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Global Lipschitz Continuity of Solutions to Parameterized Variational Inequalities

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Dedicated to the memory of Guido Stampacchia

Abstract. – *The question of Lipschitz continuity of solutions to parameterized variational inequalities with perturbed constraint sets is considered. Under the sole Lipschitz continuity assumption on data, a Lipschitz continuity result is proved which, in particular, holds for variational inequalities modeling evolutionary network equilibrium problems. Moreover, in view of real-life applications, a long-term memory is introduced and the corresponding variational inequality model is discussed.*

1. – Introduction.

The aim of this paper is to study Lipschitz continuity of the solution to the following abstract parameterized variational inequality problem:

$$\langle F(t, x^*(t)), x - x^*(t) \rangle \geq 0, \quad \forall x \in K(t), t \in [0, T], \tag{1}$$

where the constraint set $K(t)$, $t \in [0, T]$, is closed, convex and nonempty, $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a point-to-point mapping, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. Specifically, we show the subsequent result:

THEOREM 1. – *Let the following assumptions be satisfied:*

(a) *F is strongly monotone, i.e., there exists $a > 0$ such that for $t \in [0, T]$,*

$$\langle F(t, x_1) - F(t, x_2), x_1 - x_2 \rangle \geq a \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in \mathbb{R}^n;$$

(b) *F is Lipschitz continuous with respect to x, i.e., there exists $\beta > 0$ such that, for $t \in [0, T]$,*

$$\|F(t, x_1) - F(t, x_2)\| \leq \beta \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n;$$

(c) *F is Lipschitz continuous with respect to t, i.e., there exists $M > 0$ such that, for $t_1, t_2 \in [0, T]$,*

$$\|F(t_2, x) - F(t_1, x)\| \leq M \|x\| |t_2 - t_1|, \quad \forall x \in \mathbb{R}^n;$$

(d) *there exists* $\kappa \geq 0$ *such that, for* $t_1, t_2 \in [0, T]$,

$$\|P_{K(t_2)}(z) - P_{K(t_1)}(z)\| \leq \kappa |t_2 - t_1|, \quad \forall z \in \mathbb{R}^n,$$

where $P_{K(t)}(z) = \arg\min_{x \in K(t)} \|z - x\|$, $t \in [0, T]$ *denotes the projection onto the set* $K(t)$.

Then, the unique solution $x^*(t)$, $t \in [0, T]$, *to* (1) *is Lipschitz continuous on* $[0, T]$. *Moreover, for any couple* $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$, *the following estimate holds:*

$$\frac{\|x^*(t_2) - x^*(t_1)\|^2}{|t_2 - t_1|^2} \leq \gamma \left(\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} + \sup_{\substack{t_1, t_2 \in [0,T] \\ t_1 \neq t_2}} \left\| \frac{P_{K(t_2)}(z) - P_{K(t_1)}(z)}{t_2 - t_1} \right\|^2 \right), \tag{2}$$

where $\gamma = \gamma(a, \beta, M, T, L)$.

For the sake of simplifying notations we set $[\mathcal{P}_K] = \sup_{\substack{t_1, t_2 \in [0,T] \\ t_1 \neq t_2}} \left\| \frac{P_{K(t_2)}(z) - P_{K(t_1)}(z)}{t_2 - t_1} \right\|$.

As it is well-known, in virtue of Rademacher's theorem (see [24]), the existence of bounded derivatives almost everywhere in $[0, T]$ follows from the Lipschitz continuity and the subsequent estimate holds:

$$\left\| \frac{dx^*}{dt} \right\|^2_{L^\infty([0,T];\mathbb{R}^n)} \leq \gamma \left(\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} + \sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 \neq t_2}} \left\| \frac{P_{K(t_2)}(z) - P_{K(t_1)}(z)}{t_2 - t_1} \right\|^2 \right). \tag{3}$$

Moreover, it is worth mentioning that the solution to (1) is continuous on $[0, T]$. In fact, assumptions (b), (c) and (d) of Theorem 1 ensure that all the hypotheses of the following recent result (see [2]) are fulfilled:

THEOREM 2. – *Let* $K(t)$, $t \in [0, T]$, *be a family of nonempty convex closed subsets of* $\mathbb{R}^n$ *such that* $K(t_n)$ *converges to* $K(t)$ *in Mosco's sense, for any sequence* $\{t_n\}_{n \in \mathbb{N}}$, *with* $t_n \to t$, *as* $n \to +\infty$.

Let $F \in C([0, T] \times \mathbb{R}^n_+, \mathbb{R}^n_+)$ *and satisfying the following assumptions:*

i) *there exist* $B \in L^2([0, T])$ *and* $A \in L^\infty([0, T])$ *such that for* $t \in [0, T]$

$$\|F(t, x(t))\| \leq A(t)\|x(t)\| + B(t), \quad \forall x(t) \in K(t);$$

ii) *there exists* $a > 0$ *such that for* $t \in [0, T]$

$$\langle F(t, x_1(t)) - F(t, x_2(t)), x_1(t) - x_2(t) \rangle \geq a\|x_1(t) - x_2(t)\|^2, \quad \forall x_1(t), x_2(t) \in K(t).$$

Then the evolutionary variational inequality problem

$$x^*(t) \in K(t) : \langle F(t, x^*(t)), x - x^*(t) \rangle \geq 0, \forall x \in K(t), t \in [0, T],$$

admits a unique solution $x^*(t) \in K(t)$ *and* $x^*(t) \in C([0, T], \mathbb{R}^n_+)$.

We also remark that assumption (d) in Theorem 1 is always verified for the typical constraint set of dynamical network equilibrium problems (see [8, 9, 10, 11, 19, 20, 31, 32, 33, 40, 41, 42]), namely for a constraint set of the type

$$K(t) = \Big\{ x \in \mathbb{R}^n : x_j(t) \geq \lambda_j(t), j = 1, \ldots, n; \quad \sum_{j \in J_1} x_j(t) = d_1(t), \ldots, \sum_{j \in J_l} x_j(t) = d_l(t) \Big\}, t \in [0, T], \tag{4}$$

where $\bigcup_{j=1}^{l} J_j = \{1, \ldots, n\}$, $J_h \cap J_k = \emptyset$, for $h \neq k$, provided that the functions $d_j(t), \lambda_j(t) : [0, T] \to \mathbb{R}^+$ are assumed to be Lipschitz continuous and $d_h(t) - \sum_{j \in J_h} \lambda_j(t) > 0$, $h = 1, \ldots, l$. In Section 3 the variation rate of projections onto constraint set (4) will be discussed by means of a Lagrange theory approach for suitable optimization problems.

Giving a thorough analysis about the origin, the outstanding development and the well-recognized utility of variational inequalities is out of the scope of this paper, however, for the sake of completeness, we refer to the works [11, 21, 22, 26, 30, 33] and the references therein.

This paper aims to present some outcomes on sensitivity analysis and, of course, a comparison with previous and well-established results is necessary. It is impossible to list here all relevant references devoted to sensitivity analysis, we refer only to some of them that, in our view, are particularly influential or good entry points to the literature ([14, 15, 16, 17], [27, 28, 29, 30], [34, 35, 36, 37, 39, 39], [43, 44, 46, 47, 52, 53]) and, in the following, we focus our attention only on those papers which contain an updated survey.

In [29] Section 6, local differentiability results for variational inequalities with perturbed constraint sets are presented, assuming, in addition to the Lipschitz continuity on data, the existence of first and second derivatives with respect to x and the parameter. Moreover, a strong regularity condition is required. The technique is based on the fact that the variational inequality can be rewritten as an equivalent generalized equation, provided that the Lagrange multipliers exist.

The comprehensive paper [44] contains an updated review of the state of the art on sensitivity analysis and Section 5 is devoted to continuity and differentiability properties of solutions to parameterized generalized equations. Also here, in addition to a constraint qualification condition, differentiability assumptions on data are necessary in order to get local Hölder continuity and Lipschitz continuity. The results are obtained by means of a deep study of the corresponding contingent derivative multifunctions.

Papers [52] and [53] deserve particular attention. In these papers, the author, under the same assumptions as ours, but considered in a local sense, shows the local Hölder continuity of degree $\frac{1}{2}$ of the solution in [52], and the Lipschitz continuity when the constraints set is of type $K(\lambda) = \{x \geq 0 : Ax \geq \lambda\}$ in [53]. These results contributed to the outcomes of this paper, even if they are related to particular cases.

Finally, paper [35] contains a result about the directional differentiability at the point considered of the perturbed local solution set under second order regularity assumptions.

The result obtained in the present paper can also be considered from another point of view, namely as a regularization result related to the variational inequality:

$$\text{(5)} \qquad \text{Find } x^*(t) \in K \text{ such that } \int_0^T \langle F(t, x^*(t)), x - x^*(t)\rangle dt \geq 0, \quad \forall x \in K,$$

where

$$\text{(6)} \qquad \begin{aligned} K = \Big\{ & x \in L^2([0,T];\mathbb{R}^n) : x_j(t) \geq \lambda_j(t), \ \text{a.e. in } [0,T], j = 1, \dots, n; \\ & \sum_{j \in J_1} x_j(t) = d_1(t), \dots, \sum_{j \in J_l} x_j(t) = d_l(t), \ \text{a.e. in } [0,T] \Big\}. \end{aligned}$$

$\bigcup_{j=1}^{l} J_j = \{1, \dots, n\}$, $J_h \cap J_k = \emptyset$, for $h \neq k$, $d_j(t), \lambda_j(t) \geq 0$ a.e. in $[0,T], j = 1, \dots, n$, $d_h(t) - \sum_{j \in J_h} \lambda_j(t) > 0$, a.e. in $[0,T]$, $h = 1, \dots, l$.

The above variational inequality expresses in a unified manner time-dependent network equilibrium problems, such as the traffic equilibrium problem, the spatial price equilibrium problem, the financial equilibrium problem, the Walras equilibrium problem (see [11] for details).

Variational inequality (5) can be equivalently written as

$$\text{(7)} \qquad x^*(t) \in K(t) : \langle F(t, x^*(t)), x - x^*(t)\rangle \geq 0, \quad \forall x \in K(t), \text{a.e in } [0,T],$$

or, assuming data to be continuous, as follows

$$\text{(8)} \qquad x^*(t) \in K(t) : \langle F(t, x^*(t)), x - x^*(t)\rangle \geq 0, \quad \forall x \in K(t), \forall t \in [0,T].$$

Therefore our main result, on the lines of Theorem 6 in [2], shows that the solution $x^*(t) \in L^2([0,T];\mathbb{R}^n)$ belongs to $C^0([0,T],\mathbb{R}^n)$ and $\dfrac{dx^*(t)}{dt}$ belongs to $L^\infty([0,T],\mathbb{R}^n)$.

In this paper we also consider a notable extension of the time-dependent variational inequalities, including the presence of a memory term (see next section), which expresses how the current-time model is affected by the behavior in the previous time interval. In this case the variational inequality problem has the form

$$x^*(t) \in K :$$

$$\int_0^T \left(\langle C(t, x^*(t)), x - x^*(t) \rangle + \left\langle \int_0^t I(t-s) x^*(s) ds, x - x^*(t) \right\rangle \right) dt \geq 0, \forall x \in K, \tag{9}$$

or, equivalently, since K is the constraint set of the equilibrium problem,

$$x^*(t) \in K(t) :$$

$$\langle C(t, x^*(t)), x - x^*(t) \rangle + \left\langle \int_0^t I(t-s) x^*(s) ds, x - x^*(t) \right\rangle \geq 0, \tag{10}$$

$$\forall x \in K(t), \text{a.e in } [0, T].$$

Here I is a nonnegative definite $n \times n$ matrix with entries $I_{jr} \in L^2([0, T], \mathbb{R})$.

The resulting problem explicitly takes account of the contribution of the equilibrium solution from the initial time to the observation time and includes it in the operator (which can represent, for instance, traversal path cost in transportation problems; utility or risk aversion in financial equilibrium problems; supply excess or demand excess in economic market equilibrium problems) as an adjustment factor. In fact, problem (9) can be regarded as a variational inequality whose operator is the generalized function $F(t, x(t)) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, with $F_j(t, x(t)) = C_j(t, x(t)) + \int_0^t \sum_{r=1}^n I_{jr}(t-s) x_r(s) ds, \ j = 1, \ldots, n$. As a consequence, problem (9) describes a network model governed by the typical equilibrium conditions satisfied by the generalized operator. Also for this generalized problem a Lipschitz continuity result holds. In fact we are able to show the following theorem.

THEOREM 3. – *Let us suppose that assumptions (a)-(d) of Theorem* 1 *hold and, in addition, let us assume that*

(e) I is Lipschitz continuous on $[0, T]$, *namely there exists $L > 0$ such that, for* $t_1, t_2 \in [0, T]$

$$\|I(t_2) - I(t_1)\| \leq L|t_2 - t_1|.$$

Moreover, I is nonnegative definite for any $t \in [0, T]$. *Then the unique solution* $x^*(t)$ *to* (10) *is Lipschitz continuous on* $[0, T]$ *and the estimate* (2) *holds, with constant* $\gamma = \gamma(\alpha, \beta, M, T, L, \|I\|_{C^0([0,T];\mathbb{R}^{n \times n})})$.

The paper is organized as follows. Section 2 focuses on the interpretation of the memory term and its theoretical implications on model description. Section 3 is devoted to estimate the variation rate of projections onto time-dependent constraint sets describing the most common network equilibrium models. Section 4 presents our main theorems on Lipschitz continuity of the solution, and Section 5 provides a numerical example. Finally, Section 6 draws conclusions and suggests some possible research issues.

2. – The memory term in network equilibria.

The theory of an elastic body was pioneered by Boltzman [3, 5] who gave a first mathematical formulation to hereditary phenomena, where the deformations of a body are studied along with the history of the deformations under which it was subjected in the past. Later Volterra [48, 49, 50] gave his contribution to the theory of elasticity, introducing some hereditary coefficients in form of integral term in the constitutive equations for an elastic body with memory. Starting from the 1960s, see [7, 18], the principle of fading memory was advanced, suggesting that a body is able to recollect only its recent past and thus all the history before can be neglected. As a consequence, the memory term represents only the history in the time interval $(0, t)$ and all the previous events cannot affect the body behavior.

Since then, notable applications in different fields have been studied. In economics we may refer to [4], where the dynamics of market adjustment processes are described via a Volterra integral term. More recently, the integral memory term has been used in order to represent some physical characteristics of the quantities involved in mechanical and engineering problems. For instance, it may describe the relaxation tensor in viscoelastic contact models as in [1], or the conductivity of an electrolyte in electrochemical machining as in [23, 46].

Inspired by these applied problems, we suggest to introduce the memory integral term in the framework of network equilibria, thus leading to a refinement of the model. In fact, we explicitly incorporate the contribution of flows from the initial time to the observation time t, which causes the presence of the memory term. Hence, we are able to analyze how the current equilibrium solution is affected by past equilibria. We remind that the most common equilibrium problems, expressed in terms of evolutionary variational inequalities, (see [45, 51]) have the form:

$$\text{Find } x^*(t) \in K : \int_0^T \langle C(t, x^*(t)), x - x^*(t) \rangle dt \geq 0, \quad \forall x \in K.$$

Therefore, it turns out that the first effect of the presence of the integral term is the adjustment of the operator $C(t, x(t))$ which becomes $C(t, x(t)) + \int_0^t I(t-s)x(s)ds$. This means that network agents do not only incur in the current-time dependent operator, but are also subject to the impact of all previous equilibrium solutions. As a consequence, the underlying equilibrium conditions (*e.g.* Wardrop principle, Walras low, market equilibrium conditions) are required on the full operator $F_j(t, x(t)) = C_j(t, x(t)) + \int_0^t \sum_{r=1}^n I_{jr}(t-s)x_r(s)ds$, $j = 1, \ldots, n$.

It is also worth emphasizing the role of the matrix $I(t-s)$. In fact, the entries $I_{jr}(t-s)$ can be regarded as continuous weights acting on solutions and allow us to represent the history of the past equilibrium patterns and their influence on the current one. The meaning of the integral term is then justified: it expresses, by means of a relaxation over the time interval $(0, t)$, the equilibrium distribution in which network agents incur at time t, and, hence, the effect of the previous network situation on the present one.

The memory term is also strictly connected with the concept of time shifts and delay patterns. In fact, the integral term represent the displacement, namely the delay, of the equilibrium solution trajectories, due to the previous equilibrium state. Therefore, delay effects are not only regarded as perturbation factors for the constraint set, see [40] in connection with traffic network problems, but can also be interpreted as adjustment factors of operators.

3. – Estimate of the variation rate of projections onto the equilibrium problem constraint set.

This section is devoted to estimating the variation rate of projections onto time-dependent constraint sets $[\mathcal{P}_K]$ describing the most common equilibrium problems. The typical form of the set of feasible solutions in network-based models is as follows (see [11]):

$$K(t) = \Bigg\{ x(t) \in \mathbb{R}^n : x_j(t) \geq \lambda_j(t), j = 1, \ldots, n; \quad \sum_{j \in J_1} x_j(t) = d_1(t), \ldots, \sum_{j \in J_l} x_j(t) = d_l(t) \Bigg\}, t \in [0, T], \tag{11}$$

where $\bigcup_{j=1}^{l} J_j = \{1, \ldots, n\}$; $J_h \cap J_k = \emptyset$, for $h \neq k$; $d_j(t), \lambda_j(t) : [0, T] \to \mathbb{R}^+$, $j = 1, \ldots, n$; $d_h(t) - \sum_{j \in J_h} \lambda_j(t) > 0$, $h = 1, \ldots, l$, and $|J_s| = n_s$, $s = 1, \ldots, l$, being $|J_s|$ the cardinality of the set J_s.

Without loss of generality, we can assume that $d_s(t)$, $t \in [0, T]$, is stricly positive and Lipschitz continuous with Lipschitz constant L_s. It is also noteworthy that $K(t)$ can be reduced to the case where $x_j(t) \geq 0$,

$j = 1, \ldots, n$, with the transformation $x'_j(t) = x_j(t) - \lambda_j(t)$ and under condition that $\sum\limits_{j \in J_s} x'_j(t) = d_s(t) - \sum\limits_{j \in J_s} \lambda_j(t) > 0$, $s = 1, \ldots, l$.

PROPOSITION 1. – *Let z be an arbitrary point in $\mathbb{R}^n$. Then it results*

$$\left\| P_{K(t_2)}(z) - P_{K(t_1)}(z) \right\| \le \sum_{s=1}^{l} \sqrt{n_s} |d_s(t_2) - d_s(t_1)| \le \sum_{s=1}^{l} \sqrt{n_s} L_s |t_2 - t_1|.$$

PROOF. – We start with determining the projections of a point $z \in \mathbb{R}^n$ on the sets $K(t_1)$, $K(t_2)$, $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$. To this end, we use the following well-known characterization $P_{K(t)}(z) = \arg \min\limits_{x \in K(t)} \|z - x\|$.

We first project on $K(t_1)$ and observe that

$$K(t_1) = K_{d_1}(t_1) \times \cdots \times K_{d_l}(t_1), \tag{12}$$

where,

$$K_{d_s}(t_1) = \left\{ x(t_1) \in \mathbb{R}^{n_s} : x_j(t_1) \ge 0, j = 1, \ldots, n_s; \sum_{j \in J_s} x_j(t_1) = d_s(t_1) \right\}, \quad s = 1, \ldots, l.$$

Therefore, the minimization problem of our interest, $\min_{x \in K(t_1)} \|z - x\|^2$, may be written as

$$\min_{x \in K(t_1)} \|z - x\|^2 = \min_{x \in K_{d_1}(t_1)} \sum_{j \in J_1} \left(z_j - x_j(t_1) \right)^2 + \cdots + \min_{x \in K_{d_l}(t_1)} \sum_{j \in J_l} \left(z_j - x_j(t_1) \right)^2. \tag{13}$$

Thus, we are entitled to solve independently l minimization problems. We start with fixing $s \in \{1, \ldots, l\}$ and setting $J_s = \{1, \ldots, n_s\}$, then we solve the problem

$$\min_{x \in K_{d_s}(t_1)} \sum_{j \in J_s} \left(z_j - x_j(t_1) \right)^2 = \min_{x \in K_{d_s}(t_1)} \sum_{j=1}^{n_s} \left(z_j - x_j(t_1) \right)^2. \tag{14}$$

Now we make the following change of variables

$$\begin{cases} y_1(t_1) = x_1(t_1) \\ y_2(t_1) = x_2(t_1) \\ \cdots \\ y_{n_s}(t_1) = x_1(t_1) + x_2(t_1) + \cdots + x_{n_s}(t_1) = d_s(t_1), \end{cases}$$

thus we have

$$\begin{cases} x_1(t_1) = y_1(t_1) \\ x_2(t_1) = y_2(t_1) \\ \cdots \\ x_{n_s}(t_1) = y_{n_s}(t_1) - y_1(t_1) - y_2(t_1) - \cdots - y_{n_s - 1}(t_1). \end{cases}$$

For the sake of simplifying notations from now on we set $x_j(t_1) = x_j^1$, $x_j(t_2) = x_j^2$, $y_j(t_1) = y_j^1$, $y_j(t_2) = y_j^2$, $j = 1, \dots, n_s$ and $d_s(t_1) = d_s^1$.

Since $x_{n_s}^1 \geq 0$ and $y_{n_s}^1 = d_s^1$, it results that $\sum_{j=1}^{n_s-1} y_j^1 \leq d_s^1$. Hence, the constraint set becomes

$$\widetilde{K}_{d_s}(t_1) = \left\{ y^1 \in \mathbb{R}^{n_s} : y_j^1 \geq 0, j = 1, \dots, n_s - 1, y_{n_s}^1 = d_s^1; \sum_{j=1}^{n_s-1} y_j^1 \leq d_s^1 \right\}. \tag{15}$$

The minimization problem we have to solve is

$$\min_{y^1 \in \widetilde{K}_{d_s}(t_1)} \left(\sum_{j=1}^{n_s-1} \left(y_j^1 - z_j \right)^2 + \left(d_s^1 - z_{n_s} - \sum_{j=1}^{n_s-1} y_j^1 \right)^2 \right). \tag{16}$$

The Lagrangean function associated with problem (16) is

$$\begin{aligned} \mathcal{L}(y, \lambda, \mu) = & \sum_{j=1}^{n_s-1} \left(y_j^1 - z_j \right)^2 + \left(d_s^1 - z_{n_s} - \sum_{j=1}^{n_s-1} y_j^1 \right)^2 \\ & - \sum_{j=1}^{n_s-1} \lambda_j(t_1) y_j^1 + \mu(t_1) \left(\sum_{j=1}^{n_s-1} y_j^1 - d_s^1 \right), \end{aligned}$$

where $\lambda(t_1) \in \mathbb{R}_+^{n_s-1}$, and $\mu(t_1) \in \mathbb{R}_+$. We set $\lambda_j(t_1) = \lambda_j^1$ and $\mu(t_1) = \mu^1$.

Then, applying well-known results on Lagrangean multipliers (see *e.g.* [12, 13] and [25] Theorem 5.3 and pages 169-172), if y^1 denotes the minimal point of problem (16), there exist $\lambda^1 \in \mathbb{R}_+^{n_s-1}$, $\mu^1 \in \mathbb{R}_+$ such that

$$\frac{\partial \mathcal{L}}{\partial y_j^1} = 2(y_j^1 - z_j) - 2\left(d_s^1 - z_{n_s} - \sum_{j=1}^{n_s-1} y_j^1 \right) - \lambda_j^1 + \mu^1 = 0, \quad j = 1, \dots, n_s - 1,$$

and it follows that

$$y_j^1 = z_j + d_s^1 - z_{n_s} - \sum_{j=1}^{n_s-1} y_j^1 + \frac{\lambda_j^1}{2} - \frac{\mu^1}{2}, \quad j = 1, \dots, n_s - 1. \tag{17}$$

Now summing up for $j = 1 \dots, n_s - 1$, we find

$$\sum_{j=1}^{n_s-1} y_j^1 = \frac{\sum_{j=1}^{n_s-1} z_j}{n_s} + \left(1 - \frac{1}{n_s} \right) d_s^1 - \left(1 - \frac{1}{n_s} \right) z_{n_s} + \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \left(1 - \frac{1}{n_s} \right) \frac{\mu^1}{2},$$

and substituting in (17), the solution may be written as

$$y_j^1 = z_j + \frac{d_s^1}{n_s} - \frac{z_{n_s}}{n_s} - \frac{\sum_{j=1}^{n_s-1} z_j}{n_s} + \frac{\lambda_j^1}{2} - \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \frac{\mu^1}{2n_s}, \quad j = 1, \dots, n_s - 1. \tag{18}$$

In order to further simplify notations, we assume that $\frac{z_{n_s}}{n_s}$, which does not depend on j, d_s^1, d_s^2, is equal to zero and we set

$$U_j^1 = z_j + \frac{d_s^1}{n_s} - \frac{\sum_{j=1}^{n_s-1} z_j}{n_s}, \quad j = 1, \ldots, n_s - 1,$$

thus the solution becomes

$$y_j^1 = U_j^1 + \frac{\lambda_j^1}{2} - \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \frac{\mu^1}{2n_s}, \quad j = 1, \ldots, n_s - 1. \tag{19}$$

It is worth noting that the following constraints also hold: $\lambda_j^1 \geq 0$, $j = 1, \ldots, n_s - 1$; $y_j^1 \geq 0$, $j = 1, \ldots, n_s - 1$; $\lambda_j^1 y_j^1 = 0$, $j = 1, \ldots, n_s - 1$; $\sum_{j=1}^{n_s-1} y_j^1 - d_s^1 \leq 0$; $\mu^1 \geq 0$; $\mu^1 \left(\sum_{j=1}^{n_s-1} y_j^1 - d_s^1 \right) = 0$. Now, projecting the point z on the set $K(t_2)$, we find the solution

$$y_j^2 = U_j^2 + \frac{\lambda_j^2}{2} - \frac{\sum_{j=1}^{n_s-1} \lambda_j^2}{2n_s} - \frac{\mu^2}{2n_s}, \quad j = 1, \ldots, n_s - 1, \tag{20}$$

where

$$U_j^2 = z_j + \frac{d_s^2}{n_s} - \frac{\sum_{j=1}^{n_s} z_j}{n_s}, \quad j = 1, \ldots, n_s - 1,$$

$\lambda_j^2 = \lambda_j^2(t_2)$ and $\mu^2 = \mu(t_2)$ are the Lagrange multipliers associated with the constraints of the set

$$\widetilde{K}_{d_s}(t_2) = \left\{ y^2 \in \mathbb{R}^{n_s} : y_j^2 \geq 0, j = 1, \ldots, n_s - 1, y_{n_s}^2 = d_s^2; \sum_{j=1}^{n_s-1} y_j^2 \leq d_s^2 \right\}. \tag{21}$$

We observe that without any loss of generality we can suppose $d_s^1 < d_s^2$, so that $U_j^1 < U_j^2$. First, we consider the case in which there exists some index $\hat{j}$ such that $U_{\hat{j}}^2 \leq 0$ and show that $y_{\hat{j}}^1 = y_{\hat{j}}^2 = 0$. In fact, if we suppose by contradiction that $y_{\hat{j}}^1, y_{\hat{j}}^2 > 0$, then $\lambda_{\hat{j}}^1 = \lambda_{\hat{j}}^2 = 0$. Thus, by (19), (20) and being $U_{\hat{j}}^1 < U_{\hat{j}}^2$, we would have

$$y_{\hat{j}}^1 = U_{\hat{j}}^1 - \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \frac{\mu^1}{2n_s} \leq 0, \quad y_{\hat{j}}^2 = U_{\hat{j}}^2 - \frac{\sum_{j=1}^{n_s-1} \lambda_j^2}{2n_s} - \frac{\mu^2}{2n_s} \leq 0,$$

which is an absurd assertion.

Now we introduce the sets

$$I_0 = \{j : 1 \leq j \leq n_s - 1, U_j^2 \leq 0\}, \quad I_+ = \{1, 2, \ldots, n_s - 1\} \setminus I_0.$$

Thus, if $j \in I_0$, then $y_j^1 = y_j^2 = 0$. For $j \in I_+$ the projections on $K(t_1)$ and $K(t_2)$ assume the subsequent reduced forms:

$$
\begin{aligned}
y_j^1 &= U_j^1 + \frac{\lambda_j^1}{2} - \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \frac{\mu^1}{2n_s} \quad \text{if } j \in I_+, \\
y_j^2 &= U_j^2 + \frac{\lambda_j^2}{2} - \frac{\sum_{j=1}^{n_s-1} \lambda_j^2}{2n_s} - \frac{\mu^2}{2n_s} \quad \text{if } j \in I_+.
\end{aligned} \tag{22}
$$

Therefore, we can study our problem assuming $U_j^2 > 0$, $j = 1, \ldots, n_s - 1$ and obtain, as a particular case, the study of (22). Hence, we can confine our study to following cases:

$$
1. \begin{cases} y_j^1 = U_j^1 + \dfrac{\lambda_j^1}{2} - \dfrac{\sum\limits_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \dfrac{\mu^1}{2n_s}, \quad j = 1, \ldots, n_s - 1 \\ U_j^1 = z_j + \dfrac{d_s^1 - \sum\limits_{j=1}^{n_s-1} z_j}{n_s}, \quad j = 1, \ldots, n_s - 1; \end{cases} \tag{23}
$$

$$
2. \begin{cases} y_j^2 = U_j^2 + \dfrac{\lambda_j^2}{2} - \dfrac{\sum\limits_{j=1}^{n_s-1} \lambda_j^2}{2n_s} - \dfrac{\mu^2}{2n_s}, \quad j = 1, \ldots, n_s - 1 \\ U_j^2 = z_j + \dfrac{d_s^2 - \sum\limits_{j=1}^{n_s-1} z_j}{n_s}, U_j^2 > 0, \quad j = 1, \ldots, n_s - 1. \end{cases} \tag{24}
$$

Taking into account that it is impossible to have

$$
\sum_{j=1}^{n_s-1} U_j^1 \le d_s^1 \text{ and } \sum_{j=1}^{n_s-1} U_j^2 \ge d_s^2,
$$

we only have to study the following cases:

1. $\sum\limits_{j=1}^{n_s-1} U_j^1 \le d_s^1 < \sum\limits_{j=1}^{n_s-1} U_j^2 \le d_s^2$;
2. $\sum\limits_{j=1}^{n_s-1} U_j^1 < \sum\limits_{j=1}^{n_s-1} U_j^2 \le d_s^1 < d_s^2$;
3. $d_s^1 < \sum\limits_{j=1}^{n_s-1} U_j^1 < \sum\limits_{j=1}^{n_s-1} U_j^2 \le d_s^2$;
4. $d_s^1 < \sum\limits_{j=1}^{n_s-1} U_j^1$ and $\sum\limits_{j=1}^{n_s-1} U_j^2 > d_s^2$.

CASE 1. – We introduce the sets

$$J_0 = \{j : 1 \le j \le n_s - 1, \lambda_j^1 > 0\}$$

$$J_+ = \{j : 1 \le j \le n_s - 1, \lambda_j^1 = 0\} = \{1, 2, \ldots, n_s - 1\} \setminus J_0,$$

and denote by ℓ the number of elements of J_0. If $j \in J_0$ we get $y_j^1 = 0$ and summing up (23) for $j \in J_0$ we obtain

$$\sum_{j \in J_0} y_j^1 = 0 = \sum_{j \in J_0} U_j^1 + \frac{n_s - \ell}{2n_s} \sum_{j \in J_0} \lambda_j^1 - \frac{\ell}{2n_s} \mu^1,$$

and hence, being

$$-\frac{1}{2n_s} \sum_{j \in J_0} \lambda_j^1 = \frac{1}{n_s - \ell} \sum_{j \in J_0} U_j^1 - \frac{\ell}{2n_s(n_s - \ell)} \mu^1, \tag{25}$$

we get

$$y_j^1 = 0 = U_j^1 + \frac{\lambda_j^1}{2} + \frac{1}{n_s - \ell} \sum_{j \in J_0} U_j^1 - \frac{1}{2(n_s - \ell)} \mu^1, \tag{26}$$

and, also for $j \in J_+$ we have

$$y_j^1 = U_j^1 + \frac{1}{n_s - \ell} \sum_{j \in J_0} U_j^1 - \frac{1}{2(n_s - \ell)} \mu^1. \tag{27}$$

Taking into account that $y_j^2 = U_j^2$, as $(U_1^2, \ldots, U_{n_s-1}^2) \in \widetilde{K}_{d_s^2}$, we obtain for $j \in J_0$

$$y_j^2 = y_j^2 - y_j^1 = U_j^2 - U_j^1 - \frac{\lambda_j^1}{2} - \frac{1}{n_s - \ell} \sum_{j \in J_0} U_j^1 + \frac{1}{2(n_s - \ell)} \mu^1. \tag{28}$$

Now, let us remark that

$$\sum_{j \in J_0} U_j^1 = \sum_{j \in J_0} U_j^2 - \frac{\ell d_s^2}{n_s} + \frac{\ell d_s^1}{n_s} \ge -\ell \frac{d_s^2}{n_s} + \ell \frac{d_s^1}{n_s}. \tag{29}$$

Thus, if $\mu^1 = 0$ and $j \in J_0$, we get from (28) and (29)

$$y_j^2 - y_j^1 \le \frac{d_s^2}{n_s} - \frac{d_s^1}{n_s} + \frac{\ell d_s^2}{n_s(n_s - \ell)} - \frac{\ell d_s^1}{n_s(n_s - \ell)} = \frac{1}{n_s - \ell}(d_s^2 - d_s^1)$$

and

$$y_j^2 - y_j^1 = y_j^2 \ge 0.$$

Analogously, if $j \in J_+$ and $\mu^1 = 0$ we have

$$y_j^2 - y_j^1 = U_j^2 - U_j^1 - \frac{1}{n_s - \ell} \sum_{j \in J_0} U_j^1 \le \frac{1}{n_s - \ell}(d_s^2 - d_s^1),$$

and, from (23), we derive

$$y_j^2 - y_j^1 = U_j^2 - U_j^1 + \frac{\sum\limits_{j=1}^{n_s-1} \lambda_j^2}{2n_s} + \frac{\mu^1}{2n_s} > U_j^2 - U_j^1 = \frac{d_s^2 - d_s^1}{n_s}.$$

If $\mu^1 > 0$ and hence $\sum\limits_{j=1}^{n_s-1} y_j^1 = d_s^1$, from (27) we get

$$\sum_{j\in J_+} y_j^1 = \sum_{j=1}^{n_s-1} U_j^1 - \frac{\sum\limits_{j\in J_0} U_j^1}{n_s - \ell} - \frac{n_s - 1 - \ell}{2(n_s - \ell)}\mu^1,$$

and

$$d_s^1 + \frac{n_s - 1 - \ell}{2(n_s - \ell)}\mu^1 \le d_s^1 - \frac{\sum\limits_{j\in J_0} U_j^1}{n_s - \ell},$$

from which we obtain $\sum\limits_{j\in J_0} U_j^1 < 0$.

Being $\sum\limits_{j=1}^{n_s-1} y_j^1 = d_s^1$ and $\sum\limits_{j=1}^{n_s-1} U_j^1 < d_s^1$, we get

$$-\frac{\mu^1}{2(n_s - \ell)} = \frac{1}{n_s - \ell - 1}\left(d_s^1 + \frac{\sum\limits_{j\in J_0} U_j^1}{n_s - \ell} - \sum_{j=1}^{n_s-1} U_j^1\right), \tag{30}$$

and from (25), (26) and (30) we obtain

$$y_j^1 = 0 = U_j^1 + \frac{\lambda_j^1}{2} + \frac{\sum\limits_{j\in J_0} U_j^1}{n_s - \ell - 1} - \frac{\sum\limits_{j=1}^{n_s-1} U_j^1}{n_s - \ell - 1} + \frac{d_s^1}{n_s - \ell - 1}, j \in J_0 \tag{31}$$

$$y_j^1 = U_j^1 + \frac{\sum\limits_{j\in J_0} U_j^1}{n_s - \ell - 1} - \frac{\sum\limits_{j=1}^{n_s-1} U_j^1}{n_s - \ell - 1} + \frac{d_s^1}{n_s - \ell - 1}, j \in J_+. \tag{32}$$

Therefore, if $j \in J_0$, since $y_j^2 = U_j^2$ and (29) holds, we have

$$y_j^2 - y_j^1 \le \frac{n_s - 1}{n_s(n_s - \ell - 1)}(d_s^2 - d_s^1),$$

and $y_j^2 - y_j^1 = y_j^2 \ge 0$. Analogously, if $j \in J_+$, we get

$$y_j^2 - y_j^1 \le \frac{n_s - 1}{n_s(n_s - \ell - 1)}(d_s^2 - d_s^1),$$

and in virtue of (23) we find:

$$y_j^2 - y_j^1 = U_j^2 - U_j^1 + \frac{\sum\limits_{j=1}^{n_s-1} \lambda_j^1}{2n_s} + \frac{\mu^1}{2n_s} \ge \frac{d_s^2 - d_s^1}{n_s}.$$

CASE 2. – It is analogous to the previous one and therefore the proof will be omitted.

CASE 3. – Now let us suppose that

$$d_s^1 < \sum_{j=1}^{n_s-1} U_j^1 < \sum_{j=1}^{n_s-1} U_j^2 \leq d_s^2.$$

From (23) we get

$$\sum_{j=1}^{n_s-1} y_j^1 = \sum_{j=1}^{n_s-1} U_j^1 + \frac{1}{2n_s}\sum_{j=1}^{n_s-1} \lambda_j^1 - \frac{n_s-1}{2n_s}\mu^1,$$

and hence $\mu^1 > 0$, $\sum_{j=1}^{n_s-1} y_j^1 = d_s^1$. Introducing as done before the sets J_0 and J_+, we derive (29), (30), (31) and (32). Therefore if $j \in J_0$ we obtain

$$y_j^2 - y_j^1 \leq \frac{2n_s - 1}{n_s(n_s - \ell - 1)}(d_s^2 - d_s^1),$$

and $y_j^2 - y_j^1 = y_j^2 \geq 0$. Analogously, if $j \in J_+$, we get

$$y_j^2 - y_j^1 \leq \frac{2n_s - 1}{n_s(n_s - \ell - 1)}(d_s^2 - d_s^1),$$

and taking into account (23) we may write

$$y_j^2 - y_j^1 = U_j^2 - U_j^1 + \frac{\sum_{j=1}^{n_s-1} \lambda_j^1}{2n_s} + \frac{\mu^1}{2n_s} > U_j^2 - U_j^1 = \frac{d_s^2 - d_s^1}{n_s}.$$

CASE 4. – Now let us suppose that $d_s^1 < \sum_{j=1}^{n_s-1} U_j^1$ and $\sum_{j=1}^{n_s-1} U_j^2 > d_s^2$. Summing up (23) and (24) for $j = 1, \ldots, n_s - 1$ we find

$$\sum_{j=1}^{n_s-1} y_j^1 = \sum_{j=1}^{n_s-1} U_j^1 + \frac{1}{2n_s}\sum_{j=1}^{n_s-1} \lambda_j^1 - \frac{n_s-1}{2n_s}\mu^1,$$

$$\sum_{j=1}^{n_s-1} y_j^2 = \sum_{j=1}^{n_s-1} U_j^2 + \frac{1}{2n_s}\sum_{j=1}^{n_s-1} \lambda_j^2 - \frac{n_s-1}{2n_s}\mu^2,$$

and hence, by $d_s^1 < \sum_{j=1}^{n_s-1} U_j^1$ and $\sum_{j=1}^{n_s-1} U_j^2 > d_s^2$, we get $\mu^1 > 0$, $\mu^2 > 0$ and $\sum_{j=1}^{n_s-1} y_j^1 = d_s^1$, $\sum_{j=1}^{n_s-1} y_j^2 = d_s^2$.

Thus we obtain

$$\begin{cases} y_j^1 = U_j^1 + \dfrac{\lambda_j^1}{2} - \dfrac{\sum\limits_{j=1}^{n_s-1} \lambda_j^1}{2n_s} - \dfrac{\mu^1}{2n_s} \\ y_j^1 \geq 0, j = 1, \dots n_s - 1, \quad \sum\limits_{j=1}^{n_s-1} y_j^1 = d_s^1 \\ \lambda_j^1 \geq 0, \lambda_j^1 y_j^1 = 0, j = 1, \dots n_s - 1, \mu^1 > 0, \left(\sum\limits_{j=1}^{n_s-1} y_j^1 - d_s^1 \right) \mu^1 = 0 \end{cases}$$

$$\begin{cases} y_j^2 = U_j^2 + \dfrac{\lambda_j^2}{2} - \dfrac{\sum\limits_{j=1}^{n_s-1} \lambda_j^2}{2n_s} - \dfrac{\mu^2}{2n_s} \\ y_j^2 \geq 0, j = 1, \dots n_s - 1, \quad \sum\limits_{j=1}^{n_s-1} y_j^2 = d_s^2 \\ \lambda_j^2 \geq 0, \lambda_j^2 y_j^2 = 0, j = 1, \dots n_s - 1, \mu^2 > 0, \left(\sum\limits_{j=1}^{n_s-1} y_j^2 - d_s^2 \right) \mu^2 = 0, \end{cases}$$

namely vectors $y^1 = (y_1^1, \dots, y_{n_s-1}^1)$, $y^2 = (y_1^2, \dots, y_{n_s-1}^2)$ are solutions of two problems of the same type of (13), for which we can repeat the above procedure obtaining again a problem of type (19) of dimension $n_s - 2$. After $n_s - 3$ steps, we are led to a two dimensional problem that we can completely solve.

In fact, for $n_s - 1 = 2$ we find

$$y_1^1 = U_1^1 + \frac{\lambda_1^1}{2} - \frac{\lambda_1^1 + \lambda_2^1}{6} - \frac{\mu^1}{6}, \quad y_2^1 = U_2^1 + \frac{\lambda_2^1}{2} - \frac{\lambda_1^1 + \lambda_2^1}{6} - \frac{\mu^1}{6}.$$

Now, two cases may occur.

a) If $y_1^1, y_2^1 > 0$, then we have

$$y_1^1 = U_1^1 - \frac{\mu^1}{6}, \quad y_2^1 = U_2^1 - \frac{\mu^1}{6}, \quad y_1^1 + y_2^1 = d_s^1 = U_1^1 + U_2^1 - \frac{\mu^1}{3},$$

hence

$$\mu^1 = 3(U_1^1 + U_2^1 - d_s^1), y_1^1 = U_1^1 - \frac{U_1^1 + U_2^1 - d_s^1}{2}, y_2^1 = U_2^1 - \frac{U_1^1 + U_2^1 - d_s^1}{2}.$$

Now from the positivity of y_1^1 and y_2^1, we find the following conditions

$$U_1^1 - U_2^1 + d_s^1 > 0, U_2^1 - U_1^1 + d_s^1 > 0,$$

which imply that

$$U_1^2 - U_2^2 + d_s^2 > 0, U_2^2 - U_1^2 + d_s^2 > 0,$$

which in turn amounts to say that also $y_1^2, y_2^2 > 0$. In fact,

$$0 < U_1^1 - U_2^1 + d_s^1 = z_1 - z_2 + d_s^1 < z_1 - z_2 + d_s^2 = U_1^2 - U_2^2 + d_s^2,$$

$$0 < U_2^1 - U_1^1 + d_s^1 = z_2 - z_1 + d_s^1 < z_2 - z_1 + d_s^2 = U_2^2 - U_1^2 + d_s^2$$

and it is easy to see that it cannot be $y_1^2 = 0$, $y_2^2 = d_s^2$. We suppose by contradiction that $y_1^2 = 0$ and $y_2^2 = d_s^2$, then we have

$$y_1^2 = 0 = U_1^2 + \frac{\lambda_1^2}{3} - \frac{\mu^2}{6}, \quad y_2^2 = d_s^2 = U_2^2 - \frac{\lambda_1^2}{6} - \frac{\mu^2}{6},$$

$$y_1^2 + y_2^2 = d_s^2 = U_1^2 + U_2^2 + \frac{\lambda_1^2}{6} - \frac{\mu^2}{3}.$$

Therefore,

$$\mu^2 = 3\left(U_1^2 + U_2^2 + \frac{\lambda_1^2}{6} - d_s^2\right), y_1^2 = 0 = U_1^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2} + \frac{\lambda_1^2}{4}.$$

From the last inequality and the nonnegativity of λ_1^2, we get

$$\frac{\lambda_1^2}{4} = -U_1^2 + \frac{U_1^2 + U_2^2 - d_s^2}{2} > 0,$$

which contradicts with $U_1^2 - U_2^2 + d_s^2 > 0$. Thus, the solution is

$$y_1^2 = U_1^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2}, \quad y_2^2 = U_2^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2}.$$

Hence,

$$y_j^2 - y_j^1 = \frac{d_s^2 - d_s^1}{2}, \quad y_j^2 - y_j^1 \geq 0.$$

b) If $y_1^1 = 0$, y_2^1 must be equal to d_s^1 and hence we have $\mu^1 > 0$, $\lambda_2^1 = 0$ and

$$y_1^1 = 0 = U_1^1 + \frac{\lambda_1^1}{3} - \frac{\mu^1}{6}, \quad y_2^1 = d_s^1 = U_2^1 - \frac{\lambda_1}{6} - \frac{\mu^1}{6},$$

$$y_1^1 + y_2^1 = d_s^1 = U_1^1 + U_2^1 + \frac{\lambda_1}{6} - \frac{\mu^1}{3}.$$

Therefore,

$$\mu^1 = 3\left(U_1^1 + U_2^1 + \frac{\lambda_1^1}{6} - d_s^1\right),$$

$$y_1^1 = 0 = U_1^1 - \frac{U_1^1 + U_2^1 - d_s^1}{2} + \frac{\lambda_1^1}{4}, \quad y_2^1 = d_s^1 = U_2^1 - \frac{U_1^1 + U_2^1 - d_s^1}{2} - \frac{\lambda_1^1}{4}.$$

If we also assume that $y_1^2, y_2^2 > 0$, then we have

$$y_1^2 = U_1^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2}, \quad y_2^2 = U_2^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2}.$$

Consequently,

$$y_j^2 - y_j^1 < \frac{d_s^2 - d_s^1}{2}, \quad y_j^2 - y_j^1 \geq 0.$$

If we assume that $y_1^2 > 0, y_2^2 = 0$, reasoning as before, we get

$$\mu^2 = 3\left(U_1^2 + U_2^2 + \frac{\lambda_1^2}{6} - d_s^2\right),$$

$$y_1^2 = d_s^2 = U_2^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2} - \frac{\lambda_1^2}{4}, \quad y_2^2 = 0 = U_1^2 - \frac{U_1^2 + U_2^2 - d_s^2}{2} + \frac{\lambda_1^2}{4}.$$

Now, we observe that with easy calculations we find

$$\lambda_1^1 = 4\left(-U_1^1 + \frac{U_1^1 + U_2^1 - d_s^1}{2}\right) > 4\left(-U_1^2 + \frac{U_1^2 + U_2^2 - d_s^2}{2}\right) = \lambda_1^2.$$

Therefore,

$$y_j^2 - y_j^1 < \frac{d_s^2 - d_s^1}{2}, \quad y_j^2 - y_j^1 \geq 0.$$

In conclusion, since

$$x_j^1 = y_j^1, \quad j = 1, \dots, n_s - 1, \quad x_j^2 = y_j^2, \quad j = 1, \dots, n_s - 1,$$

it is proved that

$$0 \leq x_j^2 - x_j^1 < d_s^2 - d_s^1, \quad j = 1, \dots, n_s - 1.$$

For $j = n_s$, we have

$$|x_{n_s}^2 - x_{n_s}^1| \leq |y_{n_s}^2 - y_{n_s}^1| + \left|\sum_{j=1}^{n_s - 1} (y_j^2 - y_j^1)\right| < n_s |d_s^2 - d_s^1|.$$

Therefore, by (13) and Lipschitz property of d, we find

$$\left\|P_{K(t_1)}(z) - P_{K(t_2)}(z)\right\| = \left\| \arg\min_{x \in K(t_1)} \|z - x\|^2 - \arg\min_{x \in K(t_2)} \|z - x\|^2 \right\|$$

$$< \sum_{s=1}^{l} \sqrt{n_s}\, |d_s^2 - d_s^1| \leq \sum_{s=1}^{l} \sqrt{n_s}\, L_s |t_2 - t_1|.$$

□

REMARK 1. – Following Theorem 4.51 in [6], one can try to obtain the same result considering the problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|z - x\|^2 \text{ s.t. } Ax + b \in C,$$

where C is the polyhedral convex cone of the form $\{0\} \times \mathbb{R}^n_+$, A and b are, respectively, a matrix and a vector which express the constraints in K in the form $Ax + b \in C$. However, the procedure does not work in our case, since an essential assumption for its effectiveness, namely the linear independence of the rows of A, fails. In fact, we are dealing with a degenerate case and, for this reason, we need a direct proof.

4. – Main results.

In this section we state and prove our main outcomes. We first prove Theorem 1, which gives a Lipschitz continuity result for the solution to problem (8) and, subsequently, Theorem 3, which furnishes a similar relationship for the solution to the variational inequality problem with long-term memory (10).

PROOF OF THEOREM 1. – Characterizing the solution to problem (8) in terms of the projection operator on the time-dependent set of constraints $K(t)$, $t = t_1, t_2 \in [0, T]$, it results

$$x^*(t_1) = P_{K(t_1)}(x^*(t_1) - \lambda F(t_1, x^*(t_1))),$$
$$x^*(t_2) = P_{K(t_2)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))),$$

with $\lambda > 0$. In order to simplify notations, we set $\Delta x^* = x^*(t_2) - x^*(t_1)$ and $h = t_2 - t_1$. Hence we may write

$$\begin{aligned}
&\left\| \frac{\Delta x^*}{h} \right\|^2 \\
&= \left\| \frac{P_{K(t_2)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2)))}{h} \right. \\
&\quad \left. + \frac{P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_1) - \lambda F(t_1, x^*(t_1)))}{h} \right\|^2 \\
&\le \left(\left\| \frac{P_{K(t_2)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2)))}{h} \right\| \right. \\
&\quad \left. + \left\| \frac{P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_1) - \lambda F(t_1, x^*(t_1)))}{h} \right\| \right)^2.
\end{aligned}$$

Using inequality $(a+b)^2 \leq 2a^2+2b^2$, hypothesis (d) and the non expansivity of projections, we continue the inequality chain as follows

$$\left\| \frac{\Delta x^*}{h} \right\|^2 \leq 2 \left\| \frac{P_{K(t_2)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2)))}{h} \right\|^2$$

$$+2 \left\| \frac{P_{K(t_1)}(x^*(t_2) - \lambda F(t_2, x^*(t_2))) - P_{K(t_1)}(x^*(t_1) - \lambda F(t_1, x^*(t_1)))}{h} \right\|^2 \tag{33}$$

$$\leq 2[\mathcal{P}_K]^2 + 2 \left\| \frac{\Delta x^*}{h} - \lambda \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\|^2.$$

We now estimate the squared norm appearing in (33).

$$\left\| \frac{\Delta x^*}{h} - \lambda \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\|^2$$

$$= \left\| \frac{\Delta x^*}{h} \right\|^2 + \lambda^2 \left\| \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\|^2$$

$$-2\lambda \left\langle \frac{\Delta x^*}{h}, \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\rangle$$

$$= \left\| \frac{\Delta x^*}{h} \right\|^2 + \lambda^2 \left(\left\| \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_2))}{h} \right\| \right.$$

$$\left. + \left\| \frac{F(t_1, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\| \right)^2 - 2\lambda \left\langle \frac{\Delta x^*}{h}, \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_2))}{h} \right\rangle$$

$$-2\lambda \left\langle \frac{\Delta x^*}{h}, \frac{F(t_1, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\rangle.$$

Using inequality $(a+b)^2 \leq 2a^2+2b^2$, assumptions (a)-(c) and using inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, with $\varepsilon > 0$ sufficiently small, we get

$$\left\| \frac{\Delta x^*}{h} - \lambda \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_1))}{h} \right\|^2$$

$$\leq \left\| \frac{\Delta x^*}{h} \right\|^2 + \lambda^2 \left(2M^2 \|x^*\|^2 + 2\beta^2 \left\| \frac{\Delta x^*}{h} \right\|^2 \right)$$

$$+2\lambda \left\| \frac{\Delta x^*}{h} \right\| \cdot \left\| \frac{F(t_2, x^*(t_2)) - F(t_1, x^*(t_2))}{h} \right\| - 2\lambda a \left\| \frac{\Delta x^*}{h} \right\|^2$$

$$\leq \left\| \frac{\Delta x^*}{h} \right\|^2 \left(1 + 2\lambda^2 \beta^2 - 2\lambda a + 2\lambda \varepsilon \right) + \|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} M^2 \left(2\lambda^2 + \frac{\lambda}{2\varepsilon} \right).$$

In conclusion, we obtain

$$\left\|\frac{\Delta x^*}{h}\right\|^2 \leq 2[\mathcal{P}_K]^2 + 2\left[\left\|\frac{\Delta x^*}{h}\right\|^2 (1 + 2\lambda^2\beta^2 - 2\lambda a + 2\lambda\varepsilon)\right.$$

$$\left. + \|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} M^2\left(2\lambda^2 + \frac{\lambda}{2\varepsilon}\right)\right].$$

It is easy to verify that for opportune values of λ and sufficiently small values of ε it results that $2(1 + 2\lambda^2\beta^2 - 2\lambda a + 2\lambda\varepsilon) < 1$, hence we have

$$\left\|\frac{\Delta x^*}{h}\right\|^2 \leq 2c[\mathcal{P}_K]^2 + 2c\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} M^2\left(2\lambda^2 + \frac{\lambda}{2\varepsilon}\right),$$

where $c = \left(1 - 2\left(1 + 2\lambda^2\beta^2 - 2\lambda a + 2\lambda\varepsilon\right)\right)^{-1}$.

Then, setting $\overline{c} = \max\left\{c(1+\eta), c\left(1 + \frac{1}{\eta}\right)M^2\left(2\lambda^2 + \frac{\lambda}{2\varepsilon}\right)\right\}$, we get

$$\left\|\frac{\Delta x^*}{h}\right\|^2 \leq \overline{c}[\mathcal{P}_K]^2 + \overline{c}\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)}.$$

□

We are now able to extend the above result to problem (10).

PROOF OF THEOREM 3. – The proof proceed similarly to the previous one. Reasoning as before, after some steps, we obtain the following relationship:

$$\left\|\frac{\Delta x^*}{h}\right\|^2 \leq \overline{c}[\mathcal{P}_K]^2 + \overline{c}\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)} + \overline{c}\left\|\frac{G(t_2) - G(t_1)}{h}\right\|^2,$$

where $\overline{c} = \max\left\{2c, 2cM^2\left(2\lambda^2 + \frac{\lambda}{2\varepsilon_1}\right), 2c\left(\frac{\lambda}{2\varepsilon_2} + 2\lambda^2\right)\right\}$, with $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small.

By assumption (e), it results

$$\left\|\frac{G(t_2) - G(t_1)}{h}\right\| = \frac{1}{|h|}\left\|\int_0^{t_2} I(t_2 - s)x^*(s)ds - \int_0^{t_1} I(t_1 - s)x^*(s)ds\right\|$$

$$= \frac{1}{|h|}\left\|\int_0^{t_1} I(t_2 - s)x^*(s)ds + \int_{t_1}^{t_2} I(t_2 - s)x^*(s)ds - \int_0^{t_1} I(t_1 - s)x^*(s)ds\right\|$$

$$\leq \frac{1}{|h|}\left|\int_0^{t_1} \|I(t_2-s)-I(t_1-s)\|\|x^*(s)\|ds\right| + \frac{1}{|h|}\left|\int_{t_1}^{t_2} \|I(t_2-s)\|\|x^*(s)\|ds\right|$$
$$\leq L\int_0^{t_1} \|x^*(s)\|ds + \|I\|_{C^0([0,T];\mathbb{R}^{n\times n})}\|x^*\|_{C^0([0,T];\mathbb{R}^n)}$$
$$\leq (TL + \|I\|_{C^0([0,T];\mathbb{R}^{n\times n})})\|x^*\|_{C^0([0,T];\mathbb{R}^n)},$$

and finally we get

$$\left\|\frac{\Delta x^*}{h}\right\|^2 \leq \bar{c}[\mathcal{P}_K]^2 + \left(\bar{c} + \left(2T^2L^2 + 2\|I\|^2_{C^0([0,T];\mathbb{R}^{n\times n})}\right)\right)\|x^*\|^2_{C^0([0,T];\mathbb{R}^n)}.$$

□

5. – An example.

In this section we discuss a numerical example. Let us consider the time interval $[1/5, 4/5]$ and introduce the operator $C(x(t)) = (C_1(x(t)), C_2(x(t)))^T$, where

$$C_1(x(t)) = 3x_1(t) + 5x_2(t) + 4, \quad C_2(x(t)) = 5x_1(t) + 3x_2(t) + 1.$$

The integral terms are given by

$$G_1(x(t)) = \int_0^t (1+t-s)x_1(s)ds, \quad G_2(x(t)) = \int_0^t (2+t-s)x_1(s)ds,$$

with

$$I = \begin{bmatrix} 1+t-s & 0 \\ 0 & 2+t-s \end{bmatrix}.$$

It is immediate to prove that C satisfies assumptions (a)-(c) and I fulfills (e) of Theorem 3. We also suppose that $d(t) = 10t$, so that the constraint set is

$$K = \left\{x \in L^2([0,T];\mathbb{R}^2_+) : x_1(t) + x_2(t) = d(t), \text{ a.e. } t \in [1/5, 4/5]\right\}.$$

Thus, we have to solve the following problem

$$\int_0^T \left[\langle C(t, x^*(t)), x - x^*(t)\rangle + \left\langle \int_0^t I(t-s)x^*(s)ds, x - x^*(t)\right\rangle\right]dt \geq 0,$$
$$\forall x \in K.$$

The numerical treatment and the computer aided simulations are beyond the scope of this paper, however, we address the reader to [1] for a discussion on a fully discrete approximation. By applying the direct method, we find the solution

$$x_1^* = x_1^*(t) = 5t + \frac{3}{4} - \frac{20t^2 + 3t}{4 - t}, \quad x_2^* = x_2^*(t) = 5t - \frac{3}{4} + \frac{20t^2 + 3t}{4 - t}.$$

Now, we measure the rate of change of amount of flows x. Proceeding as in Section 3, we find

$$P_{K(t_1)}(x^*) = \left(\frac{x_1^* - x_2^* + 10t_1}{2}, \frac{10t_1 - x_1^* + x_2^*}{2} \right),$$

$$P_{K(t_2)}(x^*) = \left(\frac{x_1^* - x_2^* + 10t_2}{2}, \frac{10t_2 - x_1^* + x_2^*}{2} \right),$$

$$\left\| \frac{P_{K(t_2)}(x^*) - P_{K(t_1)}(x^*)}{t_2 - t_1} \right\| = 5\sqrt{2},$$

and hence also hypothesis (d) is verified. Therefore, we may apply Theorem 3 and deduce the existence of path flow derivatives.

6. – Conclusions.

In this paper we focused on Lipschitz continuity of solutions for a class of parameterized variational inequalities with perturbed constraint sets. Continuity and regularity properties are fundamental in applications as they ensure a better understanding of solution behavior and allow us to predict changes in the time horizon. This is of paramount importance especially in the study of network equilibrium problems, which, as it is well-known, can be expressed in terms of variational inequalities.

Using projection arguments we were able not only to prove that the solution belongs to $C^0([0, T], \mathbb{R}^n)$, but also that $\frac{dx^*(t)}{dt}$ belongs to $L^\infty([0, T])$. In our paper, for the first time, the memory term approach appears in the formulation of network-based equilibrium problems. As a result, the contribution of the equilibrium solution from the initial time to the observation time is integrated in the model and interpreted as an adjustment factor of the problem operator. In addition, we estimated the variation rate of projection operators, which allowed us to monitor and control the behavior of the constraint set, as well as to prove solution derivatives' existence.

Future extensions of the work include the following issues. First, the memory term model can be applied to a specific network equilibrium framework, such as the financial equilibrium problem, so as to detect new solution properties and model interpretations. Second, the variational inequality problem with integral memory term can be reformulated under a different structure of the convex set K.

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