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Schur-Finite Motives and Trace Identities (*)

ALESSIO DEL PADRONE - CARLO MAZZA

Abstract. – We provide a sufficient condition that ensures the nilpotency of endomorphisms universally of trace zero of Schur-finite objects in a category of homological type, i.e., a $\mathbb{Q}$-linear $\otimes$-category with a tensor functor to super vector spaces. We present some applications in the category of motives, where our result generalizes previous results about finite-dimensional objects, in particular by Kimura. We also present some facts which suggest that this might be the best generalization possible of this line of proof.

1. – Introduction

Let $\mathcal{A}$ be a $\mathbb{Q}$-linear tensor category in which idempotents split equipped with a functor $H$ to super vector spaces. The endomorphisms universally of trace zero of an object are the endomorphisms whose compositions with any other endomorphism have all trace zero. In the case $\mathcal{A}$ is a category of motives, the endomorphisms $\mathcal{N}(A)$ universally of trace zero of an object $A$ are a subset of the numerically trivial ones. According to a result by Kimura in [Kim05], if an object $A$ is finite-dimensional, then every numerically trivial endomorphism of $A$ is nilpotent. A still open question is whether the same result holds for Schur-finite objects (see [DPM05]).

Finiteness conditions for motives are related to part of the standard conjectures: in particular, if $\mathcal{A}$ is the category of Chow motives, then the finiteness of the motive of a surface with $p_g = 0$ is equivalent to Bloch’s Conjecture (see [GP03, Theorem 7]). In this paper we show (Theorem 3.1) that if $\mathcal{A}$ has the sign property (Definition 2.3) and $S_2(A) = 0$, where $\lambda$ is a partition which is not too big, i.e., it does not contain the rectangle with $a + 2$ rows and $b + 2$ columns, $a$ and $b$ being the dimensions of the even and the odd part of $H(A)$, then every endomorphism in $\mathcal{N}(A)$ is nilpotent.

We start by recalling the definitions for the different finiteness notions and their main properties. We then recall the nilpotency conjecture and how this

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relates to the various finiteness notions. Then the main result is stated and
d proved (modulo a combinatorial result) and we dedicate the rest of the section to
analyzing some related facts and reasons why this might be the sharpest result
possible using this particular line of proof. The last section of the paper will
present the applications of the theorem to the category of motives.

We keep the notation and the terminology of [DPM05], except that we will
write Tr for the categorical trace in the sense of [JSV96], tr for the ordinary trace
of matrices, and we will write $V_0|V_1 (d_0|d_1)$ for (the dimension of) the super vector
space having the $d_i$-dimensional vector space $V_i$ in degree $i$.

Let us now recall some basic facts and notations from combinatorics: a part-
tion $\lambda$ of $n$ is a sequence of integers $(\lambda_1, \ldots, \lambda_r)$ such that $\lambda_i \geq \lambda_{i+1} > 0$ for all
$i = 1, \ldots, r - 1$ and $\sum_{i} \lambda_i = n$. We will often confuse a partition with its asso-
ciated Young diagram and, e.g., we say that the partition $(b^a)$ is the rectangle with $a$ rows and $b$ columns. If $\lambda$ is a partition, we write $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ for the
transposed partition. We say that $(i, j) \in \lambda$ if $\lambda_i \geq j$. The maximal hook of $\lambda$ is the
hook $(\lambda_1, 1^{r-1})$; the maximal skew hook is the set $\{(i, j) \in \lambda \text{ s.t. } (i + 1, j + 1) \not\in \lambda\}$. If $\nu$ is the maximal (skew or not) hook, then $\lambda \setminus \nu$ is the partition
$(\lambda_2 - 1, \ldots, \lambda_r - 1)$. Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$ be two partitions, we
say that $\mu \subseteq \lambda$ if $s \leq r$ and $\mu_i \leq \lambda_i$ for all $i = 1, \ldots, s$.

2. – Definitions.

Let $\mathcal{A}$ be a pseudo-abelian $\otimes$-category, i.e., a “$\otimes$-catégorie rigide sur $F$” as
in [And04, 2.2.2] in which idempotents split. We assume that $F = \text{End}_A(1)$ is a
field and it contains $\mathbb{Q}$. The partitions $\lambda$ of an integer $n$ give a complete set of
mutually orthogonal central idempotents

$$d_\lambda := \frac{\dim V_\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma)\sigma$$

in the group algebra $\mathbb{Q}\Sigma_n$ (see [FH91]). We define an endofunctor on $\mathcal{A}$ by
setting $S_\lambda(A) = d_\lambda(A^{\otimes n})$. This is a multiple of the classical Schur functor cor-
responding to $\lambda$. In particular, we define $\text{Sym}^n(A) = S_{(n)}(A)$ and $\mathcal{A}^n(A) = S_{(1^n)}(A)$.
The following definitions are directly inspired by [Del02] and [Kim05] (see
[AK02], [GP03], and [Maz04] for further reference).

**Definition 2.1.** – An object $A$ of $\mathcal{A}$ is **Schur-finite** if there is a partition $\lambda$
such that $S_\lambda(A) = 0$. If $S_\lambda(A) = 0$ with $\lambda$ of the form $(n)$ (respectively, $\lambda = (1^n)$)
then $A$ is called **odd** (respectively, **even**). We say that $A$ is **finite-di-
men-sional** (in the sense of Kimura-O’Sullivan) if $A = A_+ \oplus A_-$ with $A_+$ even and
$A_-$ odd.
Both finite-dimensionality and Schur-finiteness are stable under direct sums, tensor products, duals, and taking direct summands. Hence every finite-dimensional object is Schur-finite, but the converse does not hold (see [DP06, 2.6.5.1]). On the other hand, this weaker condition is compatible with triangulated structures on the category while finite-dimensionality is not (see [Maz04, 3.6 and 3.8]).

One of the most important consequences of finite-dimensionality is the nilpotency of endomorphisms universally of trace zero.

**Definition 2.2.** — Recall that we have $F$-linear trace maps $\text{Tr}: \text{End}_A(A) \to \text{End}_A(1)$ compatible with $\otimes$-functors. We define the $F$-submodule of endomorphisms universally of trace zero as

$$\mathcal{N}(A) := \{ f \in \text{Hom}_A(A, A) \mid \text{Tr}(f \circ g) = 0, \text{ for all } g \in \text{Hom}_A(A, A) \}.$$ 

We say an object $A$ is a **phantom** if $\text{Id}_A \in \mathcal{N}(A)$.

André and Kahn proved in [AK02, 9.1.14] that if $A$ is a finite-dimensional object, then any $f \in \mathcal{N}(A)$ is nilpotent. In particular, if all objects of $\mathcal{A}$ are finite-dimensional, then the projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{N}$ lifts idempotents and is conservative (hence “there are no phantom objects”). In general $\otimes$-categories, Schur-finiteness is not sufficient to get the nilpotency of $\mathcal{N}(A)$; see [AK02, 10.1.1] for an example of a phantom non-zero Schur-finite object, i.e., whose identity is universally of trace zero.

In order to extend the André-Kahn result to a larger subclass of Schur-finite objects, we need to find a peculiar feature which forces the nilpotency and is expected to be true in the category of motives. We will show that the sign property is such a feature.

From now on, $\mathcal{A}$ will also be a category of **homological type** (see [Kah, 4]), i.e., a category with a $\otimes$-functor to super vector spaces $H : \mathcal{A} \to s\text{Vect}$ which we will call “cohomology” by abuse of notation.

**Definition 2.3.** — We say that an object $A$ in a category of homological type has the **sign property** if the projections on the even and the odd part of the cohomology $H(A) = H(A)_0 \oplus H(A)_1$ lift to endomorphisms in $\mathcal{A}$ (cf. [Kah, 4.8]).

The next theorem is our main result: its proof relies on a combinatorial result from [DPF]. The rest of the section will be dedicated to some related remarks some of which suggest that this might be the best generalization possible of this line of proof.
3. – The main result.

THEOREM 3.1. – Let $A$ be a category of homological type. Suppose $A$ has the sign property and let $H(A)$ be of dimension $d_0|d_1$. Let $S_\lambda(A) = 0$ for a partition $\lambda$ of $n \geq 2$ such that $(d_1 + 2)^{d_0+2} \not\subseteq \lambda$ and let $s$ be the length of the biggest hook in $\lambda$. Then for any $f \in \mathcal{N}(A)$ we have $f^{\omega(s-1)} = 0$.

PROOF. – For $\sigma \in \Sigma_n$, we index the corresponding decomposition of $\{1, \ldots, n\}$ into disjoint cycles $\gamma_1, \ldots, \gamma_n$ so that the support of $\gamma_1$ contains $1$; moreover we define $l_i$ to be the order of the cycle $\gamma_i$, and $L = L(\sigma) := \max\{l_i\}$ to be the maximum length of the cycles of $\sigma$.

As $S_\lambda(A) = 0$ we have $\sum_{\sigma} \chi_\lambda(\sigma) \cdot t_\sigma \cdot f_{\gamma_1} = 0$ for any $f_1, \ldots, f_n \in \text{End}_A(A)$. By the Murnaghan-Nakayama rule (see [FH91, Problem 4.45]) $\chi_\lambda(\sigma) = 0$ if $L(\sigma) > s$. Hence [AK02, 7.2.6] with $A_1 = \cdots = A_n = A$, gives that in $\text{End}_A(A)$

$$\sum_{\sigma \in \Sigma_n: L(\sigma) \leq s} \chi_\lambda(\sigma) \cdot t_\sigma \cdot f_{\gamma_1} = 0,$$

where $f_{\gamma_1} := f_{\gamma_1^{-1}} \circ \cdots \circ f_{\gamma_1} \circ f_1$, $t_\sigma := \prod_{j=2}^q t_{\sigma,j}$, and $t_{\sigma,j} := \text{tr}(f_{\gamma_1^{-1}(k_j)} \circ \cdots \circ f_{k_j(k_j)} \circ f_{k_j})$ with $k_j$ any element in the support of $\gamma_j$ (if $l_1 = n$, i.e. $q = 1$, then $t_{\sigma} = 1$).

Set $f_1 := \text{Id}_A$ and $f_1 = \ldots = f_s := f$ (still no restrictions on $f_{s+1}, \ldots, f_n$). If $\text{Supp}(\gamma_1) \subseteq \{1, \ldots, s\}$, not all of the $f_i$'s are in the composition $f_{\gamma_1}$, hence at least one of them must appear in a trace $\text{tr}(f_{\gamma_1^{-1}(k_j)} \circ \cdots \circ f_{k_j(k_j)} \circ f_{k_j})$. But $f$ is numerically trivial, so $t_\sigma = 0$ for any such $\sigma$, and

$$0 = \sum_{\sigma \in \Sigma_n: \text{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \cdot f_{\gamma_1} =$$

$$\left( \sum_{\sigma \in \Sigma_n: \text{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \right) F^{\omega(s-1)} = x \cdot F^{\omega(s-1)},$$

where $x := \sum_{\sigma \in \Sigma_n: \text{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \in F$. It is enough to show $x \neq 0$ for some choice of the $f_i$'s.

If $r = 0$ then $\lambda = \nu = (n - j, 1^j)$ is itself a hook, $t_\sigma = 1$ for any $\sigma$ with $l_1 = n$ and by [FH91, Exercise 4.16] $x$ is just $(n-1)!(-1)^{j} \neq 0$, hence $\mathcal{N}(A)$ is nilpotent.

If $\lambda$ is not a hook let $\delta := \lambda \setminus \nu$. The element $x \in F$ is a sum over $\sigma = \gamma_1 \circ \sigma'$ such that $\gamma_1$ is an $s$-cycle of $\{1, \ldots, s\}$ and $\sigma'$ is a permutation of $\{s + 1, \ldots, n\}$, so by Murnaghan-Nakayama $\chi_\lambda(\sigma) = \chi_{\lambda \setminus \nu}(\sigma')$, and $x = (-1)^{a_\delta} |\{s \text{ cycles of } \Sigma_n\}| \sum_{\sigma' \in \Sigma, \chi_{\delta}(\sigma') \cdot t_\sigma$. Thus we are reduced to study elements of the form

$$y(\delta; g_1, \ldots, g_r) := \sum_{\sigma \in \Sigma_r} \chi_{\delta}(\sigma) \cdot \prod_{j=1}^q t_{\sigma,j},$$

where we can choose freely $g_1, \ldots, g_r \in \text{End}_A(A)$. 
Since $A$ has the sign property then there exist two endomorphisms $\pi_0$ and $\pi_1$ such that $t_i := \text{Tr}(H(\pi_i)) = (-1)^id_i$ ($i = 0, 1$), and we have the following trace identities:

(T1) $t_i = \text{Tr}(\pi_i) = \text{Tr}(\pi_i^l) \in \mathbb{Z}$ for all $l > 0$, and

(T2) $\text{Tr}(\pi_j \circ \cdots \circ \pi_{j_k}) = 0$ for any $k > 1$ and any non-constant map $j: \{1, \ldots, k\} \to \{0, 1\}$.

Let us choose $g_1 = \ldots = g_r = g := a_0\pi_0 + a_1\pi_1$, then it is an immediate consequence of the properties (T1) and (T2) that $y$ becomes a polynomial in $a_0, a_1, t_0, t_1$

$$y(\delta; g) := y(\delta; g, \ldots, g) = \frac{\dim V_\delta}{|\delta|!} \sum_{\sigma \in S_\delta} \chi_\delta(\sigma) \prod_{j=1}^q (a_0^jt_0 + a_1^jt_1)$$

Since $H$ is a tensor functor, $S_\lambda(H(A)) = H(S_\lambda(A))$. But $S_\lambda(H(A)) = 0$ if and only if $(d_0 + 1, d_1 + 1) \in \lambda$ (see [Del02, 1.9]) and therefore $S_\lambda(A) = 0$ implies that $(d_0 + 1, d_1 + 1) \in \lambda$, and, in particular, $(d_0, d_1) \in \delta$. But by hypothesis, we have that $(d_0 + 2, d_1 + 2) \notin \lambda$ and then $(d_0 + 1, d_1 + 1) \notin \delta$. So $(d_0, d_1)$ is in the maximal skew hook of $\delta$.

From [DPM, Cor 5.3], it follows that $y(\delta; g)$ is the polynomial $P(\delta; a_0, a_1; t_0, t_1)$ in $\mathbb{Q}[a_0, a_1, t_0, t_1]$ which, when computed for $t_0 = d_0$ and $t_1 = -d_1$, is a non-zero polynomial in $a_0$ and $a_1$. Since the coefficients of this polynomial are in a field of characteristic zero, this proves the theorem.

**Remark 3.2.** – Proving that the function $y(\lambda \setminus \nu; -\ldots,-)$ is not zero is a combinatorial problem because it does not depend on the choice of the category. In particular, it can be calculated on a super-vector space.

**Remark 3.3.** – If $S_\lambda(A) = 0$ and its cohomology is of super dimension $d_0\mid d_1$, then $\lambda \supseteq ((d_1 + 1)^{d_0+1})$, and there exist $f_1, \ldots, f_r$ such that $y(\lambda \setminus \nu; f_1, \ldots, f_r) \neq 0$ only if $\lambda \supseteq ((d_1 + 2)^{d_0+2})$ (see the statement of Berele’s result in [Ber88, 3.1]).

4. – Motives and nilpotency.

For a general reference, see [And04, Ch. 4]. For any admissible equivalence ~ on algebraic cycles, motives of smooth projective varieties over a field $k$ with coefficients in $F$ form a pseudo-abelian $\otimes$-category $\mathcal{M}_\sim(k)_F$. If $X$ is a variety, we write $h(X)$ for its motive. For any $f \in \text{End}_k(h(X))$, $\text{tr}(f) = \deg(T_f \cdot \Delta_X)$ and therefore $\mathcal{N}(h(X)) = \mathcal{Z}_\sim^{\dim(X)}(X \times X)_{F,\text{num}}$ (numerically trivial correspondences of degree zero).

In the special case of Chow motives, the André-Kahn result [AK02b, 9.1.14] generalizes a previous result by Kimura ([Kim05, 7.5]) who also conjectured in
loc. cit. that all Chow motives are finite-dimensional, and hence all $N(b(X))$ are
nilpotent. Moreover, the conjectures of Bloch-Beilinson-Murre (together with the
$N(b(X))$ standard conjecture) imply the nilpotency of all endomorphism algebras and this implies the finite-dimensionality of each object.

The category of motives is of homological type for any Weil cohomology $H$
when the adequate equivalence relation is finer than the homological equivalence
generated by $H$. In this case, $tr(f) = \sum_j (-1)^j Tr(f|H^j(X))$ by the Lefschetz
formula. The sign property in this case is known as the sign conjecture ([And04,
5.1.3] and [Jan07, p. 426]): we say that $X$ satisfies the conjecture $C^+(X)$ if the
projections on the even and the odd part of the cohomology are algebraic. This is
a part of the conjecture on the algebraicity of the Chow-Kühneth decomposition
of the diagonal. The main difference is that we do not require the lifts to be
idempotents or orthogonal, although they are so in cohomology. It can be shown
that $C^+(X)$ is equivalent to the finite-dimensionality of the motive of $X$ modulo
homological equivalence.

Let $H$ be any Weil cohomology, and let $X$ be a smooth projective variety. The
cohomology $H(X)$ is a super vector space of dimension $(d_{ev}, d_{odd})$, and we set
$\lambda_{H(X)} := (d_{odd} + 1)^{d_{ev}+1} (\text{the rectangle with } d_{odd} + 1 \text{ columns and } d_{ev} + 1 \text{ rows}).$
By [Del02, 1.9], $S_\lambda(H(X)) \neq 0$ if and only if $\lambda \not\supset \lambda_{H(X)}$. Hence, $S_\lambda(b(X)) \neq 0$ if
$\lambda \not\supset \lambda_{H(X)}$. So $S_\lambda(b(X)) = 0$ implies that $\lambda \supset \lambda_{H(X)}$.

**Proposition 4.1.** – Let $X$ be a smooth projective variety, and let $\lambda$ be a
partition such that $\lambda \not\supset (d_{odd}(X) + 2)^{d_{ev}(X)+2}$. If $S_\lambda(b(X)) = 0$ (and hence $\lambda \supset \lambda_{H(X)}$) and $C^+(X)$ holds, then any $f$ in $N(b(X))$ is nilpotent. Moreover, if
$X$ is a surface with $p_g = 0$, Bloch’s conjecture holds for $X$.

**Proof.** – The motive $b(X)$ satisfies the conditions of Theorem 3.1. Bloch’s
conjecture for surfaces with $p_g = 0$ is now a formal consequence of [Kim05, 7.6
and 7.7].

**Remark 4.2.** – By 3.1, we see that the index of nilpotency of an $f \in N(A)$
depends on the (minimal) partition whose Schur-functor kills $A$, which in turn
depends on the super-dimension of $H(A)$ as a $\mathbb{Q}$-vector space. However, for any
pure motive $b(X)$, the global nilpotency bound for $N(b(X))$ should be $\dim(X) + 1$
by Bloch-Beilinson-Murre’s conjectures ([Jan94, Conjecture 2.1 (strong e)]). As a
partial evidence, Morihiko Saito proved in [Sai] that for smooth complex surfaces
$S$ with $p_g = 0$, Bloch’s conjecture ([Jan94, Conjecture 1.8]) is equivalent to
$(CH^2(S \times S)_{\text{hom}})^3 = 0$.

**Theorem 4.3.** – Let $X$ be a smooth projective variety. Under $C^+(X)$ the fol-
lowing are equivalent:

1. $b(X)$ is Kimura-finite;
(2) $S_{dH}(b(X)) = 0$;
(3) $\mathcal{N}(b(X^n))$ is nilpotent for every $n \geq 1$.

**Proof.** – It is easy to show that 1 $\Rightarrow$ 2. For 3 $\Rightarrow$ 1 we proceed as follows. As $C^+(X)$ holds and $\mathcal{N}(b(X))$ is nilpotent, then there exist two motives $X_+$ and $X_-$ whose cohomologies are exactly the even and the odd part of $H(X)$. It is now easy to prove that $b(X) = M_+ \oplus M_-$ with $M_+$ even and $M_-$ odd because it will be enough to check it in cohomology. We need to verify 2 $\Rightarrow$ 3. Assume that $S_{dH,c}(b(X)) = 0$. From the proof of [Del02, Cor. 1.13], we find that $S_{dH,c}(b(X^n)) = S_{dH,c}(b(X)^{\otimes n}) = 0$. Since $C^+(X^n)$ holds true, Proposition 4.1 gives that $\mathcal{N}(b(X^n))$ is nilpotent. 

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