \eta \in C_c^\infty(\Omega)\,.$$

To prove this, it is sufficient to take the same test functions used in Theorem 2.2, multiplied by a cut-off function $\eta^2 \in C_c^\infty(\Omega)$ and use the local estimates available from the previous Step 3 (estimates (5.25), (5.26)).

STEP 5 In this step we prove that for every $C \Subset \Omega$, we have

$$\lim_{m\to 0} \int_{C\cap\{u_n\le m\}} b_n(x, u_n, Du_n) = 0 \quad \text{uniformly in } n\,.$$

Note that we cannot use the same arguments of Step 5 in Theorem 2.2 since now we do not have $\gamma(s) \ge 0$ and bounded. We choose as test function in (3.1)

$$v = -(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- \varphi^2 \in H^1_0(\Omega), \quad \varphi \in C_c^\infty(\Omega)\,.$$

By (2.4), noticing that $v \le 0$:

$$\begin{aligned} &-2\int_\Omega a(x, u_n, Du_n)\varphi D\varphi (e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- \\ &+ C_2 \int_{\{u_n\le m\}} |Du_n|^2 \varphi^2 e^{\gamma_n(u_n)-\gamma_n(m)} \beta_n(u_n) \\ \le \lambda &\int_{\{u_n\le m\}} u_n (e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- \varphi^2 \\ &- \int_\Omega b_n(x, u_n, Du_n)(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- \varphi^2\,. \end{aligned}$$

We use condition (2.8) to cancel similar terms; observing that $(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- \le 1$, we have, by Holder inequality

$$\int_{\{u_n \le m\}} b_n(x, u_n, Du_n)\varphi^2 \le \lambda \int_{\{u_n \le m\}} u_n \varphi^2$$

$$+ 2\left(\int_{\{u_n \le m\}} |a_n(x, u_n, Du_n)|^2 \varphi^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} \varphi^2 |D\varphi|^2 \right)^{\frac{1}{2}} \le \text{cst } m\,.$$

The last inequality follows by (4.21) which still holds true.

This concludes the proof of Step 5.

Step 6 and Step 7 can then be achieved in the same way as the corresponding ones in the proof of Theorem 2.2. This concludes the proof of Theorem 2.3.

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