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Some Nonlinear Evolution Problems in Mixed Form

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Dedicated to the memory of Guido Stampacchia

Abstract. – *This work deals with some abstract equations, either linear or nonlinear, arising from the so-called mixed formulation of PDEs of elliptic and parabolic type. This class of variational formulations turns out to be particularly relevant in connection with the development of finite elements approximations. We prove the well-posedness of both the stationary and the evolution problems.*

1. – Introduction.

This paper is concerned with either linear or nonlinear abstract equations which directly arise from the so-called *mixed formulation* of some elliptic and parabolic PDEs. Let us assume that we are given two real Hilbert spaces, V and Q, and two operators: $A : V \to V'$ possibly nonlinear and multivalued, and $B : V \to Q$ linear and continuous. Our basic stationary problem reads as follows: for any $(f, g) \in V' \times Q$, we search for $(u, p) \in V \times Q$ such that

$$\begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.1}$$

In particular this system includes appropriate reformulations of the classical Stokes and Poisson problems:

$$\begin{gathered} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \ \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{0} \ \text{ on } \partial\Omega, \\ -\Delta p = g \ \text{ in } \Omega, \quad p = 0 \ \text{ on } \partial\Omega \end{gathered} \tag{1.2}$$

Mixed variational formulations are relevant for the numerics, and are at the basis of a class of effective finite elements discretization procedures. In particular, problem (1.1) is the archetype of the so-called *mixed finite element methods*, which have been considered since the 70s [Bab73, Bre74, BF91] in both the continuous and space-discrete frameworks. We shall make no attempt to review the huge literature on mixed formulations and mixed finite elements methods. Let us however mention that the case of a linear operator A has been intensively investigated whereas the nonlinear theory for (1.1) is much less settled (we just refer to [Sch77, BN90, Le82, Gat02, GM75, MF01]).

It seems natural to associate to the stationary problem (1.1) a first-order time-relaxation dynamics of the form:

$$\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} + \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.3}$$

This equation may model e.g. a slightly compressible fluid-flow in a poro-elastic porous medium [KSWK06, SS04]. We shall be mainly concerned with the well-posedness of the initial-value problem associated to (1.3). The idea of combining mixed variational formulations with time-evolution is not new, and has been pursued in the frame of mixed finite element methods since the 80s'. Again, the literature on this subject is extensive, see e.g. the pioneering papers by QUARTERONI [Qua80] and JOHNSON & THOMÉE [JT81], the monograph by THOMÉE [Tho06], and the papers [BG04, CW06, CFA06, HR82, Pan98]. These works mainly focus on convergence and error estimates for space-time discrete methods. To our knowledge, even in the linear case the discussion on the abstract continuous flow problem in the full generality of (1.3) is still missing.

PLAN OF WORK. – In the next section, first we define our abstract functional framework, and review some equivalent formulations of the property of closure of the range in terms of *inf-sup conditions,* via the Banach Closed Range Theorem. We then illustrate some simple examples leading to the abstract scheme (1.1), briefly review some well-posedness results for the linear equation, and extend them to the case of a nonlinear operator A.

Afterwards we prove that the evolution problem is well-posed. As the linearity of A actually plays little role here, from the beginning we focus upon the case of a maximal monotone operator $A : V \to V'$, and then show how some statements may be improved in the linear case.

2. – The stationary case.

We shall start our discussion with the analysis of (1.1). In this section first we recall the general functional frame and review the classical results of the linear theory, namely for $A : V \to V'$ linear and continuous. We then extend these results to some relevant nonlinear situations. The analysis of the stationary case will clearly be the starting point for the development of the evolution theory.

2.1 – *The general functional frame.*

The following assumptions and notation will be implicitly assumed throughout. V and H will be real Hilbert spaces with $V \subset H$ densely and continuously, so that by identifying H with its topological dual, H', we get a standard Hilbert triplet $V \subset H = H' \subset V'$. By Q we denote another Hilbert space, that we identify with its dual Q'. We let the corresponding scalar products be $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$, the duality pairing between V' and V be $\langle\cdot,\cdot\rangle_{V'\times V}$, and the norm of the any normed space E be $\|\cdot\|_E$. For any set $W\subset V$, we shall denote its *annihilator* by $W^0\subset V'$:

$$W^0=\{v\in V' \ : \ \langle v,w\rangle_{V'\times V}=0 \ \ \forall w\in W\}.$$

For any convex, proper, and lower semicontinuous function $\psi: V\to(-\infty,+\infty]$, we let its *effective domain* be $D(\psi)=\{v\in V: \psi(v)<+\infty\}$, and its (possibly multivalued) *subdifferential* (in the sense of Convex Analysis) $\partial\psi: V\to V'$ be defined as

$$y\in\partial\psi(v) \quad \text{iff} \quad v\in D(\psi) \ \text{ and } \ \langle y,w-v\rangle_{V'\times V}\le\psi(w)-\psi(v) \quad \forall w\in V.$$

We shall assume that we are given two operators: $A: V\to V'$ (possibly nonlinear and multivalued), and $B: V\to Q'$ linear and continuous with adjoint $B': Q\to V'$. In the following the operator B will often be required to fulfill the *closed range condition*

$$\mathrm{Rg}(B) \ \text{ is closed.} \tag{2.1}$$

This assumption turns out to be crucial for the theory of variational problems in mixed form [BF91]. Moreover by Banach's Closed Range Theorem [Yos80, Sec. VII.5, p. 205], one has the chain of equivalences

$$\begin{aligned} &\mathrm{Rg}(B) \ \text{ is closed} \ \Leftrightarrow \ \mathrm{Rg}(B') \ \text{ is closed} \\ \Leftrightarrow \ &\mathrm{Rg}(B)=(\mathrm{Ker}\,B')^{\perp} \ \Leftrightarrow \ \mathrm{Rg}(B')=(\mathrm{Ker}\,B)^0. \end{aligned} \tag{2.2}$$

(The reader will note that here $(\mathrm{Ker}\,B)^0$ occurs rather than $(\mathrm{Ker}\,B)^{\perp}$, for we have not identified the dual space V' with V.)

The condition (2.1) (i.e., any of the conditions in (2.2)) is usually equivalently expressed by means of the so-called *inf-sup condition*

$$\exists k_0>0 \ : \ \inf_{q\in Q}\sup_{v\in V}\frac{(Bv,q)_Q}{\|v\|_V\|q\|_{Q/\mathrm{Ker}\,B'}}\ge k_0,$$

and by (2.2) this is in turn equivalent to

$$\exists h_0>0 \ : \ \inf_{v\in V}\sup_{q\in Q}\frac{\langle B'q,v\rangle_{V'\times V}}{\|v\|_{V/\mathrm{Ker}\,B}\|q\|_Q}\ge h_0,$$

(throughout we imply that suprema and infima are taken out of the null element, whenever needed). The above inf-sup conditions may also be represented in the equivalent forms

$$\exists k_0>0 \ : \ \|q\|_{Q/\mathrm{Ker}\,B'}\le\frac{1}{k_0}\|B'q\|_{V'} \quad \forall q\in Q, \tag{2.3}$$

$$\exists h_0>0 \ : \ \|v\|_{V/\mathrm{Ker}\,B}\le\frac{1}{h_0}\|Bv\|_{Q'} \quad \forall v\in V. \tag{2.4}$$

These conditions are often referred to by saying that B' (B, respectively) is *bounding* [Rud91, Thm. 4.15].

Finally, note that the dual space $(\operatorname{Ker} B)'$ is canonically isomorphic to $((\operatorname{Ker} B)^{\perp})^0$ $(\subset V')$. We shall then identify these two spaces, and denote them by $(\operatorname{Ker} B)'$. We let $\pi : V' \to (\operatorname{Ker} B)'$ be the orthogonal projection and $\pi' : V \to \operatorname{Ker} B$ be the corresponding adjoint (i.e., the orthogonal projection on $\operatorname{Ker} B$).

2.2 – *The stationary problem setup,*

Abstract problems like (1.1) represent several PDEs. Here we point out two examples in the special case of a linear and continuous operator $A : V \to V'$. First we consider the *Stokes flow* with homogeneous Dirichlet conditions

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \ \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{0} \ \text{ on } \partial\Omega, \tag{2.5}$$

whose variational formulation corresponds to (1.1) with

$$\begin{gathered} V = (H_0^1(\Omega))^3, \quad H = (L^2(\Omega))^3, \quad Q = \left\{ p \in L^2(\Omega) \int_\Omega p = 0 \right\}, \\ A = -\Delta, \quad B = \operatorname{div}, \quad B' = -\nabla, \quad g \equiv 0. \end{gathered} \tag{2.6}$$

Thus $\operatorname{Rg}(B) = Q$ (see, e.g., [Tem77]), whence $\operatorname{Ker}(B') = \{0\}$ by (2.2). Moreover, $\operatorname{Ker} B = \{\boldsymbol{u} \in (H_0^1(\Omega))^3 \ : \ \operatorname{div} \boldsymbol{u} = 0\}$ whence $\pi' \boldsymbol{u} = \nabla\phi$, where $\boldsymbol{u} = \nabla\phi + \operatorname{curl} \boldsymbol{a}$ is the classical Helmholtz decomposition. For any $f \in V'$ we then have $\langle \pi f, \boldsymbol{u} \rangle_{V' \times V} = \langle f, \nabla\phi \rangle_{V' \times V}$.

Our second example is the *Poisson problem* with homogeneous Dirichlet conditions

$$-\Delta p = g \ \text{ in } \Omega, \quad p = 0 \ \text{ on } \partial\Omega, \tag{2.7}$$

which may be decoupled as $\boldsymbol{u} = \nabla p$ and $-\operatorname{div} \boldsymbol{u} = g$, and may be written in variational form as (1.1) for

$$\begin{gathered} H = (L^2(\Omega))^3, \quad V = \{\boldsymbol{u} \in (L^2(\Omega))^3 \ : \ \operatorname{div} \boldsymbol{u} \in L^2(\Omega)\}, \quad Q = L^2(\Omega), \\ A = I, \quad B = \operatorname{div}, \quad B' = -\nabla, \quad f \equiv 0, \end{gathered} \tag{2.8}$$

where I is the restriction to V of the injection $H \subset V'$. Here also $\operatorname{Rg}(B) = Q$ and hence $\operatorname{Ker}(B') = 0$. Moreover, $\operatorname{Ker} B = \{\boldsymbol{u} \in V \ : \ \operatorname{div} \boldsymbol{u} = 0\}$ and $\pi' \boldsymbol{u} = \nabla\phi$, where $\boldsymbol{u} = \nabla\phi + \operatorname{curl} \boldsymbol{a}$ again is the Helmholtz decomposition.

Further standard applications of variational problems in mixed form occur in linearized elastostatics, in time-harmonic Maxwell's equations, and others [BF91].

If $A = \partial\psi$ (which is symmetric whenever A is linear) where ψ is convex and lower semicontinuous, then the pair (u, p) solves (1.1) iff it is a saddle-point of the convex-concave functional

$$\phi(u, p) = \psi(u) + (Bu, p)_Q - \langle f, u\rangle_{V'\times V} + (g, p)_Q, \tag{2.9}$$

or equivalently iff u minimizes the convex functional $u \mapsto \psi(u) - \langle f, u\rangle_{V'\times V}$ under the linear constraint $-Bu = g$:

$$\min_{-Bu=g} \big(\psi(u) - \langle f, u\rangle_{V'\times V}\big). \tag{2.10}$$

There are some advantages in finding a saddle-point for ϕ via (1.1), rather than solving the constrained minimum problem (2.10). First, the mixed form of (1.1) often is better suited than (2.10) for numerical investigations. The occurrence of a Lagrangian multiplier simplifies the numerical treatment of the problem, for it is generally hard to build finite element approximations fulfilling an (albeit linear) constraint. Moreover, mixed formulations may need a less regular functional setting, and thus allow for more freedom and flexibility.

Mixed formulations are also well-suited in case one is interested in computing accurately *gradients*. For the Poisson problem, for instance, the gradient ∇u can in principle be retrieved from u by numerical differentiation, this procedure necessarily entailing some accuracy loss. On the other hand, the mixed formulation of the problem allows instead the direct computation of $p = \nabla u$.

In several cases the operator A is non-symmetric; for instance this may occur in presence of convection. In the non-symmetric case, problem (1.1) does not then correspond to a constrained minimization.

2.3 – *The linear case.*

In the functional framework of Subsection 2.1, let us now assume that

$$A : V \to V' \quad \text{is linear and continuous (and possibly non-symmetric).} \tag{2.11}$$

The next classical theorem is the main result in the linear case.

THEOREM 2.1. (BREZZI, [Bre74]). – *Let* (2.11) *hold and* $(f, g) \in V' \times \mathrm{Rg}(B)$. *Moreover, assume that* (2.1) *is fulfilled and that*

$$\pi A \ \textit{ is an isomorphism between } \mathrm{Ker}\ B \textit{ and } (\mathrm{Ker}\ B)'. \tag{2.12}$$

Then there exists a unique solution $(u, p) \in V \times Q/\mathrm{Ker}\, B'$ *of* (1.1). *Moreover, this solution is unique for any* $(f, g) \in V' \times \mathrm{Rg}(B)$ *only if* (2.1) *and* (2.12) *are fulfilled.*

As clearly uniqueness of p may only hold up to elements of $\mathrm{Ker}\,B'$, we may restrict our analysis to the quotient space $Q/\mathrm{Ker}\,B'$ from the beginning. The isomorphism condition (2.12) may equivalently be rewritten as

$$\exists a_0 > 0 \ : \ \begin{cases} \displaystyle\inf_{u_0 \in \mathrm{Ker}\,B} \sup_{v_0 \in \mathrm{Ker}\,B} \frac{\langle Au_0, v_0 \rangle_{V' \times V}}{\|u_0\|_V \|v_0\|_V} \geq a_0, \\ \displaystyle\inf_{v_0 \in \mathrm{Ker}\,B} \sup_{u_0 \in \mathrm{Ker}\,B} \frac{\langle Au_0, v_0 \rangle_{V' \times V}}{\|u_0\|_V \|v_0\|_V} \geq a_0, \end{cases} \tag{2.13}$$

which, in case A is symmetric, is reduced to a single condition, namely

$$\exists a_0 > 0 \ : \ \inf_{u_0 \in \mathrm{Ker}\,B} \sup_{v_0 \in \mathrm{Ker}\,B} \frac{\langle Au_0, v_0 \rangle_{V' \times V}}{\|u_0\|_V \|v_0\|_V} \geq a_0. \tag{2.14}$$

A sufficient condition for (2.13) (equivalently, for (2.12)) to hold is that

$$\exists a_0 > 0 \ : \ \langle Au, u \rangle_{V' \times V} \geq a_0 \|u\|_V^2 \quad \forall u \in \mathrm{Ker}\,B. \tag{2.15}$$

In passing note that the latter is equivalent to (2.14) whenever A is symmetric and has a countable spectrum, see e.g. [Rud91, Thm. 12.29.d, p. 328]. We shall also consider the stronger property

$$\exists a_0 > 0 \ : \ \langle Au, u \rangle_{V' \times V} \geq a_0 \|u\|_V^2 \ \ \forall u \in V. \tag{2.16}$$

Both (2.15) and the latter turn out to be relevant for applications; for instance, (2.15) applies to the Poisson problem (2.7)-(2.8), whereas the Stokes flow (2.5)-(2.6) fulfills the stronger condition (2.16). Finally, in view of the evolution theory, we mention a weaker (but global) coercivity condition for A:

$$\exists a_0, \beta_0 > 0 \ : \ \langle Au, u \rangle_{V' \times V} + \beta_0 \|u\|_H^2 \geq a_0 \|u\|_V^2 \ \ \forall u \in V. \tag{2.17}$$

We also point out a result for the following variant of the problem (1.1) (see [Arn81]):

$$\begin{pmatrix} A & B' \\ -B & \lambda I_Q \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.18}$$

where $\lambda > 0$ and I_Q is the identity in Q.

THEOREM 2.2. (Thm. 1.2, p. 47 [BF91]). – *Let* (2.11) *hold, A be positive, $\lambda > 0$, and* (2.1) *and* (2.15) *be fulfilled. Then there exists a unique solution $(u, p) \in V \times Q$ of* (2.18) *for any $(f, g) \in V' \times Q$.*

We refer e.g. to [CHZ03] for a comprehensive discussion of (linear) generalizations of (1.1) and (2.18).

2.4 – *The nonlinear case.*

Next we deal with the existence of a solution of (1.1) in the case of a nonlinear operator A. More precisely, we assume that

(2.19) $\quad A : V \to V'$ is maximal monotone (possibly multivalued).

See e.g. [Bre73, Bar76] for classical results on maximal monotone operators.

LEMMA 2.3. (Maximal monotonicity). – *If* (2.19) *is fufilled, then*

$$M = \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} : V \times Q \to V' \times Q \quad \textit{is maximal monotone.}$$

PROOF. – Let us denote by $J : V \to V'$ the Riesz isomorphim and set

$$F = \begin{pmatrix} J & 0 \\ 0 & I_Q \end{pmatrix} : V \times Q \to V' \times Q.$$

As the operator M is clearly monotone, it suffices to show that $\mathrm{Rg}(\lambda F + M) = V' \times Q$ for some (hence all) $\lambda > 0$. For any $(f, g) \in V' \times Q$, defining $p = (g + Bu)/\lambda$, one must thus find $u \in V$ such that

$$(\lambda J + A)(u) + \frac{1}{\lambda} B'Bu \ni f - \frac{1}{\lambda} B'g.$$

Now, the operator $B' \circ B : V \to V'$ is (symmetric) positive, linear, and continuous, hence maximal monotone. Moreover, $\lambda J + A$ is maximal monotone and coercive. By applying [Bar76, Cor. 1.3, p. 48] we than infer that

$$(\lambda J + A) + \frac{1}{\lambda} B' \circ B : V \to V'$$

is onto. The assertion then follows. □

We are now able to state our first existence result for the nonlinear case.

THEOREM 2.4. (A nonlinear). – *Let $A : V \to V'$ be a possibly nonlinear and multivalued operator; assume that* (2.1) *is fulfilled and that*

(2.20) $\forall v \in \mathrm{Ker}\, B^{\perp}, \quad u \mapsto \pi A(u + v) : \mathrm{Ker}\, B \to (\mathrm{Ker}\, B)'$ *is maximal monotone*,

(2.21) $$\forall r \geq 0, \quad \lim_{\substack{\|Bu\|_Q \leq r,\ \xi \in A(u) \\ \|u\|_V \to +\infty}} \frac{\langle \xi, u \rangle_{V' \times V}}{\|u\|_V} = +\infty.$$

For any $(f, g) \in V' \times \mathrm{Rg}(B)$ then there exists a solution $(u, p) \in V \times Q/\mathrm{Ker}\, B'$ of (1.1).

The reader may compare this statement with the nonlinear well-posedness theory of SCHEURER [Sch77], who deals with approximation issues, including also the case where the second equation in (1.1) is replaced by the inequality $Bu \leq g$, assuming that the space Q is ordered. Note that the first part of Theorem 2.1 follows from Theorem 2.4 for A linear. Some further results on nonlinear versions of (1.1) may be found in [BN90, Le82, Gat02, GM75, MF01] and in other works.

PROOF OF THEOREM 2.4. – This argument follows the lines of the original proof of [Bre74, Thm. 1.1], here adapted to the nonlinear case. There exists a unique $u_g \in \operatorname{Ker} B^\perp$ such that $Bu_g = g$. By (2.20)-(2.21) the operator $u \mapsto \pi A(u + u_g) : \operatorname{Ker} B \to (\operatorname{Ker} B)'$ is maximal monotone and coercive. This operator is then onto, so that for any $f \in V'$ there exists $u_0 \in \operatorname{Ker} B$ such that $\pi A(u_0 + u_g) \ni \pi f$. Thus there exists $\xi \in A(u_0 + u_g)$ such that $\pi(f - \xi) = 0$, namely

$$f - \xi \in ((\operatorname{Ker} B)')^\perp = (\operatorname{Ker} B)^0.$$

We saw that the closed range assumption (2.1) is equivalent to $(\operatorname{Ker} B)^0 = \operatorname{Rg}(B')$, cf. (2.2). Hence, there exists $p \in Q/\operatorname{Ker} B'$ (unique, for any u_0) such that $f - \xi = B'p$. The thesis then follows by letting $u = u_0 + u_g$. □

REMARK 2.5. – In the latter argument the assumptions (2.20)-(2.21) grant that

$$v \mapsto \pi\, A(v + u_g) \text{ restricted to Ker } B \text{ is onto (Ker } B)', \tag{2.22}$$

where u_g is the only element in $\operatorname{Ker} B^\perp$ such that $Bu_g = g$. The latter condition is weaker than (2.20)-(2.21), and would suffice to reproduce the existence argument (note that for A linear (2.12) and (2.22) are equivalent). On the other hand, the stronger conditions (2.20) and (2.21) are independent of the selection of g.

For instance, (2.21) holds whenever $A : V \to V'$ is coercive on the whole space V, namely,

$$\lim_{\substack{\xi \in A(u) \\ \|u\|_V \to +\infty}} \frac{\langle \xi, u \rangle_{V' \times V}}{\|u\|_V} = +\infty. \tag{2.23}$$

Note that the latter is implied by (2.16). More generally, we shall also consider the weaker condition

$$\exists \beta_0 \geq 0 \;:\; \lim_{\substack{\xi \in A(u) \\ \|u\|_V \to +\infty}} \frac{\langle \xi, u \rangle_{V' \times V} + \beta_0 \|u\|_H 2}{\|u\|_V} = +\infty. \tag{2.24}$$

Next we state a nonlinear version of Theorem 2.2, in view of the analysis of evolution. For any $\lambda > 0$ let us consider the problem

$$\begin{pmatrix} A & B' \\ -B & \lambda I_Q \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.25}$$

THEOREM 2.6. (A nonlinear, $\lambda > 0$). – *Let* $\lambda > 0$, $(f, g) \in V' \times Q$, (2.19) *be fulfilled, and assume that*

- (2.23) *be fulfilled, or*
- (2.1), (2.21), $f \in \mathrm{Rg}(B')$.

Then, there exists a solution $(u, p) \in V \times Q$ *of* (2.25).

PROOF. – Let us first assume that (2.23) is fulfilled. By letting $p = (g + Bu)/\lambda$, we are left with the problem

$$A(u) + \frac{1}{\lambda} B'Bu \ni f - \frac{1}{\lambda} B'g.$$

As $A + B' \circ B/\lambda : V \to V'$ is maximal monotone and coercive, this equation has a solution $u \in V$.

Let us now turn to the case of (2.1) and (2.21) for $f \in \mathrm{Rg}(B')$. We shall first decompose g and p as follows

$$\begin{aligned} &g = g^0 + g^1 \in \mathrm{Ker}\, B' \oplus (\mathrm{Ker}\, B')^{\perp} \overset{(2.2)}{=} \mathrm{Ker}\, B' \oplus \mathrm{Rg}(B), \\ &p = p^0 + p^1 \in \mathrm{Ker}\, B' \oplus \mathrm{Rg}(B), \end{aligned} \tag{2.26}$$

and rewrite (2.18) as

$$\begin{pmatrix} A & B' & 0 \\ -B & \lambda I_{\mathrm{Rg}(B)} & 0 \\ 0 & 0 & \lambda I_{\mathrm{Ker}\, B'} \end{pmatrix} \begin{pmatrix} u \\ p^1 \\ p^0 \end{pmatrix} \ni \begin{pmatrix} f \\ g^1 \\ g^0 \end{pmatrix}.$$

Hence $p^0 = g^0/\lambda$, and one is left with a problem in $(u, p^1) \in V \times \mathrm{Rg}(B)$ only. We may thus assume that $g \in \mathrm{Rg}(B)$ with no loss of generality.

Let us now fix any $\varepsilon > 0$ and consider the regularized problem

$$M_\varepsilon \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \varepsilon J + A & B' \\ -B & \lambda I_Q \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.27}$$

which has a solution, for $M_\varepsilon : V \times Q \to V' \times Q$ is maximal monotone (by Lemma 2.3) and coercive. Thus there exists a triple $(u_\varepsilon, p_\varepsilon, \xi_\varepsilon)$ such that

$$\varepsilon J u_\varepsilon + \xi_\varepsilon + B' p_\varepsilon = f, \quad -Bu_\varepsilon + \lambda p_\varepsilon = g, \quad \xi_\varepsilon \in A(u_\varepsilon). \tag{2.28}$$

Next we shall provide estimates on $(u_\varepsilon, p_\varepsilon, \xi_\varepsilon)$ independent of ε, and then pass to the limit. After decomposing u_ε as

$$u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^1 \in \mathrm{Ker}\, B \oplus \mathrm{Ker}\, B^{\perp},$$

by (2.4) and (2.27) we get that

$$\begin{aligned}\|u^1_\varepsilon\|_V &\overset{u^1_\varepsilon\in \mathrm{Ker}\,B^\perp}{=} \|u^1_\varepsilon\|_{V/\mathrm{Ker}\,B} \overset{(2.4)}{\le} \frac{1}{h_0}\|Bu^1_\varepsilon\|_Q \\ &\overset{u^0_\varepsilon\in \mathrm{Ker}\,B}{=} \frac{1}{h_0}\|Bu_\varepsilon\|_Q \overset{(2.27)}{=} \frac{1}{h_0}\|\lambda p_\varepsilon - g\|_Q.\end{aligned} \tag{2.29}$$

Moreover, again by (2.28) and as $f\in \mathrm{Rg}(B') = (\mathrm{Ker}\,B)^0$,

$$\begin{aligned}&\varepsilon\|u_\varepsilon\|^2_V + \langle \xi_\varepsilon, u_\varepsilon\rangle_{V'\times V} + \lambda\|p_\varepsilon\|^2_Q \le \langle f, u^1_\varepsilon\rangle_{V'\times V} + (g, p_\varepsilon)_Q \\ &\overset{(2.4)}{\le} \|f\|_{V'}\frac{1}{h_0}\|\lambda p_\varepsilon - g\|_Q + \|g\|_Q\|p_\varepsilon\|_Q.\end{aligned}$$

Hence, p_ε and u^1_ε are uniformly bounded in $Q/\mathrm{Ker}\,B'$ and V, respectively. By using (2.21) and (2.26) we also infer that u_ε is bounded in V, independently of ε. Finally, by comparing the terms of (2.27) we see that ξ_ε is also uniformly bounded in V'. We then obtain that, possibly extracting (not relabeled) subsequences,

$$(u_\varepsilon, p_\varepsilon, \xi_\varepsilon) \to (u, p, \xi) \quad \text{weakly in} \quad V\times Q/\mathrm{Ker}\,B' \times V',$$

so that the equations

$$\xi + B'p = f, \quad -Bu + \lambda p = g$$

are established. In order to show that $\xi\in A(u)$, let us test (2.26) on $(u_\varepsilon, p_\varepsilon)$ and pass to the superior limit; by lower semicontinuity we get

$$\begin{aligned}\limsup_{\varepsilon\to 0}\langle \xi_\varepsilon, u_\varepsilon\rangle_{V'\times V} &= \limsup_{\varepsilon\to 0}\Big(\langle f, u_\varepsilon\rangle_{V'\times V} + (g, p_\varepsilon)_Q - \varepsilon\|u_\varepsilon\|^2_V - \lambda\|p_\varepsilon\|^2_Q\Big) \\ &\le \langle f, u\rangle_{V'\times V} + (g,p)_Q - \lambda\|p\|^2_Q = \langle \xi, u\rangle_{V'\times V}.\end{aligned}$$

A standard result on maximal monotone operators [Bar76, Lemma 1.3, p. 42] then yields the inclusion $\xi\in A(u)$. □

PROPOSITION 2.7 (Uniqueness and continuous dependence on data). – *If A is strictly monotone and single-valued, then Problem* (1.1) *has at most one solution in $V\times Q/\mathrm{Ker}\,B'$. Moreover, if* (2.1) *holds and A is strongly monotone and Lipschitz continuous, then the solution depends continuously on $(f,g)\in V\times \mathrm{Rg}(B)$.*

PROOF. – For $i = 1,2$, let $(u_i, p_i)\in V\times Q/\mathrm{Ker}\,B'$ solve problem (1.1). Hence

$$\begin{aligned}\langle A(u_1) - A(u_2), u_1 - u_2\rangle_{V'\times V} &= -\langle B'(p_1 - p_2), u_1 - u_2\rangle_{V'\times V} \\ &= -(p_1 - p_2, B(u_1 - u_2))_Q \\ &= -(p_1 - p_2, g - g)_Q = 0.\end{aligned}$$

As A is strictly monotone, we then infer that $u_1 \equiv u_2$. Hence $B'(p_1 - p_2) = 0$, whence $p_1 = p_2$. for $p_1, p_2 \in Q/\mathrm{Ker}\, B'$.

For $i = 1, 2$, let now $(f_i, g_i) \in V' \times \mathrm{Rg}(B)$ be given, and $(u_i, p_i) \in V \times Q/\mathrm{Ker}\, B'$ be corresponding solutions of (1.1). Thus there exists $\xi_i \in A(u_i)$ such that

$$\xi_i + B'p_i = f_i$$

$$-Bu_i = g_i.$$

Taking the difference between the above system for $i = 1, 2$, and testing the result by the pair $(u_1 - u_2, p_1 - p_2)$, we get

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle_{V' \times V} \leq \langle f_1 - f_2, u_1 - u_2 \rangle_{V' \times V} + (g_1 - g_2, p_1 - p_2)_Q.$$

As

$$\|p_1 - p_2\|_Q \overset{(2.3)}{\leq} \frac{1}{k_0} \|B'(p_1 - p_2)\|_{V'} = \frac{1}{k_0} \|f_1 - f_2 - A(u_1) + A(u_2)\|_{V'},$$

by the strong monotonicity (with constant a_0) and the Lipschitz-continuity (with constant Λ) of A, we infer that

$$\begin{aligned} a_0 \|u_1 - u_2\|_V^2 &\leq \|f_1 - f_2\|_{V'} \|u_1 - u_2\|_V + \|g_1 - g_2\|_Q \|p_1 - p_2\|_Q \\ &\overset{(2.3)}{\leq} \|f_1 - f_2\|_{V'} \|u_1 - u_2\|_V + \|g_1 - g_2\|_Q \frac{1}{k_0} \|f_1 - f_2 - A(u_1) + A(u_2)\|_{V'} \\ &\leq \|f_1 - f_2\|_{V'} \|u_1 - u_2\|_V + \|g_1 - g_2\|_Q \frac{1}{k_0} \left(\|f_1 - f_2\|_{V'} + \Lambda \|u_1 - u_2\|_V \right). \end{aligned}$$

The continuous dependence of (u, p) on the data then easily follows. □

3. – The evolution problem.

We shall start by a well-posedness result, which rests upon a direct extension of the analysis of the stationary case to evolution. As the linearity of A actually plays a little role here, from the very beginning we focus our attention on the case of a maximal monotone operator $A : V \to V'$, and postpone the discussion of the linear case to Subsection 3.1.

We shall consider two cases. First we deal with an operator A that is coercive on $\mathrm{Ker}\, B$ and assume that the range of B is closed (Theorem 3.1). In particular we extend Theorem 2.1 to the (nonlinear and) evolution case. We then represent the classical nonlinear parabolic theory in the mixed framework by assuming that A is weakly coercive, and without requiring the range of B to be closed (Theorem 3.2).

THEOREM 3.1. (A coercive on $\operatorname{Ker} B$). – *Let* (2.1), (2.19), *and* (2.21) *hold. Then, for all* $(f, g) \in W^{1,1}(0,T; H \times Q)$, *and* $(u_0, p_0) \in D(A) \times Q$ *with* $f(0) - \xi_0 - B'p_0 \in H$ *for some* $\xi_0 \in A(u_0)$, *there exists a pair*

$$(u, p) \in \big(W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)\big) \times W^{1,\infty}(0,T;Q)$$

such that

$$\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} + \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix} \qquad \text{a.e. in } (0,T), \tag{3.1}$$

$$(u(0), p(0)) = (u_0, p_0). \tag{3.2}$$

PROOF. – We proceed by approximation and passage to the limit. For any $\varepsilon > 0$ let us consider the regularized problem

$$\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} + \begin{pmatrix} A + \varepsilon J & B' \\ -B & \varepsilon I_Q \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix}, \tag{3.3}$$

where $J : V \to V'$ is the Riesz isomorphism. The operator

$$\begin{pmatrix} A + \varepsilon J & B' \\ -B & \varepsilon I_Q \end{pmatrix} : V \times Q \to V' \times Q$$

is maximal, strongly monotone (see Lemma 2.3), and coercive. By classical well-posedness results of [Bre71, Bar76], (3.3) then has a unique solution

$$(u_\varepsilon, p_\varepsilon) \in \big(W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)\big) \times W^{1,\infty}(0,T;Q).$$

By testing this system on $(u_\varepsilon, p_\varepsilon)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}(\|u_\varepsilon\|_H^2 + \|p_\varepsilon\|_Q^2) + \langle \xi_\varepsilon, u_\varepsilon \rangle_{V' \times V} + \varepsilon \|u_\varepsilon\|_V^2 + \varepsilon \|p_\varepsilon\|_Q^2 = (f, u_\varepsilon)_H + (g, p_\varepsilon)_Q, \tag{3.4}$$

where

$$\xi_\varepsilon = f - \dot{u}_\varepsilon - \varepsilon J u_\varepsilon - B' p_\varepsilon \in A(u_\varepsilon).$$

Next we proceed formally, by a technique that might easily be justified via an appropriate approximation. Let us take the time-derivative of (3.3), and test it on $(\dot{u}_\varepsilon, \dot{p}_\varepsilon)$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}(\|\dot{u}_\varepsilon\|_H^2 + \|\dot{p}_\varepsilon\|_Q^2) + \langle \dot{\xi}_\varepsilon, \dot{u}_\varepsilon \rangle_{V' \times V} + \varepsilon \|\dot{u}_\varepsilon\|_V^2 + \varepsilon \|\dot{p}_\varepsilon\|_Q^2 = (\dot{f}, \dot{u}_\varepsilon)_H + (\dot{g}, \dot{p}_\varepsilon)_Q.$$

By the hypotheses on the initial data and by the system (3.3), we see that $(\dot{u}_\varepsilon(0), \dot{p}_\varepsilon(0))$ is uniformly bounded in $H \times Q$. By the latter equality, via the classical Gronwall lemma we then infer that

$$(u_\varepsilon, p_\varepsilon) \quad \text{is bounded in } W^{1,\infty}(0,T; H \times Q) \text{ independently of } \varepsilon. \tag{3.5}$$

As A is not assumed to be coercive on V, a uniform estimate is needed for u_ε in V. To this aim, let us decompose u_ε:

$$u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^1 \in \operatorname{Ker} B \oplus \operatorname{Ker} B^\perp; \tag{3.6}$$

note that, a.e. in $(0, T)$,

$$\begin{aligned} \|u_\varepsilon^1\|_V &\overset{u_\varepsilon^1 \in \operatorname{Ker} B^\perp}{=} \|u_\varepsilon^1\|_{V/\operatorname{Ker} B} \overset{(2.4)}{\leq} \frac{1}{h_0}\|Bu_\varepsilon^1\|_Q \\ &\overset{u_\varepsilon^0 \in \operatorname{Ker} B}{=} \frac{1}{h_0}\|Bu_\varepsilon\|_Q \overset{(3.3)}{=} \frac{1}{h_0}\|\dot{p}_\varepsilon + \varepsilon p_\varepsilon - g\|_Q. \end{aligned} \tag{3.7}$$

By (3.5) we easily see that

$$\langle \xi_\varepsilon, u_\varepsilon \rangle_{V' \times V} \quad \text{is bounded in } L^\infty\ (0, T) \text{ independently of } \varepsilon\,.$$

Now, as by (3.7) all the u_ε^1's belong to a fixed ball in V for a.e. t, and

$$\|Bu_\varepsilon\|_{L^\infty(0,T;Q)} \leq \|\dot{p}_\varepsilon + \varepsilon p_\varepsilon - g\|_{L^\infty(0,T;Q)},$$

by (2.21) and (3.5), we then infer that u_ε is bounded in $L^\infty(0, T; V)$, independently of ε. Moreover, by comparing the terms of the first equation of (3.3), we see that ξ_ε is uniformly bounded in $L^\infty(0, T; V')$ as well. Hence there exist (u, p, ξ) such that, by possibly passing to (not relabelled) subsequences,

$$\begin{aligned} u_\varepsilon &\to u \quad \text{weakly star in } \ W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \\ p_\varepsilon &\to p \quad \text{weakly star in } \ W^{1,\infty}(0, T; Q), \\ \xi_\varepsilon &\to \xi \quad \text{weakly star in } \ L^\infty(0, T; V'). \end{aligned}$$

The initial conditions (3.2) are then fulfilled, and

$$\dot{u} + \xi + B'p = f, \quad \dot{p} - Bu = g \quad \text{a.e.in } \ (0, T).$$

In view of identifying the limit ξ, let us integrate (3.4) in time and then pass to the superior limit. By exploiting the limit equations and by lower semicontinuity, we get

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \int_0^T \langle \xi_\varepsilon, u_\varepsilon \rangle_{V' \times V} \leq \limsup_{\varepsilon \to 0} \left(\int_0^T \Big(\langle f, u_\varepsilon \rangle_{V' \times V} + (g, p_\varepsilon)_Q - \varepsilon \|u_\varepsilon\|_V^2 - \varepsilon \|p_\varepsilon\|_Q^2 \Big) \right. \\ &\left. - \frac{1}{2}\|u_\varepsilon(T)\|_H^2 - \frac{1}{2}\|p_\varepsilon(T)\|_Q^2 + \frac{1}{2}\|u_0\|_H^2 + \frac{1}{2}\|p_0\|_Q^2 \right) \\ &\leq \int_0^T \langle f, u \rangle_{V' \times V} + (g, p)_Q - \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|p(T)\|_Q^2 + \frac{1}{2}\|u_0\|_H^2 + \frac{1}{2}\|p_0\|_Q^2 \\ &= \int_0^T \langle \xi, u \rangle_{V' \times V} \end{aligned}$$

so that $\xi \in A(u)$ a.e. in $(0, T)$, by a classical result [Bar76, Lemma 1.3, p. 42]. □

Next we provide a well-posedness result for the case of A being weakly coercive (2.24). Here, as it may be expected, neither the closure of $\mathrm{Rg}(B)$ (2.1) nor the absolute continuity of the data f and g is required.

THEOREM 3.2. (A weakly coercive on V). – *Let* (2.19) *and* (2.24) *hold. Then:*

1. *If* $A : V \to V'$ *is linearly bounded, namely there exists* $c > 0$ *such that*

$$\|\xi\|_{V'} \leq c\big(1 + \|u\|_V\big) \quad \forall \xi \in A(u),$$

then for any $(f, g) \in L^2(0, T; V' \times Q)$ *and* $(u_0, p_0) \in H \times Q$*, there exists a pair*

$$(u, p) \in \big(H^1(0, T; V') \cap L^2(0, T; V)\big) \times H^1(0, T; Q)$$

that solves (3.1) *and* (3.2).

2. *If* $(f, g) \in W^{1,1}(0, T; H \times Q)$*,* $u_0 \in D(A)$*, and* $f(0) - \xi_0 - B'p_0 \in H$ *for some* $\xi_0 \in A(u_0)$*, then the solution* (u, p) *belongs to* $W^{1,\infty}(0, T; H \times Q)$*.*

PROOF. – 1. As the operator

$$M = \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} : V \times Q \to V' \times Q$$

is maximal monotone (see Lemma 2.3), linearly bounded, and weakly coercive on $V \times Q$, one may easily prove the stated result by amending the existence theory of [Bre71, Bar76].

2. This second result stems from the application of [Sho97, Thm. IV.6.1]. Here just the *range condition*

$$\mathrm{Rg}(I_{H\times Q} + M) = H \times Q$$

must be checked. This clearly holds if $\beta_0 \leq 1$ as $\beta_0 I_{H\times Q} + M$ is maximal monotone (cf. Lemma 2.3) and coercive by (2.24). Let us now consider the case of $\beta_0 > 1$. As we already know that $\mathrm{Rg}(\beta_0 I_{H\times Q} + M) = H \times Q$, for all $(v, q) \in H \times Q$ we may define $(u, p) = S(v, p)$ to be the unique solution of

$$\begin{pmatrix} A + \beta_0 I_H & B' \\ -B & \beta_0 I_Q \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f + (\beta_0 - 1)v \\ g + (\beta_0 - 1)q \end{pmatrix}.$$

By letting $(u_i, p_i) = S(v_i, q_i)$, $i = 1, 2$, we readily check that

$$\begin{aligned}
&\beta_0 \|(u_1, p_1) - (u_2, p_2)\|^2_{H\times Q} \\
&\leq (\beta_0 - 1)(v_1 - v_2, u_1 - u_2)_H + (\beta_0 - 1)(q_1 - q_2, p_1 - p_2)_Q \\
&\leq \frac{\beta_0}{2} \|(u_1, p_1) - (u_2, p_2)\|^2_{H\times Q} + \frac{(\beta_0 - 1)^2}{2\beta_0} \|v_1 - v_2\|^2_H + \frac{(\beta_0 - 1)^2}{2\beta_0} \|q_1 - q_2\|^2_Q.
\end{aligned}$$

Hence

$$\|(u_1,p_1)-(u_2,p_2)\|^2_{H\times Q}$$

$$\leq \frac{(\beta_0-1)^2}{\beta_0^2}\|(v_1,q_1)-(v_2,q_2)\|^2_{H\times Q} < \|(v_1,q_1)-(v_2,q_2)\|^2_{H\times Q}.$$

Thus S is a contraction in $H \times Q$. Its fixed point (u,p) solves $(I_{H\times Q}+M)\cdot (u,p)=(f,g)$. □

Next we state a result of continuous dependence on the data for the solution of (3.1).

LEMMA 3.3. (Continuous dependence on data). – *For* $i=1,2$ *let* (u_i,p_i) *solve* (3.1) *for data* $(f_i,g_i):(0,T)\to V'\times Q$*, and set* $\xi_i = f_i - \dot{u}_i - B'p_i$*. Then*

$$\|(u_1,p_1)-(u_2,p_2)\|^2_{H\times Q}(t)+\int_0^t \langle \xi_1-\xi_2, u_1-u_2\rangle_{V'\times V}$$

$$\leq c\Big(\|(u_1,p_1)-(u_2,p_2)\|^2_{H\times Q}(0)+\|f_1-f_2\|^2_{L^1(0,t;H)}+\|g_1-g_2\|^2_{L^1(0,t;H)}\Big),$$

for any $t\in[0,T]$*, for some constant* $c>0$ *that may depend on* T.

3.1 – *The linear subcase.*

If A is linear and positive, the above results of existence and continuous dependence on the data may be sharpened by weakening the coercivity assumptions. In particular we have the following statement.

COROLLARY 3.4 (A linear and coercive on $\operatorname{Ker} B$). – *Let* (2.1), (2.11), (2.15) *hold. Then, for all* $(f,g)\in W^{1,1}(0,T;H\times Q)$*, and* $(u_0,p_0)\in V\times Q$ *that fulfill the compatibility condition* $f(0)-Au_0-B'p_0\in H$*, there exists a unique pair*

$$(u,p)\in \big(W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)\big)\times W^{1,\infty}(0,T;Q)$$

that solves (3.1) *and* (3.2)*. Moreover,* (u,p) *depends Lipschitz-continuously on the data.*

COROLLARY 3.5 (A linear and weakly coercive on V). – *Let* (2.11) *and* (2.17) *hold and A be positive. Then,*

1. *for all* $(f,g)\in L^2(0,T;V'\times Q)$ *and* $(u_0,p_0)\in H\times Q$*, there exists a unique pair*

$$(u,p)\in \big(H^1(0,T;V')\cap L^2(0,T;V)\big)\times H^1(0,T;Q)$$

that solves (3.1) *and* (3.2). *Moreover,* (u, p) *depends Lipschitz-continuously on the data.*

2. *If* $(f, g) \in W^{1,1}(0, T; H \times Q)$, $u_0 \in V$, *and* $f(0) - Au_0 - B'p_0 \in H$, *then the solution* (u, p) *belongs to* $(W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V)) \times W^{1,\infty}(0, T; Q)$, *and depends Lipschitz-continuously on the data with respect to the topologies of these spaces.*

Note that, if the time-derivative of (f, g) is not integrable, some weak coercivity is needed for A, since (3.1) includes abstract parabolic equations as subproblems along with the choice $Q = \{0\}$. On the other hand, no closednesss of $\mathrm{Rg}(B)$ is needed whenever A is weakly coercive: the parabolic setting is better behaved than the corresponding stationary one.

SOME FURTHER ISSUES. – (i) In a forthcoming paper we shall deal with the two degenerate relaxation dynamics

$$\begin{pmatrix} \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ \dot{p} \end{pmatrix} + \begin{pmatrix} A & B' \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \ni \begin{pmatrix} f \\ g \end{pmatrix},$$

that respectively include the evolutive Stokes problem and the heat equation.

(ii) The behavior of the solution of (1.3) and of the two latter systems as $t \to +\infty$ is clearly relevant for the study of the phenomena that are represented by these models, and will be addressed apart. This issue is of paramount importance whenever (either continuous or discrete) time-relaxation is used as a mean for approximating the stationary problem. This is at the basis of the classical algorithms of Uzawa, [Uza58], and Arrow-Hurwicz, [AH58], that apply to saddle problems (whereas the present framework is nonvariational).

(iii) The analysis may also be extended in other directions. For instance a positive symmetric linear and continuous operator might be inserted in the left side of (1.3). More generally, one might deal with a relaxation term of the form

$$\partial\psi\big(\dot{x}(t), \dot{p}(t)\big) \qquad \text{or} \qquad \frac{\partial}{\partial t}\big[\partial\psi\big(x(t), p(t)\big)\big],$$

ψ being a lower semicontinuous convex potential. This may be compared with the doubly-nonlinear equations studied e.g. in [AL83], [CV90], [DS81].

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