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Danilo Percivale, Paolo Podio-Guidugli

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A General Linear Theory of Elastic Plates and its Variational Validation

DANILO PERCIVALE - PAOLO PODIO-GUIDUGLI

Dedicated to the memory of Guido Stampacchia.

Abstract. – *We provide a variational justification for shearable-plate models that generalize the classic Reissner-Mindlin model. Firstly, we give an argument leading to choose a fairly general linearly elastic monoclinic material response. Secondly, we prove that, for materials in such constitutive class, the variational limit of certain suitably scaled 3D energies is a functional whose minimum over a maximal subspace of admissible functions coincides with the minimum of the generalized Reissner-Mindlin functional.*

1. – Introduction.

All classical theories of thin elastic structures admit a variational formulation, consisting in the minimization of a peculiar energy functional on a peculiar function space. An attractive manner to capture structure thinness is to establish the position of such a 'lower-dimensional' energy functional with respect to the energy functional of some three-dimensional *parent theory* of elasticity. A fashionable analytical approach to achieve this is based on the notion of $\Gamma-$*convergence* [2, 1].

Roughly speaking, given a family of minimum problems $\min_{u \in X_\varepsilon} F_\varepsilon(u)$ with ε a thickness parameter, it may happen that a minimum problem $\min_{u \in X} F_0(u)$ is found, such that minimizers and minima of the problems ruled by the functionals F_ε converge to minimizers and minima of the problem ruled by the functional F_0. To establish the relationship with three-dimensional elasticity of a given structure model through the notion of $\Gamma-$convergence, one has to find a family of three-dimensional functionals that, as $\varepsilon \to 0+$, $\Gamma-$converges to a three-dimensional target functional (see our Theorem 1 below) in tight kinship with the lower-dimensional structure model at hand (our Theorem 2). Thus, $\Gamma-$convergence is basically a method to justify and validate a given structure model, not to deduce it, and even less to propose a new one.

When the parent theory is linear elasticity, a method to arrive at an exahustive list of $\varepsilon-$families and target functionals associated to both two- and onedimensional structure problems has been proposed by Miara and Podio-Guidugli in [6, 7]. Prompted by those results, Paroni, Podio-Guidugli, and Tomassetti, have produced a three-dimensional energy functional that reduces to the Reissner-Mindlin plate functional via thickness integration when evaluated over a Reissner-Mindlin displacement field [8, 9]; and they have proven that this functional is the $\Gamma-$limit of a parametric family of energy functionals of three-dimensional linear elasticity. As suggested in [6, 7], all functionals in the family have two key features in common: (i) the material response they reflect is a constrained type of *transverse isotropy*; (ii) they include a *second-gradient* contribution to the stored-energy density, which proves expedient to recover the Reissner-Mindlin form of the minimizing displacement.

While the mathematical techniques we here use do not differ much from those in [9], our setting does. To begin with, we take time to motivate and construct the energy functionals of both a general parent theory of linear elasticity and the related plate theories; this is a delicate task, carried over in Section 2. The material response our three-dimensional parent theory embodies is a *constrained* type of *monoclinic* response; more importantly, it does *not* involve any second-gradient contribution. Needless to say, the Reissner-Mindlin plate functional is nothing but a special case of the general plate functional we obtain. This may seem surprising; it certainly surprised us: over and above having, as we had, a faultless proof of variational convergence, we wanted to understand the mechanical significance of the corresponding, but different, ingredients in the proof given in [9] and in ours. Here is what we found, in short.

A short premiss helps putting us on the right track. As best exemplified by the classic work of Saint-Venant on the problem that bears his name, an efficient method to solve linear elasticity problems is the *semi-inverse method*, consisting in searching for exact solutions in a class specified by a short list of parameter fields, under assumptions on the data consistent with the parametrization. As one of us advocated since long (see e.g. [11] and the literature cited therein), any such a priori partial representation of candidate solutions may be regarded as the stipulation of certain internal constraints, to be maintained by suitable reactive stress fields. Consider, for example, the standard representation (2.14) of the Reissner-Mindlin displacement field:

$$\boldsymbol{u}_{RM}(x, x_3) = (v_a(x) + x_3\varphi_a(x))\boldsymbol{e}_a + w(x)\boldsymbol{e}_3,$$

and note that, at a point x of the mid cross section of a plate-like body, $\boldsymbol{u}_{RM}$ is linear in the transverse fiber coordinate x_3. This representation – which is quickly shown equivalent to requiring that transverse fibers are inextensible, remain straight, and deform homogeneously – may be seen as the combination of two *internal constraints*:

$$u_{3,3} = 0 \quad \text{and} \quad \boldsymbol{u}_{,33} = \boldsymbol{0},$$

of the first and second order, respectively, both imposed pointwise. Now, in variational elasticity, there are two not mutually exclusive ways to take a constraint into account: either it is made part of the definition of the function space over which the functional to minimize is defined or that functional is penalized by addition of a term that vanishes only if the constraint in question is enforced. In [9], the latter way has been followed to take into account the second-order point constraint implicit in the Reissner-Mindlin representation of the displacement field. In this paper, we follow the former way as well and, at distinct variance with [9], we replace that second-order point constraint with a suitable *first-order constraint imposed fiberwise* (see Subsection 2.3)[1].

Our variational convergence results are expounded in Section 3. The family of functionals F_ε we select and its candidate $\Gamma-$limit F_0 are introduced in Subsection 3.1, where a detailed proof of $\Gamma-$convergence of F_ε to F_0 is given (Theorem 1). In the next subsection we make precise the relation of F_0 to our target, the *shearable-plate functional*, defined to be a generalization of the classic Reissner-Mindlin plate functional consistent with the constitutive theory developed in Subsection 2.1; in particular, we show that F_0 takes the same minimum value as the shearable-plate functional at a displacement field that can be represented in the Reissner-Mindlin form (Theorem 2). Interestingly, if the fiberwise constraint is dropped as we do in Subsection 3.3, a $\Gamma-$convergence result (Theorem 3) can be proven along the same lines of Theorem 1 for a family of functionals J_ε that we regard as more general than family F_ε in that the functionals J_ε do not feature any constraint-related penalization; the minimum of the unconstrained limit functional J_0 corresponding to the family J_ε is strictly greater than the minimum of the shearable-plate functional (unless shearing deformations are constitutively banished, so that the sherable-plate functional reduces to the generalized Kirchhoff-Love functional for unshearable plates). In fact, our last result (Theorem 4) is that the fiberwise-constraint space is the largest subspace of admissible functions over which J_0 has the same minimum as the shearable-plate functional.

2. – The parent theory.

All structure theories should be mathematically simpler than their three-dimensional parent theories: were they not, they would miss their goal, which is

([1]) We believe that fiber constraints are the 'natural' ones to use in the case of plate and shell theories (and that, likewise, cross-section constraints are 'natural' in the case of straight and curved rod theories). There are various reasons to support this views: for one, it can be shown that fiber constraints of higher and higher order are at the roots of the so-called *hierarchical* theories of plates.

to furnish a low-cost fair approximation to the solution of a three-dimensional problem, a solution one chooses not to bother and find. Such relative simplicity results from a number of *ad hoc* assumptions, reflecting certain specific symmetries in the data and including certain crucial kinematical *Ansätze*, that is, certain peculiar representations of the solutions to the original three-dimensional problem. The purpose of this section is to describe the most general assumptions of this sort.

2.1 – *Symmetries built in the data.*

a. Domain. The domain on which the three-dimensional problem is posed has to be *plate-like*, i.e., a right cylinder $\mathcal{C}$ with constant cross section. It is convenient to choose a cartesian frame with origin o and x_a-axes ($a = 1, 2$) in the plane of the middle cross section $\mathcal{P}$, a simply-connected regular region; the x_3-axis is then aligned with the axis of $\mathcal{C}$, with $x_3 \in (-h, h)$, $h > 0$. With the use of the orthonormal unit vectors $\boldsymbol{e}_i\,(i = 1, 2, 3)$, the components of $\boldsymbol{x} := x - o$, the position vector with respect to o of a typical point x of $\mathcal{P}$, are the cartesian coordinates of x itself: $x = o + x_a \boldsymbol{e}_a$; likewise, a typical point p of $\mathcal{C}$, has position vector $\boldsymbol{p} = \boldsymbol{x} + x_3 \boldsymbol{e}_3$; the *transverse fiber* through the point $x \in \mathcal{P}$ is the set of points $\mathcal{F}(x) := \{p \in \mathcal{C}\,|(p - x) \cdot \boldsymbol{e}_3 \in (-h, h)\}$.

b. Material Response. Recall that the response to deformation of a linearly elastic material is described by an *elasticity tensor* $\mathbb{C}$, a symmetric linear transformation of the space of symmetric second-order tensors into itself; as such, $\mathbb{C}$ has at most 21 independent components

$$\mathbb{C}_{ijhk} := \operatorname{sym}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot \mathbb{C}[\operatorname{sym}(\boldsymbol{e}_h \otimes \boldsymbol{e}_k)], \quad \mathbb{C}_{ijhk} = \mathbb{C}_{hkij}, \tag{2.1}$$

a number that becomes smaller and smaller as the material response to deformation becomes less and less *anisotropic* (a *transversely isotropic* material has 5 independent material moduli, an *isotropic* material has 2). When it comes to deriving plate theories, the elasticity tensor of the material of which $\mathcal{C}$ is comprised should agree with the symmetries intrinsic to the geometry of $\mathcal{C}$, in two ways.

Firstly, $\mathbb{C}$ should be such that

$$\mathbb{C}(x, x_3) = \mathbb{C}(x, -x_3) \quad \text{for all } x \in \mathcal{P},\ x_3 \in (-h, h); \tag{2.2}$$

that is to say, the material should be distributed in $\mathcal{C}$ in a mirror-symmetric manner with respect to the plane $x_3 = 0$. Secondly, at all points of $\mathcal{C}$, the material should be *monoclinic with respect to the direction* $\boldsymbol{e}_3$, i.e., for $\boldsymbol{R}$ the mirror reflection about the middle plane and $\boldsymbol{P}$ the orthogonal projection on the axis of $\mathcal{C}$:

$$\boldsymbol{R} := \boldsymbol{1} - 2\boldsymbol{P}, \quad \boldsymbol{P} := \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \tag{2.3}$$

$\mathbb{C}$ should satisfy

$$\mathbb{C}[\boldsymbol{R}\boldsymbol{E}\boldsymbol{R}^T] = \boldsymbol{R}\mathbb{C}[\boldsymbol{E}]\boldsymbol{R}^T \quad \text{for all symmetric second-order tensors } \boldsymbol{E}; \tag{2.4}$$

(2.4) implies that the whole monoclinic class is parameterized by 13 independent material moduli.[2]

REMARK. – The classic plate theories of Kirchhoff-Love and Reissner-Mindlin are derived under the more special assumptions that $\mathbb{C}$ is *constant* and *coherently transversely isotropic* (that is to say, transversely isotropic with respect to the direction $\boldsymbol{e}_3$). Assumptions more general than (2.2) and (2.4) would lead to plate theories unnecessarily complicated for most of applications (but not all, see [4]).

We shall make use of an implication of (2.4) that we now derive. Let

$$\boldsymbol{C}_a := \frac{1}{\sqrt{2}}(\boldsymbol{e}_a \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_a) = \sqrt{2}\,\mathrm{sym}\,(\boldsymbol{e}_a \otimes \boldsymbol{e}_3),$$

and let the space Sym of all symmetric second-order tensors be split in the direct sum of the two-dimensional subspace $\widehat{\mathrm{Sym}} := \mathrm{span}(\boldsymbol{C}_1, \boldsymbol{C}_2)$ and its four-dimensional complement $\overline{\mathrm{Sym}}$:

$$\mathrm{Sym} = \overline{\mathrm{Sym}} \oplus \widehat{\mathrm{Sym}};$$

accordingly, each $\boldsymbol{E} \in \mathrm{Sym}$ is split into two orthogonal addenda:

$$\boldsymbol{E} = \bar{\boldsymbol{E}} + \hat{\boldsymbol{E}}, \quad \bar{\boldsymbol{E}} \in \overline{\mathrm{Sym}}, \quad \hat{\boldsymbol{E}} \in \widehat{\mathrm{Sym}}. \tag{2.5}$$

As is easy to verify,

$$\boldsymbol{R}\boldsymbol{C}_a\boldsymbol{R}^T = -\boldsymbol{C}_a,$$

and, in view of (2.4),

$$\mathbb{C}[\boldsymbol{C}_a]\boldsymbol{R} = -\boldsymbol{R}\mathbb{C}[\boldsymbol{C}_a] \quad \Rightarrow \quad \mathbb{C}[\boldsymbol{C}_a] \in \widehat{\mathrm{Sym}};$$

hence,

$$\mathbb{C}[\hat{\boldsymbol{E}}] \in \widehat{\mathrm{Sym}}. \tag{2.6}$$

Moreover, if $\bar{\boldsymbol{E}} \in \overline{\mathrm{Sym}}$ is split into the orthogonal addenda

$$\bar{\boldsymbol{E}} = \check{\boldsymbol{E}} + E_{33}\boldsymbol{P},$$

([2]) A version of the corresponding general representation of $\mathbb{C}$, that dates back to the pioneering work of Voigt in crystallography, can be found at p. 88 of [3].

then (2.4) implies that both $\mathbb{C}[\check{\boldsymbol{E}}]$ and $\mathbb{C}[\boldsymbol{P}]$ belong to $\overline{\mathrm{Sym}}$, and hence that

$$\mathbb{C}[\bar{\boldsymbol{E}}] \in \overline{\mathrm{Sym}}\,. \tag{2.7}$$

Relations (2.6) and (2.7) yield the following partial representation result for a $\boldsymbol{e}_3-$monoclinic elasticity tensor:

$$\mathbb{C} = \bar{\mathbb{C}} + \hat{\mathbb{C}}, \quad \bar{\mathbb{C}} := \mathbb{C}|_{\overline{\mathrm{Sym}}}\,, \quad \hat{\mathbb{C}} := \mathbb{C}|_{\widehat{\mathrm{Sym}}}\,, \tag{2.8}$$

with

$$\hat{\mathbb{C}} = \frac{1}{2}G_{\alpha\beta}(\boldsymbol{C}_\alpha \otimes \boldsymbol{C}_\beta + \boldsymbol{C}_\beta \otimes \boldsymbol{C}_\alpha), \quad G_{\alpha\beta} = G_{\beta\alpha} =: \boldsymbol{G} \cdot \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta. \tag{2.9}$$

Consequently, the stress $\boldsymbol{S} = \mathbb{C}[\boldsymbol{E}]$ associated with the strain $\boldsymbol{E}$ also consists of two orthogonal parts:

$$\boldsymbol{S} = \bar{\boldsymbol{S}} + \hat{\boldsymbol{S}}, \quad \bar{\boldsymbol{S}} = \bar{\mathbb{C}}[\bar{\boldsymbol{E}}], \quad \hat{\boldsymbol{S}} = \hat{\mathbb{C}}[\hat{\boldsymbol{E}}] = G_{\alpha\beta}(\boldsymbol{C}_\beta \cdot \hat{\boldsymbol{E}})\boldsymbol{C}_\alpha. \tag{2.10}$$

Finally, the stored-energy density per unit reference volume takes the form

$$\sigma(\boldsymbol{E}) := \frac{1}{2}\,\boldsymbol{E} \cdot \mathbb{C}[\boldsymbol{E}] = \bar{\sigma}(\bar{\boldsymbol{E}}) + \hat{\sigma}(\hat{\boldsymbol{E}}), \tag{2.11}$$

with

$$\bar{\sigma}(\bar{\boldsymbol{E}}) = \frac{1}{2}\,\bar{\boldsymbol{E}} \cdot \bar{\mathbb{C}}[\bar{\boldsymbol{E}}], \quad \hat{\sigma}(\hat{\boldsymbol{E}}) = \frac{1}{2}\hat{\boldsymbol{E}} \cdot \hat{\mathbb{C}}[\hat{\boldsymbol{E}}]. \tag{2.12}$$

In linear elasticity, it is standard to require that the stored-energy density be *convex*; in the case of (2.11), this holds true if and only if both $\bar{\sigma}$ and $\hat{\sigma}$ are convex:

$$\bar{\boldsymbol{E}} \cdot \bar{\mathbb{C}}[\bar{\boldsymbol{E}}] \geq \bar{\gamma}\,|\bar{\boldsymbol{E}}|^2 \;\; (\bar{\gamma} > 0) \qquad \hat{\boldsymbol{E}} \cdot \hat{\mathbb{C}}[\hat{\boldsymbol{E}}] \geq \hat{\gamma}\,|\hat{\boldsymbol{E}}|^2 \;\; (\hat{\gamma} > 0). \tag{2.13}$$

A glance to the rightmost side of formula (2.11), in which the energy due to *transverse shearing* is split from the rest, makes clear that a three-dimensional parent theory based on a stored-energy functional with density specified by relations ((2.2), (2.9), and) (2.11)-(2.12) accomodates effortlessly two families of plate theories, that of *shearable plates*, to which the Reissner-Mindlin theory belongs, and that of *unshearable plates*, exemplified by the Kirchhoff-Love theory.[3]

([3]) In particular, the Reissner-Mindlin theory obtains when both $\bar{\mathbb{C}}$ and $\hat{\mathbb{C}}$ satisfy (2.4) over the continuous group of all rotations about an axis parallel to $\boldsymbol{e}_3$, a requirement that reduces the 10 independent components of $\bar{\mathbb{C}}$ to 4 and the 3 independent components of $\hat{\mathbb{C}}$ to 1. The rotations in question may be written in the form

$$\boldsymbol{R} = \mathbf{1} + \sin\varphi(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) - (1 - \cos\varphi)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2), \quad \varphi \in [0, 2\pi).$$

As independent components of $\bar{\mathbb{C}}$ and $\hat{\mathbb{C}}$ one can take, respectively, $\mathbb{C}_{1111}$, $\mathbb{C}_{1122}$, $\mathbb{C}_{1112}$, $\mathbb{C}_{1133}$ and $\mathbb{C}_{1313}$.

2.2 – *Kinematical* Ansätze *and pointwise-powerless internal constraints.*

We already remarked that an essential ingredient of classic plate theories is the kinematical *Ansatz* they begin with, consisting in one or another partial representation of the admissible displacement fields in $\mathcal{C}$. For example, the Reissner-Mindlin theory begins by presuming that the displacement field in a three-dimensional plate-like body can be well approximated by a field of the form

$$\boldsymbol{u}_{RM}(x, x_3) = (v_a(x) + x_3\varphi_a(x))\boldsymbol{e}_a + w(x)\boldsymbol{e}_3, \tag{2.14}$$

while the Kirchhoff-Love theory confines attention to

$$\boldsymbol{u}_{KL}(x, x_3) = (v_a(x) - x_3 w,_a(x))\boldsymbol{e}_a + w(x)\boldsymbol{e}_3; \tag{2.15}$$

the Reissner-Mindlin's *Ansatz* is richer: it features 5 scalar parameter fields over $\mathcal{P}$, while Kirchhoff-Love's has 3 and may be seen as the subcase where one takes

$$\varphi_a = -w,_a\,.$$

Interestingly, (2.15) is the general solution of the following system of first-order PDEs:

$$u_{3,3} = 0 \quad \& \quad u_{a,3} + u_{3,a} = 0, \tag{2.16}$$

in all respects a *standard* ($\equiv$ imposed pointwise in terms of the linear strain measure $\boldsymbol{E}(\boldsymbol{u}) = \operatorname{sym}\nabla\boldsymbol{u}$) *internal constraint*; such constraint precludes both transverse stretching and transverse shearing. Similarly, (2.14) is the general solution of

$$u_{3,3} = 0 \quad \& \quad u_{a,33} = 0, \tag{2.17}$$

again a point constraint – this time nonstandard, due to the presence of the second-order PDEs – imposing that transverse fibers be inextensible and remain straight. In terms of $\boldsymbol{E}$, these constraints read, respectively,

$$\boldsymbol{P}\cdot\boldsymbol{E}(\boldsymbol{u}(x, x_3)) = 0 \quad \& \quad \boldsymbol{C}_a\cdot\boldsymbol{E}(\boldsymbol{u}(x, x_3)) = 0\,, \tag{2.18}$$

and

$$\boldsymbol{P}\cdot\boldsymbol{E}(\boldsymbol{u}(x, x_3)) = 0 \quad \& \quad \boldsymbol{C}_a\cdot\boldsymbol{E},_3(\boldsymbol{u}(x, x_3)) = 0\,. \tag{2.19}$$

Recall now that an internal constraint is generally thought of as maintained by a *nondissipative reactive stress field*. In a first-gradient theory such as standard linear elasticity, the reactive stress $\boldsymbol{S}^{(R)}$ that maintains a given point constraint is whatever field is *pointwise powerless*, i.e., satisfies the orthogonality condition

$$\boldsymbol{S}^{(R)}\cdot\boldsymbol{E} = 0 \tag{2.20}$$

at each point for all admissible strain fields $\boldsymbol{E}$. Accordingly, as its form (2.18) makes evident, the Kirchhoff-Love constraint may be thought of as maintained by a pointwise powerless reactive stress field of the form

$$\boldsymbol{S}^{(R)}(x, x_3) = s_3^{(R)}(x, x_3)\boldsymbol{P} + s_a^{(R)}(x, x_3)\boldsymbol{C}_a \quad \text{in } \mathcal{C}, \tag{2.21}$$

where the constitutively undetermined fields $s_i^{(R)}$ are to be chosen in such a way as to satisfy the balance equations [10]. However, strictly speaking, *a point constraint involving the second gradient of displacement*, such as $(2.17)_2$, *makes sense only if some second (or higher)-gradient theory of elasticity is taken as a parent theory* (see [5] and the literature cited therein); indeed, within such a theory, a formal deduction of the Reissner-Mindlin plate theory can be assembled with little effort [7]. We here adopt a different course of action.

Briefly, we stay within the framework of standard linear elasticity, and we replace the point constraint $(2.17)_2$ by a suitable *fiber constraint*, maintained by reactive stress fields being *fiberwise powerless*, i.e., such as to satisfy the following weak-orthogonality condition:

$$\int_{\mathcal{F}(x)} \boldsymbol{S}^{(R)} \cdot \boldsymbol{E} = 0, \quad \text{at each point } x \in \mathcal{P} \text{ and for all admissible strain fields } \boldsymbol{E}.$$

We discuss the fiber constraint in question in the next subsection.

2.3 – *The fiberwise-powerless constraint for shearable plate theories.*

Recall that we have regarded the form of the in-plane components of the Reissner-Mindlin displacement field (2.14), namely,

$$(\boldsymbol{u}_{RM})_a = v_a(x) + x_3\varphi_a(x), \tag{2.22}$$

as dictated by the imposition of the internal constraint $(2.17)_2$ *at each point* $p = x + x_3\boldsymbol{e}_3$ *of* $\mathcal{C}$. We now introduce an internal constraint which, if required to hold *at each point* x *of* $\mathcal{P}$ (that is to say, *fiberwise*), has the same mathematical consequences as far as variational convergence is concerned (we shall prove that this is the case in Section 3.1).

For $u \in H^1(\mathcal{C})$, consider for a.e. $x \in \mathcal{P}$ the following integral condition:

$$\int_{-h}^{+h} (3h^{-1}x_3 u - h u_{,3})\, dx_3 = 0. \tag{2.23}$$

It is the matter of a simple calculation to show that condition (2.23) is equivalent to

$$\int_{-h}^{+h} (1 - 3h^{-2}{x_3}^2) u_{,3}\, dx_3 = 0 \tag{2.24}$$

as well as, granted regularity, to

$$\int_{-h}^{+h} (x_3 - h^{-3}x_3^3)u_{,33}dx_3 = 0. \tag{2.25}$$

Clearly, all functions u which are even with respect to x_3 satisfy (2.23); the odd function

$$v(x, x_3) = b(x)x_3,$$

also does, whatever the function $b \in H^1(\mathcal{P})$. Thus,

$$f(t) = a(x) + b(x)x_3 \quad (a, b \in H^1(\mathcal{P})),$$

satisfies (2.23); needless to say, such a u is the general solution of

$$u_{,33} = 0,$$

the differential equation corresponding to the second-gradient constraint $(2.17)_2$. These are the reasons why we regard an a priori condition of the form (2.23) on each in-plane component of the displacement field as an internal constraint, weaker than if it had the representation (2.22), but of the same nature; and we regard (2.24) as a characterization of the corresponding fiberwise powerless reactive stress fields that, on taking into account $(2.17)_1$, can be written as

$$\int_{\mathcal{F}(x)} \boldsymbol{S}^{(R)} \cdot \boldsymbol{E}\, dx_3 = 0, \quad \text{with} \quad \boldsymbol{S}^{(R)}(x, x_3) = (1 - 3x_3^2)\boldsymbol{C}_a\,.$$

One may ask what *odd* functions satisfy (2.23). We offer the following characterization.

PROPOSITION. – *Let $w \in H^1(\mathcal{C})$ such that*

$$w(x, x_3) = w(x, -x_3), \quad \int_0^h w(x, x_3)dx_3 = 0 \tag{2.26}$$

for a.e. $x \in \mathcal{P}$. Then, the function $v \in H^1(\mathcal{C})$ defined by

$$v(x, x_3) = \frac{1}{h}e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2} \int_0^{x_3} w(x, s)e^{-\frac{3}{2}\left(\frac{s}{h}\right)^2}\, ds \tag{2.27}$$

is odd with respect to x_3 and satisfies condition (2.23). Conversely, if $v \in H^1(\mathcal{C})$ is odd with respect to x_3 for a.e. $x \in \mathcal{P}$ and satisfies (2.23), then there is $w \in H^1(\mathcal{C})$ satisfying condition (2.26) such that (2.27) holds.

PROOF. – That v in (2.27) is odd, if w satisfies (2.26) , may be verified by direct inspection; in particular, then, $v(x,0)=0$ for a.e. $x\in\mathcal{P}$. Formula (2.27) is nothing but a representation of the unique solution, for a.e. $x\in\mathcal{P}$, of the following Cauchy problem in $[0,h]$:

$$\begin{cases} hv_{,3}-3h^{-1}x_3v=w, \\ v(x,0)=0. \end{cases} \tag{2.28}$$

To prove the converse assertion, it is enough to take w as defined by $(2.28)_1$, for v odd with respect to x_3 and compliant with (2.23).

COROLLARY. – *For every $u\in H^1(\mathcal{C})$ there exist $v\in H^1(\mathcal{C})$ satisfying* (2.23) *and $\zeta\in H^1(\mathcal{P})$ such that*

$$u(x,x_3)=v(x,x_3)+\zeta(x)m_h(x_3), \tag{2.29}$$

where

$$m_h(x_3)=\frac{1}{h}e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2}\int_0^{x_3}e^{-\frac{3}{2}\left(\frac{s}{h}\right)^2}ds. \tag{2.30}$$

PROOF. – If $u\in H^1(\mathcal{C})$, let u_{odd} be its odd part with respect to x_3, and u_e its even one. Then, on setting

$$v(x,x_3)=h\,e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2}\frac{\partial}{\partial x_3}\left(e^{-\frac{3}{2}\left(\frac{x_3}{h}\right)^2}u_{odd}\right),$$

a direct inspection shows that

$$u_{odd}(x,x_3)=\frac{1}{h}e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2}\int_0^{x_3}v(x,s)e^{-\frac{3}{2}\left(\frac{s}{h}\right)^2},ds$$

and hence

$$\begin{aligned} u_{odd}(x,x_3) &= \frac{1}{h}e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2}\int_0^{x_3}\left(v(x,s)-\int_0^h v(x,\tau)d\tau\right)e^{-\frac{3}{2}\left(\frac{s}{h}\right)^2}ds \\ &\quad +\frac{1}{h}e^{\frac{3}{2}\left(\frac{x_3}{h}\right)^2}\int_0^{x_3}e^{-\frac{3}{2}\left(\frac{s}{h}\right)^2}ds\int_0^h v(x,\tau)d\tau\,. \end{aligned}$$

Relation (2.29) is recovered by taking

$$v(x,x_3)=u_e(x,x_3)+e^{\frac{3}{2}t^2}\int_0^{x_3}\left(v(x,s)-\int_0^h v(x,\tau)d\tau\right)e^{-\frac{3}{2}s^2}\,ds$$

and

$$\zeta(x) = \int_0^h \nu(x, \tau) d\tau.$$

3. – The variational validation.

To set our three-dimensional stage, we identify our three-dimensional domain, the cylinder $\mathcal{C}$, with the cartesian product $\mathcal{P} \times (-h, h)$, we assume that the two-dimensional flat domain $\mathcal{P}$ has a Lipschitz boundary $\partial\mathcal{P}$, and we introduce the function space:

$$\mathcal{V} = \{\boldsymbol{v} \in H^1(\mathcal{C}) : \boldsymbol{v} = \boldsymbol{0} \text{ in } \partial\mathcal{P} \times (-h, h)\};$$

next, given $\boldsymbol{f} \in L^2(\mathcal{C})$, we consider the variational problem:

$$\min_{\boldsymbol{v}\in\mathcal{V}} F(\boldsymbol{v}), \quad F(\boldsymbol{v}) := \int_{\mathcal{C}} \sigma(\boldsymbol{E}(\boldsymbol{v}))\, dx dx_3 - \int_{\mathcal{C}} \boldsymbol{f} \cdot \boldsymbol{v}\, dx dx_3, \tag{3.1}$$

for σ the *monoclinic* stored-energy density (2.11) discussed in Subsection 2.1. For brevity, here and henceforth we write $H^1(\mathcal{C})$ for $H^1(\mathcal{C}; \mathbb{R}^3)$, $L^2(\mathcal{C})$ for $L^2(\mathcal{C}; \mathbb{R}^3)$ *et sim.*; we also denote by dx the integration measure over $\mathcal{P}$.

With a view toward constructing pairs consisting of a continuous family of functionals of this type and its Γ–limit, we recall, again from Subsection 2.1, that any $\boldsymbol{E} \in \mathrm{Sym}$ can be split as follows:

$$\boldsymbol{E} = \bar{\boldsymbol{E}} + \hat{\boldsymbol{E}},$$

with

$$\begin{cases} \bar{\boldsymbol{E}} = \check{\boldsymbol{E}} + E_{33}\boldsymbol{P}, \\ \check{\boldsymbol{E}} = E_{11}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + E_{12}\, (\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + E_{22}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \\ \hat{\boldsymbol{E}} = \sqrt{2}\, E_{\alpha 3} \boldsymbol{C}_\alpha, \end{cases} \tag{3.2}$$

and that, consistently, σ can be split into two convex addenda:

$$\sigma(\boldsymbol{E}) = \bar{\sigma}(\bar{\boldsymbol{E}}) + \hat{\sigma}(\hat{\boldsymbol{E}}).$$

Moreover, we introduce here for later use another monoclinic and convex quadratic form in $\boldsymbol{E}$, namely, the *relaxed* stored energy density

$$\sigma^*(\boldsymbol{E}) := \min_{\psi\in\mathbb{R}} \sigma(\boldsymbol{E} + \psi\boldsymbol{P}) = \bar{\sigma}^*(\check{\boldsymbol{E}}) + \hat{\sigma}(\hat{\boldsymbol{E}}), \quad \bar{\sigma}^*(\check{\boldsymbol{E}}) := \min_{\psi\in\mathbb{R}} \bar{\sigma}(\check{\boldsymbol{E}} + \psi\boldsymbol{P}). \tag{3.3}$$ [4]

(4) It can be shown that $\bar{\sigma}^*$ satisfies the following relation:

$$\bar{\sigma}^* = \sigma|_{\mathcal{A}}, \quad \mathcal{A} := \{\boldsymbol{E} \in \mathrm{Sym} \,|\, \mathbb{C}[\boldsymbol{E}] \cdot \boldsymbol{P} = 0\}$$

(see [9] for a discussion of the nonstandard internal constraint that enters the definition of $\mathcal{A}$).

It is to be noticed that

$$F(\boldsymbol{v}) + \int_{\mathcal{C}} \boldsymbol{f} \cdot \boldsymbol{v}\, dx dx_3$$
$$= \int_{\mathcal{C}} \Big(\bar{\sigma}\big(\check{\boldsymbol{E}}(\boldsymbol{v})\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{v})\big) \Big) dx dx_3 =: \Sigma(\boldsymbol{v}) \geq \Sigma^*(\boldsymbol{v}) := \int_{\mathcal{C}} \Big(\bar{\sigma}^*\big(\check{\boldsymbol{E}}(\boldsymbol{v})\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{v})\big) \Big) dx dx_3.$$

3.1 – *A family of functionals and its Γ–limit.*

For each fixed $\varepsilon \in (0,1]$, we set

$$x_3 = (\varepsilon h)t, \quad t \in (-1,1)\,,$$

so that

$$(x, x_3) \in \mathcal{C}_\varepsilon := \mathcal{P} \times (-\varepsilon h, \varepsilon h), \quad (x,t) \in \mathcal{D} := \mathcal{P} \times (-1,1);$$

and we scale the data $\boldsymbol{f}$ and σ as follows:

$$\begin{aligned} &\boldsymbol{f}_\varepsilon(x, x_3) = x_3\, g_a(x)\boldsymbol{e}_a + (\varepsilon h)^2 g_3(x)\boldsymbol{e}_3, \quad g_a, g_3 \in L^2(\mathcal{P}); \\ &\sigma_\varepsilon(\boldsymbol{E}) = \bar{\sigma}(\bar{\boldsymbol{E}}) + \varepsilon^2 \hat{\sigma}(\hat{\boldsymbol{E}}). \end{aligned} \tag{3.4}$$

Next, we introduce the function spaces:

$$\mathcal{V}_\varepsilon := \{\boldsymbol{v} \in H^1(\mathcal{C}_\varepsilon) : \boldsymbol{v} = \boldsymbol{0} \text{ in } \partial\mathcal{P} \times (-\varepsilon h, \varepsilon h)\},$$
$$\mathcal{U} := \{\boldsymbol{u} \in H^1(\mathcal{D}) : \boldsymbol{u} = \boldsymbol{0} \text{ in } \partial\mathcal{P} \times (-1,1)\},$$

whose typical elements $\boldsymbol{v}$ and $\boldsymbol{u}$ we mutually scale as follows:

$$\boldsymbol{v}(x, x_3) = \varepsilon h\, u_a(x,t)\boldsymbol{e}_a + u_t(x,t)\boldsymbol{e}_3; \tag{3.5}$$

and we provisionally consider the following ε–family of functionals G_ε defined over $\mathcal{V}_\varepsilon$:

$$\begin{aligned} G_\varepsilon(\boldsymbol{v}) :=\ &\varepsilon^{-3} \Bigg(\frac{1}{h} \int_{\mathcal{C}_\varepsilon} \sigma_\varepsilon(\boldsymbol{E}(\boldsymbol{v}))\, dx dx_3 - \frac{1}{h^3} \int_{\mathcal{C}_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{v}\, dx dx_3 \Bigg) \\ &+ \varepsilon^{-4} \sum_{a=1}^{2} \int_{\mathcal{P}} \Bigg(\frac{1}{h} \int_{-\varepsilon h}^{\varepsilon h} (1 - 3(\varepsilon h)^{-2} {x_3}^2) E_{a3}(\boldsymbol{v}) dx_3 \Bigg)^2 dx. \end{aligned} \tag{3.6}$$

We now proceed to show how this definition induces the definition of the family F_ε of functionals over $\mathcal{U}$ we make the object of our Γ–convergence theorem below.

We have from $(3.4)_1$ and (3.5) that

$$\boldsymbol{f}_\varepsilon \cdot \boldsymbol{v} = (\varepsilon h)^2 (t g_a u_a + g_3 u_t)\,. \tag{3.7}$$

We also have from (3.5) that

$$v_{a,\beta} = (\varepsilon h) u_{a,\beta}\,, \quad v_{a,3} = u_{a,t}\,, \quad v_{3,a} = u_{t,a}\,, \quad v_{3,3} = (\varepsilon h)^{-1} u_{t,t}\,;$$

hence, by $(3.2)_2$,

$$E_{a\beta}(\boldsymbol{v}) = (\varepsilon h) E_{a\beta}(\boldsymbol{u}), \quad E_{a3}(\boldsymbol{v}) = E_{at}(\boldsymbol{u}), \quad E_{33}(\boldsymbol{v}) = (\varepsilon h)^{-1} E_{tt}(\boldsymbol{u}),$$

or rather

$$\bar{\boldsymbol{E}}(\boldsymbol{v}) = (\varepsilon h)\check{\boldsymbol{E}}(\boldsymbol{u}) + (\varepsilon h)^{-1} E_{tt}(\boldsymbol{u})\boldsymbol{P}\big), \quad \hat{\boldsymbol{E}}(\boldsymbol{v}) = \hat{\boldsymbol{E}}(\boldsymbol{u}).$$

Thus, on taking into account the fact that $\bar{\sigma}$ is homogeneous of degree 2, $(3.4)_2$ becomes:

$$\sigma_\varepsilon(\boldsymbol{E}(\boldsymbol{v})) = \varepsilon^2 \Big(\bar{\sigma}\big(h\check{\boldsymbol{E}}(\boldsymbol{u}) + \varepsilon^{-2} h^{-1} E_{tt}(\boldsymbol{u})\boldsymbol{P}\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{u})\big)\Big). \tag{3.8}$$

In conclusion, in view of (3.7) and (3.8), we have from (3.6) that

$$\begin{aligned} G_\varepsilon(\boldsymbol{v}) = & \int_{\mathcal{D}} \Big(\bar{\sigma}(h\check{\boldsymbol{E}}(\boldsymbol{u}) + \varepsilon^{-2} h^{-1} E_{tt}(\boldsymbol{u})\boldsymbol{P}) + \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\Big) dxdt \\ & - \int_{\mathcal{D}} (t g_a u_a + g_3 u_3)\, dxdt + \varepsilon^{-2} \sum_{a=1}^{2} \int_{\mathcal{P}} \left(\int_{-1}^{1} (1 - 3t^2) E_{at}(\boldsymbol{u}) dt \right)^2 d\boldsymbol{x} =: F_\varepsilon(\boldsymbol{u}). \end{aligned} \tag{3.9}$$

With this definition, we are in a position to formulate our main result.

THEOREM 1. – *Let F_ε be the family of functionals over $\mathcal{U}$ defined in* (3.9); *moreover, let*

$$\mathcal{U}_0 := \left\{ \boldsymbol{u} \in \mathcal{U} : u_t = u_t(x) \quad \& \quad \int_{-1}^{1} (3t u_a - u_{a,t})\, dt = 0 \right\}$$

and let

$$F_0(\boldsymbol{u}) = \begin{cases} \displaystyle\int_{\mathcal{D}} \Big(\bar{\sigma}^*\big(h\check{\boldsymbol{E}}(\boldsymbol{u})\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{u})\big)\Big)\, dxdt - \int_{\mathcal{D}} (t g_a u_a + g_3 u_t)\, dxdt & \text{if } \boldsymbol{u} \in \mathcal{U}_0, \\ +\infty \quad \text{otherwise in } H^1(\mathcal{D}). \end{cases}$$

Then,

$$\Gamma_{w-H^1} - \lim F_\varepsilon(\boldsymbol{u}) = F_0(\boldsymbol{u}). \tag{3.10}$$

PROOF. – (*Lower-bound inequality*) Let $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in $w - H^1$. Then, either $\liminf F_\varepsilon(\boldsymbol{u}_\varepsilon) = +\infty$ or $F_\varepsilon(\boldsymbol{u}_\varepsilon) \leq C$ (here and henceforth C denotes a positive constant, whose value may vary from case to case). In the former instance, there is nothing to prove; in the latter, due to the definition of F_ε and to the convexity of $\bar\sigma$, we have that

$$\text{both}\quad \int_{\mathcal{D}} |E_{3t}(\boldsymbol{u}_\varepsilon)|^2 dx dt \le C(\varepsilon h)^4 \quad \text{and} \quad \int_{\mathcal{P}} \left(\int_{-1}^{1} (1-3t^2) E_{at}(\boldsymbol{u}) dt \right)^2 dx \le C(\varepsilon h)^2,$$

whence, by letting $\varepsilon \to 0$ and by appealing to the lower semicontinuity of the L^2-norm with respect to the weak topology, we deduce that

$$\begin{cases} E_{tt}(\boldsymbol{u}) = 0 \ \text{ a.e. in } \mathcal{D}, \\ \displaystyle\int_{-1}^{1} (1-3t^2) E_{at}(\boldsymbol{u}) dt = 0 \ \ (a = 1,2) \ \text{ a.e. in } \mathcal{P}. \end{cases} \tag{3.11}$$

Then, not only the first of (3.11) implies that

$$u_t = u_t(x), \tag{3.12}$$

but also, given that

$$\int_{-1}^{1} (1-3t^2)\, dt = 0, \tag{3.13}$$

integration by parts of the second yields:

$$\boldsymbol{u} \in \mathcal{U}_0.^5 \tag{3.14}$$

On the other hand, in view of a definitional property of the relaxed stored energy,

$$F_\varepsilon(\boldsymbol{u}_\varepsilon) \ge \int_{\mathcal{D}} \Big(\bar\sigma^*(h\check{\boldsymbol{E}}(\boldsymbol{u}_\varepsilon)) + \hat\sigma(E_{at}(\boldsymbol{u}_\varepsilon)\boldsymbol{C}_a) \Big)\, dx dt - \int_{\mathcal{D}} (t g_a(\boldsymbol{u}_\varepsilon)_a + g_3(\boldsymbol{u}_\varepsilon)_3)\, dx dt\,.$$

Given that all stored-energy densities involved are convex, we conclude that the following lower-bound inequality holds true:

$$\liminf F_\varepsilon(\boldsymbol{u}_\varepsilon) \ge F_0(\boldsymbol{u}). \tag{3.15}$$

(*Recovery sequence*) We have to prove that, for every $\boldsymbol{u} \in H^1(\mathcal{D})$, there is a sequence $\boldsymbol{u}_\varepsilon \in H^1(\mathcal{D})$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in $w - H^1$ and that, moreover,

$$\limsup F_\varepsilon(\boldsymbol{u}_\varepsilon) \le F_0(\boldsymbol{u}). \tag{3.16}$$

([5]) To reach this conclusion, it is enough to make use of (3.12) to verify that $0 = \int_{-1}^{+1} (1-3t^2) E_{at}(\boldsymbol{u}) dt = 2 \int_{-1}^{+1} (1-3t^2) u_{a,t} dt = \int_{-1}^{+1} (3t u_a - u_{a,t}) dt\,.$

We begin by assuming that $\boldsymbol{u} \in C^2(\bar{\mathcal{D}}) \cap \mathcal{U}_0$. Then, there is a unique $\psi \in C^1(\bar{\mathcal{D}})$ such that

$$\bar{\sigma}(\check{\boldsymbol{E}}(\boldsymbol{u}) + \psi \boldsymbol{P}) = \bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{u})). \tag{3.17}$$

For

$$\Psi(x,t) = \int_0^t \psi(x,\tau)\, d\tau,$$

and for

$$\vartheta_\varepsilon \in C_0^1(\mathcal{P}), \quad 0 \le \vartheta_\varepsilon \le 1, \quad \vartheta_\varepsilon(x) \equiv 1 \;\text{ if }\; \operatorname{dist}(x, \partial\mathcal{P}) \ge \sqrt{\varepsilon h}\,,$$

we choose

$$\boldsymbol{u}_\varepsilon(x,t) = u_a(x,t)\boldsymbol{e}_a + \big(u_3(x) + \varepsilon^2 h\, \vartheta_\varepsilon(x)\Psi(x,t)\big)\boldsymbol{e}_3 \tag{3.18}$$

which, by recalling that $\psi \in C^1(\bar{\mathcal{D}})$, belongs to $C^1(\bar{\mathcal{D}})$. Consequently, we have that

$$\begin{aligned} \bar{\sigma}\big(\bar{\boldsymbol{E}}(\boldsymbol{u}_\varepsilon)) &= \bar{\sigma}(\check{\boldsymbol{E}}(\boldsymbol{u}) + \varepsilon\vartheta_\varepsilon\psi \boldsymbol{P}\big), \\ \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}_\varepsilon)) &= \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{u}) + \frac{1}{\sqrt{2}}\varepsilon^2 h(\vartheta_\varepsilon\Psi)_{,a}\, \boldsymbol{C}_a\big). \end{aligned} \tag{3.19}$$

By (3.19), we have that $\sigma(\boldsymbol{E}(\boldsymbol{u}_\varepsilon)) \to \sigma(\boldsymbol{E}(\boldsymbol{u}))$ a.e. in $\mathcal{D}$; moreover, given that σ is a quadratic form, there exists $\Lambda > 0$ such that

$$\sigma(\boldsymbol{E}(\boldsymbol{u}_\varepsilon)) \le \Lambda \|\boldsymbol{E}(\boldsymbol{u}_\varepsilon)\|^2; \tag{3.20}$$

therefore,

$$\int_{\mathcal{D}} \sigma(\boldsymbol{E}(\boldsymbol{u}_\varepsilon))\, dxdt \quad \to \int_{\mathcal{D}} \Big(\bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{u})) + \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\Big) dxdt. \tag{3.21}$$

As a further consequence of (3.18), by recalling (3.13) and by noticing that $|\vartheta_{\varepsilon,a}(x)| \le C/\sqrt{\varepsilon h}$, we get:

$$\int_{-1}^{1} (1-3t^2)E_{at}(\boldsymbol{u}_\varepsilon)dt = \frac{1}{2}\int_{-1}^{1} (1-3t^2)\varepsilon^2 h(\vartheta_\varepsilon\Psi)_{,a}\, dt, \tag{3.22}$$

whence

$$\int_{\mathcal{P}} \bigg(\int_{-1}^{1} (1-3t^2)E_{at}(\boldsymbol{u}_\varepsilon)dt\bigg)^2 dx \le C'\varepsilon^3 h \int_{\mathcal{D}} |\nabla\Psi|^2. \tag{3.23}$$

Then, with the use of (3.21) and (3.23), we easily get:

$$\limsup F_\varepsilon(\boldsymbol{u}_\varepsilon) \le F_0(\boldsymbol{u}). \tag{3.24}$$

By observing that F_0 is continuous with respect to the strong $H^1(\mathcal{D})$ topology, a standard density argument shows that, for every $\boldsymbol{u} \in H^1(\mathcal{D})$, there is a sequence $\bar{\boldsymbol{u}}_\varepsilon$ such that $\bar{\boldsymbol{u}}_\varepsilon \to \boldsymbol{u}$ in $w - H^1$ and that, moreover,

$$\limsup F_\varepsilon(\boldsymbol{u}_\varepsilon) \leq F_0(\boldsymbol{u}). \tag{3.25}$$

This concludes the proof.

3.2 – *The relationship between the Γ–limit and the Reissner-Mindlin functional.*

The equilibria of a Reissner-Mindlin plate are determined by minimizing an energy functional whose quadratic principal part is

$$\Sigma_{RM}(\boldsymbol{\varphi}, w) = \frac{1}{2}\int_{\mathcal{P}}\Big(S|\boldsymbol{\varphi} + \nabla w|^2 + B((1-\nu)|\boldsymbol{E}(\boldsymbol{\varphi})|^2 + \nu(\mathrm{div}\boldsymbol{\varphi})^2)\Big)\,dx; \tag{3.26}$$

$S = S_0\, h$, $B = B_0\, h^3$ and ν are, respectively, the *shearing stiffness*, the *bending stiffness* and the *lateral contraction modulus*; the constitutive parameters S_0, B_0, and ν are chosen so as to guarantee positivity of the integrand of (3.26).

In order to show the link between functional F_0 and shearable plate theories more general than Reissner-Mindlin's, we introduce the *shearable-plate functional*

$$\mathcal{RM}(\boldsymbol{\varphi}, w) := \int_{\mathcal{P}}\left(\frac{2}{3}\bar{\sigma}^*(h\check{\boldsymbol{E}}(\boldsymbol{\varphi})) + 2\,\hat{\sigma}\left(\frac{1}{\sqrt{2}}(\varphi_a + w_{,a})\boldsymbol{C}_a\right)\right)dx - \int_{\mathcal{P}}\left(\frac{2}{3}g_a\varphi_a + 2\,g_3 w\right)dx,$$

defined over the space for $H^1_0(\mathcal{P}; \mathbb{R}^2 \times \mathbb{R})$; for each admissible pair for $(\boldsymbol{\varphi}, w)$,

$$F_0(t\boldsymbol{\varphi} + w\boldsymbol{e}_3) = \mathcal{RM}(\boldsymbol{\varphi}, w), \tag{3.27}$$

an observation crucial to the proof of the following theorem.

THEOREM 2. – *Let $\boldsymbol{u} \in \mathrm{argmin}\, F_0$. Then, there exists $\overline{\boldsymbol{\varphi}} \in H^1_0(\mathcal{P}; \mathbb{R}^2)$ such that both $\boldsymbol{u} = t\overline{\boldsymbol{\varphi}} + \overline{w}\boldsymbol{e}_3$ and $(\overline{\boldsymbol{\varphi}}, \overline{w}) \in \mathrm{argmin}\, \mathcal{RM}$; moreover, $\mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}) = F_0(\boldsymbol{u})$.*

PROOF. – Let $\boldsymbol{u} = \bar{\boldsymbol{u}} + \overline{w}\boldsymbol{e}_3 \in \mathrm{argmin}\, F_0$, with $\bar{\boldsymbol{u}} = \bar{u}_a\boldsymbol{e}_a$. Set

$$\overline{\boldsymbol{\varphi}} = \frac{3}{2}\int_{-1}^{1} t\bar{\boldsymbol{u}}\,dt \quad \text{and} \quad \boldsymbol{z} = t\overline{\boldsymbol{\varphi}} + \overline{w}\boldsymbol{e}_3,$$

so that

$$\check{\boldsymbol{E}}(\boldsymbol{z}) = t\check{\boldsymbol{E}}(\overline{\boldsymbol{\varphi}}), \quad \hat{\boldsymbol{E}}(\boldsymbol{z}) = \frac{1}{\sqrt{2}}(\overline{w}_{,a} + \overline{\varphi}_a)\boldsymbol{C}_a;$$

$$\check{\boldsymbol{E}}(\boldsymbol{u}) - \check{\boldsymbol{E}}(\boldsymbol{z}) = \check{\boldsymbol{E}}(\boldsymbol{u} - \boldsymbol{z}) = \check{\boldsymbol{E}}(\bar{\boldsymbol{u}} - t\overline{\boldsymbol{\varphi}}), \quad \hat{\boldsymbol{E}}(\bar{\boldsymbol{u}} - t\overline{\boldsymbol{\varphi}}) = \frac{1}{\sqrt{2}}(\bar{u}_{a,t} - \overline{\varphi}_a)\boldsymbol{C}_a\,.$$

Note that

$$F_0(\boldsymbol{u}) - F_0(\boldsymbol{z}) = \int_{\mathcal{D}} \Big(\bar{\sigma}^*\big(h\check{\boldsymbol{E}}(\boldsymbol{u})\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{u})\big)\Big)\, dxdt - \int_{\mathcal{D}} \Big(\bar{\sigma}^*\big(h\check{\boldsymbol{E}}(\boldsymbol{z})\big) + \hat{\sigma}\big(\hat{\boldsymbol{E}}(\boldsymbol{z})\big)\Big)\, dxdt.$$

Then, by convexity,

$$F_0(\boldsymbol{u}) \geq F_0(\boldsymbol{z}) + \int_{\mathcal{D}} D\bar{\sigma}^*(h\check{\boldsymbol{E}}(\boldsymbol{z}))\cdot\check{\boldsymbol{E}}(\boldsymbol{u}-\boldsymbol{z})dxdt + \int_{\mathcal{D}} D\hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{z}))\cdot\hat{\boldsymbol{E}}(\boldsymbol{u}-\boldsymbol{z})dxdt$$

$$= F_0(\boldsymbol{z}) + \int_{\mathcal{P}} \left(D\bar{\sigma}^*(h\check{\boldsymbol{E}}(\overline{\boldsymbol{\varphi}}))\cdot\check{\boldsymbol{E}}\left(\int_{-1}^{1} t\bar{\boldsymbol{u}}\, dt - \frac{2}{3}\,\overline{\boldsymbol{\varphi}}\right)\right) dx$$

$$+ \frac{1}{\sqrt{2}}\boldsymbol{C}_a \cdot \int_{\mathcal{P}} \left(\int_{-1}^{1} \bar{u}_{a,t}\, dt - 2\overline{\varphi}_a \right) D\hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{z}))dx = F_0(\boldsymbol{z})$$

(here $D\bar{\sigma}^*$ denotes differentiation of the mapping $\bar{\sigma}^*$ with respect to its tensor argument, and similarly for $D\hat{\sigma}$; the last addendum vanishes because

$$\int_{-1}^{1} \bar{u}_{a,t}\, dt - 2\overline{\varphi}_a = \int_{-1}^{1} (\bar{u}_{a,t} - 3t\,\bar{u}_a)dt$$

and $\boldsymbol{u} \in \mathcal{U}$). Taking into account (3.27) and the fact that $\boldsymbol{u}$ is a minimizer, we have:

$$\mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}) = F_0(\boldsymbol{z}) \geq F_0(\boldsymbol{u});$$

this completes the proof.

3.3 – *A more general family of functionals and its Γ-limit.*

For every $\boldsymbol{v} \in \mathcal{V}_\varepsilon$, we define

$$K_\varepsilon(\boldsymbol{v}) = \varepsilon^{-3}\left(\frac{1}{h}\int_{\mathcal{C}_\varepsilon} \sigma_\varepsilon(\boldsymbol{E}(\boldsymbol{v}))\, dxdx_3 - \frac{1}{h^3}\int_{\mathcal{C}_\varepsilon} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{v} dxdx_3\right), \tag{3.28}$$

and we note that the scalings (3.4)-(3.5) yield:

$$\begin{aligned} K_\varepsilon(\boldsymbol{v}) = &\int_{\mathcal{D}} \Big(\bar{\sigma}(h\check{\boldsymbol{E}}(\boldsymbol{u}) + \varepsilon^{-2}h^{-1}E_{tt}(\boldsymbol{u})\boldsymbol{P}) + \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\Big)dxdt \\ &- \int_{\mathcal{D}} (tg_a u_a + g_3 u_t)\, dxdt =: J_\varepsilon(\boldsymbol{u}). \end{aligned} \tag{3.29}$$

THEOREM 3. – *J_ε be the family of functionals over $\mathcal{U}$ defined in* (3.29); *moreover, let*

$$\widetilde{\mathcal{U}_0} = \{\boldsymbol{u} \in \mathcal{U} : u_t = u_t(x)\} \tag{3.30}$$

and let

$$J_0(\boldsymbol{u}) = \begin{cases} \int\limits_{\mathcal{D}} \Big(\bar{\sigma}^*(h\check{\boldsymbol{E}}(\boldsymbol{u})) + \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\Big)\,dxdt - \int\limits_{\mathcal{D}} (tg_a u_a + g_3 u_t)\,dxdt & \textit{if } \boldsymbol{u} \in \widetilde{\mathcal{U}_0}, \\ +\infty \qquad \textit{otherwise in } H^1(\mathcal{D}). \end{cases}$$

Then,

$$\Gamma_{w-H^1} \lim J_\varepsilon(\boldsymbol{u}) = J_0(\boldsymbol{u})\,. \tag{3.31}$$

PROOF. – The proof is essentially the same of Theorem 1, both for the lower bound estimate and for the construction of the recovery sequence.

It is of some interest to compare the families of functionals F_ε and J_ε, both defined over the space $\mathcal{U}$, and their respective $\Gamma-$limits F_0 and J_0, the former being defined over the subspace $\mathcal{U}_0$ of the space $\widetilde{\mathcal{U}_0}$ where the latter is defined. Clearly,

$$F_\varepsilon(\boldsymbol{u}) \geq J_\varepsilon(\boldsymbol{u});$$

equality holds over $\mathcal{U}_0$. In the situations covered by Theorem 3, one cannot expect that the functionals $\mathcal{RM}$ and J_0 attain the same minimum and that a simple relation between minimum points holds. Nevertheless, the gap between the two minima can be estimated, as shown in the theorem to follow.

THEOREM 4. – *Let* $(\overline{\boldsymbol{\varphi}}, \overline{w}) \in \operatorname{argmin} \mathcal{RM}$. *Then,*

$$\min_{\mathcal{U}} J_0 \leq \mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}) + \min_{\boldsymbol{p}} \widetilde{W}(\boldsymbol{p}; \overline{\boldsymbol{\varphi}}, \overline{w}) \leq \mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}), \tag{3.32}$$

where

$$\widetilde{W}(\boldsymbol{p}; \overline{\boldsymbol{\varphi}}, \overline{w}) = \sqrt{2}\, m_1(1)\, \boldsymbol{C}_a \cdot \int\limits_{\mathcal{P}} p_a D\hat{\sigma}\Big(\frac{1}{\sqrt{2}}(\overline{\varphi}_a + \overline{w}_{,a})\boldsymbol{C}_a\Big)\,dx + W(\boldsymbol{p}),$$

$$W(\boldsymbol{p}) = \Big(\int\limits_{-1}^{1} m_1^2(t)\,dt\Big)\int\limits_{\mathcal{P}} \bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{p}))\,dx + \Big(\int\limits_{-1}^{1} |m_1'(t)|^2\,dt\Big)\int\limits_{\mathcal{P}} \hat{\sigma}\Big(\frac{1}{\sqrt{2}}\,p_a\boldsymbol{C}_a\Big)dx$$
$$-\Big(\int\limits_{-1}^{1} t\, m_1(t)\,dt\Big)\int\limits_{\mathcal{P}} g_a p_a\,dx\,, \quad \forall \boldsymbol{p} \in H_0^1(\mathcal{P}; \mathbb{R}^2)\,,$$

and m_1 *is defined as in* (2.30). *Moreover, unless* $\overline{\varphi}_a + \overline{w},_a \equiv 0$, *the last inequality in* (3.32) *is strict.*

PROOF. – Let $\boldsymbol{u} \in \mathcal{U}$; then, in view of the Corollary, there are $\boldsymbol{v} \in \mathcal{U}_0$ and $\boldsymbol{p} \in H_0^1(\mathcal{P}; \mathbb{R}^2)$ such that $\boldsymbol{u} = \boldsymbol{v} + m_1(t) p_a \boldsymbol{e}_a$, with the function m_1 defined as in (2.30). Then,

$$\check{\boldsymbol{E}}(\boldsymbol{u}) = \check{\boldsymbol{E}}(\boldsymbol{v}) + m_1 \check{\boldsymbol{E}}(\boldsymbol{p}), \quad \hat{\boldsymbol{E}}(\boldsymbol{u}) = \hat{\boldsymbol{E}}(\boldsymbol{v}) + \frac{1}{\sqrt{2}} m_1' p_a \boldsymbol{C}_a \,;$$

furthermore, on recalling that m is odd, we have that

$$\int_{\mathcal{C}} \bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{u}))\, dx dt = \int_{\mathcal{C}} \bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{v}))\, dx dt + \left(\int_{-1}^{1} m_1^2 \, dt \right) \int_{\mathcal{P}} \bar{\sigma}^*(\check{\boldsymbol{E}}(\boldsymbol{p}))\, dx,$$

and

$$\int_{\mathcal{C}} \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\, dx dt = \int_{\mathcal{C}} \hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{v}))\, dx dt$$

$$+ \left(\int_{-1}^{1} |m_1'|^2 \, dt \right) \int_{\mathcal{P}} \hat{\sigma}(\frac{1}{\sqrt{2}}\, p_a \boldsymbol{C}_a)\, dx + \frac{1}{\sqrt{2}} \boldsymbol{C}_a \cdot \int_{\mathcal{C}} m_1'\, p_a D\hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{v}))\, dx dt \,.$$

Hence,

$$\min_{\mathcal{U}} J_0 = \min_{\boldsymbol{p}} \min_{\boldsymbol{v}} \left\{ F_0(\boldsymbol{v}) + \frac{1}{\sqrt{2}} \boldsymbol{C}_a \cdot \int_{\mathcal{C}} m_1'\, p_a D\hat{\sigma}(\hat{\boldsymbol{E}}(\boldsymbol{v}))\, dx dt + W(\boldsymbol{p}) \right\} . \tag{3.33}$$

Now, given that $\boldsymbol{v}$ is a Reissner-Mindlin flexure displacement if it has the form $\boldsymbol{v}(x,t) = t\boldsymbol{\varphi}(x) + w(x)\boldsymbol{e}_3$, in which case $\hat{\boldsymbol{E}}(\boldsymbol{v}) = \dfrac{1}{\sqrt{2}}(\varphi_a + w,_a)\boldsymbol{C}_a$, we get

$$\min_{\mathcal{U}} J_0 \le \min_{\boldsymbol{p}} \min_{(\boldsymbol{\varphi}, w)} \left\{ F_0(t\boldsymbol{\varphi} + w\boldsymbol{e}_3) + \sqrt{2}\, m_1(1)\, \boldsymbol{C}_a \right.$$

$$\left. \cdot \int_{\mathcal{P}} p_a D\hat{\sigma}\left(\frac{1}{\sqrt{2}}(\varphi_a + w,_a)\boldsymbol{C}_a \right) dx + W(\boldsymbol{p}) \right\}$$

$$\le \min_{\boldsymbol{p}} \min_{(\boldsymbol{\varphi}, w)} \left\{ \mathcal{RM}(\boldsymbol{\varphi}, w) + \sqrt{2}\, m_1(1)\, \boldsymbol{C}_a \cdot \int_{\mathcal{P}} p_a D\hat{\sigma}\left(\frac{1}{\sqrt{2}}(\varphi_a + w,_a)\boldsymbol{C}_a \right) dx + W(\boldsymbol{p}) \right\}$$

$$\le \mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}) + \min_{\boldsymbol{p}} \widetilde{W}(\boldsymbol{p}; \overline{\boldsymbol{\varphi}}, \overline{w}) \le \mathcal{RM}(\overline{\boldsymbol{\varphi}}, \overline{w}).$$

It is readily seen that, for each fixed $(\overline{\boldsymbol{\varphi}},\overline{w})$, the functional

$$\boldsymbol{p}\mapsto\widetilde{W}(\boldsymbol{p};\overline{\boldsymbol{\varphi}},\overline{w})$$

has a unique minimum, at $\bar{\boldsymbol{p}}$, say; and that $\bar{\boldsymbol{p}}\equiv\mathbf{0}$ if and only if $\overline{\varphi}_a+\overline{w}_{,a}\equiv 0$. If this is the case, we have from (3.33) that

$$\min_{\mathcal{U}} J_0=\mathcal{RM}(\overline{\boldsymbol{\varphi}},\overline{w});$$

otherwise,

$$\mathcal{RM}(\bar{w},\bar{\boldsymbol{\varphi}})+\min_{\boldsymbol{p}}\widetilde{W}(\boldsymbol{p};\overline{\boldsymbol{\varphi}},\overline{w})<\mathcal{RM}(\overline{\boldsymbol{\varphi}},\overline{w}). \tag{3.34}$$

REMARK. – The variational theory of Kirchhoff-Love plates is based on a functional whose quadratic principal part obtains by letting $\boldsymbol{\varphi}=-\nabla w$ in (3.26), and hence has the following form:

$$\Sigma_{KL}(w)=\frac{1}{2}\int_{\mathcal{P}}B\Big((\Delta w)^2-2(1-\nu)(w_{,11}\,w_{,22}-(w_{,12})^2)\Big)\,dx; \tag{3.35}$$

just as we did with the Reissner-Mindlin plate functional, we generalize Kirchhoff-Love's into the following *unshearable-plate functional*:

$$\mathcal{KL}(w):=\int_{\mathcal{P}}\frac{2}{3}\bar{\sigma}^*(h\nabla\nabla w)\,dx-\int_{\mathcal{P}}\Big(\frac{2}{3}g_{a,a}+2g_3\Big)w\,dx,\quad w\in H^1_0(\mathcal{P};\mathbb{R})\,.$$

As expected,

$$F_0(-t\nabla w+w\boldsymbol{e}_3)=\mathcal{RM}(\nabla w,w)=\mathcal{KL}(w)=J_0(-t\nabla w+w\boldsymbol{e}_3)\,. \tag{3.36}$$

Theorem 4 tells us that $\mathcal{U}_0$ is the greatest subspace of $\mathcal{U}$ on which the minimum of J_0 coincides with the minimum of the generalized Reissner-Mindlin functional . . . unless the latter coincides with the Kirchhoff-Love one! Moreover, inequality (3.34) suggests that minimization of J_0 over the displacement class

$$\widetilde{\mathcal{U}}=\{\boldsymbol{u}\in\mathcal{U}:\boldsymbol{u}=(t\varphi_a+m_1(t)p_a)\boldsymbol{e}_a+w\boldsymbol{e}_3;\ \boldsymbol{\varphi},\boldsymbol{p}\in H^1_0(\mathcal{P};\mathbb{R}^2),\quad w\in H^1_0(\mathcal{P});\ t\in(-1,+1)\}$$

is strictly better than minimization over $\mathcal{U}_0$.

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Danilo Percivale: Dipartimento di Ingegneria della Produzione, Università di Genova
Piazzale Kennedy, Fiera del Mare, Padiglione D - 16129 Genova, Italy
E-mail: percivale@dimet.unige.it

Paolo Podio-Guidugli: Dipartimento di Ingegneria Civile, Università di Roma TorVergata
Viale Politecnico 1 - 00133 Roma, Italy
E-mail: ppg@uniroma2.it

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