Bollettino
Unione Matematica Italiana

Alessio Del Padrone, Carlo Mazza

Schur-Finite Motives and Trace Identities

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **2** (2009), n.1, p. 37–44.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2009_9_2_1_37_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2009.

Bollettino U. M. I.
(9) **II** (2009), 37-44

Schur-Finite Motives and Trace Identities (*)

Alessio Del Padrone - Carlo Mazza

Abstract. – *We provide a sufficient condition that ensures the nilpotency of endomorphisms universally of trace zero of Schur-finite objects in a category of homological type, i.e., a $\mathbb{Q}$-linear $\otimes$-category with a tensor functor to super vector spaces. We present some applications in the category of motives, where our result generalizes previous results about finite-dimensional objects, in particular by Kimura. We also present some facts which suggest that this might be the best generalization possible of this line of proof.*

1. – Introduction

Let $\mathcal{A}$ be a $\mathbb{Q}$-linear tensor category in which idempotents split equipped with a functor H to super vector spaces. The endomorphisms universally of trace zero of an object are the endomorphisms whose compositions with any other endomorphism have all trace zero. In the case $\mathcal{A}$ is a category of motives, the endomorphisms $\mathcal{N}(A)$ universally of trace zero of an object A are a subset of the numerically trivial ones. According to a result by Kimura in [Kim05], if an object A is finite-dimensional, then every numerically trivial endomorphism of A is nilpotent. A still open question is whether the same result holds for Schur-finite objects (see [DPM05]).

Finiteness conditions for motives are related to part of the standard conjectures: in particular, if $\mathcal{A}$ is the category of Chow motives, then the finiteness of the motive of a surface with $p_g = 0$ is equivalent to Bloch's Conjecture (see [GP03, Theorem 7]). In this paper we show (Theorem 3.1) that if A has the sign property (Definition 2.3) and $S_\lambda(A) = 0$, where λ is a partition which is not too big, i.e., it does not contain the rectangle with $a+2$ rows and $b+2$ columns, a and b being the dimensions of the even and the odd part of $H(A)$, then every endomorphism in $\mathcal{N}(A)$ is nilpotent.

We start by recalling the definitions for the different finiteness notions and their main properties. We then recall the nilpotency conjecture and how this

(*) Comunicazione tenuta a Bari il 26 settembre 2007 in occasione del XVIII Congresso dell'Unione Matematica Italiana.

relates to the various finiteness notions. Then the main result is stated and proved (modulo a combinatorial result) and we dedicate the rest of the section to analyzing some related facts and reasons why this might be the sharpest result possible using this particular line of proof. The last section of the paper will present the applications of the theorem to the category of motives.

We keep the notation and the terminology of [DPM05], except that we will write Tr for the categorical trace in the sense of [JSV96], *tr* for the ordinary trace of matrices, and we will write $V_0|V_1$ $(d_0|d_1)$ for (the dimension of) the super vector space having the d_i-dimensional vector space V_i in degree i.

Let us now recall some basic facts and notations from combinatorics: a partition λ of n is a sequence of integers $(\lambda_1, \dots, \lambda_r)$ such that $\lambda_i \geq \lambda_{i+1} > 0$ for all $i = 1, \dots, r-1$ and $\sum_i \lambda_i = n$. We will often confuse a partition with its associated Young diagram and, e.g., we say that the partition (b^a) is the rectangle with a rows and b columns. If λ is a partition, we write $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ for the transposed partition. We say that $(i,j) \in \lambda$ if $\lambda_i \geq j$. The maximal hook of λ is the hook $(\lambda_1, 1^{r-1})$; the maximal skew hook is the set $\{(i,j) \in \lambda \text{ s.t. } (i+1, j+1) \notin \lambda\}$. If ν is the maximal (skew or not) hook, then $\lambda \setminus \nu$ is the partition $(\lambda_2 - 1, \dots, \lambda_r - 1)$. Let $\mu = (\mu_1, \dots, \mu_s)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$ be two partitions, we say that $\mu \subseteq \lambda$ if $s \leq r$ and $\mu_i \leq \lambda_i$ for all $i = 1, \dots, s$.

2. – Definitions.

Let $\mathcal{A}$ be a **pseudo-abelian $\otimes$-category**, i.e., a "$\otimes$-catégorie rigide sur F" as in [And04, 2.2.2] in which idempotents split. We assume that $F = \mathrm{End}_{\mathcal{A}}(\mathbb{1})$ is a field and it contains $\mathbb{Q}$. The partitions λ of an integer n give a complete set of mutually orthogonal central idempotents

$$d_\lambda := \frac{\dim V_\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma)\sigma$$

in the group algebra $\mathbb{Q}\Sigma_n$ (see [FH91]). We define an endofunctor on $\mathcal{A}$ by setting $S_\lambda(A) = d_\lambda(A^{\otimes n})$. This is a multiple of the classical Schur functor corresponding to λ. In particular, we define $\mathrm{Sym}^n(A) = S_{(n)}(A)$ and $\Lambda^n(A) = S_{(1^n)}(A)$. The following definitions are directly inspired by [Del02] and [Kim05] (see [AK02], [GP03], and [Maz04] for further reference).

DEFINITION 2.1. – *An object A of $\mathcal{A}$ is* **Schur-finite** *if there is a partition λ such that $S_\lambda(A) = 0$. If $S_\lambda(A) = 0$ with λ of the form (n) (respectively, $\lambda = (1^n)$) then A is called* **odd** *(respectively,* **even***). We say that A is* **finite-dimensional** *(in the sense of Kimura-O'Sullivan) if $A = A_+ \oplus A_-$ with A_+ even and A_- odd.*

Both finite-dimensionality and Schur-finiteness are stable under direct sums, tensor products, duals, and taking direct summands. Hence every finite-dimensional object is Schur-finite, but the converse does not hold (see [DP06, 2.6.5.1]). On the other hand, this weaker condition is compatible with triangulated structures on the category while finite-dimensionality is not (see [Maz04, 3.6 and 3.8]).

One of the most important consequences of finite-dimensionality is the nilpotency of endomorphisms universally of trace zero.

DEFINITION 2.2. – *Recall that we have F-linear* **trace** *maps* $\mathsf{Tr}:\mathrm{End}_{\mathcal{A}}(A)\longrightarrow \mathrm{End}_{\mathcal{A}}(\mathbb{1})$ *compatible with $\otimes$-functors. We define the F-submodule of* **endomorphisms universally of trace zero** *as*

$$\mathcal{N}(A) := \{f \in \mathrm{Hom}_{\mathcal{A}}(A,A) \mid \mathsf{Tr}(f\circ g) = 0, \ \text{ for all } \ g\in \mathrm{Hom}_{\mathcal{A}}(A,A)\}.$$

We say an object A is a **phantom** *if* $\mathrm{Id}_A \in \mathcal{N}(A)$.

André and Kahn proved in [AK02, 9.1.14] that if A is a finite-dimensional object, then any $f\in\mathcal{N}(A)$ is nilpotent. In particular, if all objects of $\mathcal{A}$ are finite-dimensional, then the projection functor $\mathcal{A}\to\mathcal{A}/\mathcal{N}$ lifts idempotents and is conservative (hence “there are no phantom objects”). In general $\otimes$-categories, Schur-finiteness is not sufficient to get the nilpotency of $\mathcal{N}(A)$; see [AK02, 10.1.1] for an example of a phantom non-zero Schur-finite object, i.e., whose identity is universally of trace zero.

In order to extend the André-Kahn result to a larger subclass of Schur-finite objects, we need to find a peculiar feature which forces the nilpotency and is expected to be true in the category of motives. We will show that the sign property is such a feature.

From now on, $\mathcal{A}$ will also be a category of **homological type** (see [Kah, 4]), i.e., a category with a $\otimes$-functor to super vector spaces $H:\mathcal{A}\to sVect$ which we will call “cohomology” by abuse of notation.

DEFINITION 2.3. – *We say that an object A in a category of homological type has the* **sign property** *if the projections on the even and the odd part of the cohomology $H(A) = H(A)_0\oplus H(A)_1$ lift to endomorphisms in $\mathcal{A}$ (cf.* [Kah, 4.8]).

The next theorem is our main result: its proof relies on a combinatorial result from [DPM]. The rest of the section will be dedicated to some related remarks some of which suggest that this might the best generalization possible of this line of proof.

3. – The main result.

THEOREM 3.1. – *Let $\mathcal{A}$ be a category of homological type. Suppose A has the sign property and let $H(A)$ be of dimension $d_0|d_1$. Let $S_\lambda(A) = 0$ for a partition λ of $n \geq 2$ such that $(d_1+2)^{(d_0+2)} \not\subseteq \lambda$ and let s be the length of the biggest hook in λ. Then for any $f \in \mathcal{N}(A)$ we have $f^{\circ(s-1)} = 0$.*

PROOF. – For $\sigma \in \Sigma_n$, we index the corresponding decomposition of $\{1, \ldots, n\}$ into disjoint cycles $\gamma_1, \ldots, \gamma_n$ so that the support of γ_1 contains 1; moreover we define l_i to be the order of the cycle γ_i, and $L = L(\sigma) := \max_i\{l_i\}$ to be the maximum length of the cycles of σ.

As $S_\lambda(A) = 0$ we have $\sum_\sigma \chi_\lambda(\sigma) \cdot \sigma \circ f_1 \otimes \cdots \otimes f_n = 0$ for any $f_1, \ldots, f_n \in \mathrm{End}_{\mathcal{A}}(A)$. By the Murnaghan-Nakayama rule (see [FH91, Problem 4.45]) $\chi_\lambda(\sigma) = 0$ if $L(\sigma) > s$. Hence [AK02, 7.2.6] with $A_1 = \cdots = A_n = A$, gives that in $\mathrm{End}_{\mathcal{A}}(A)$

$$\sum_{\sigma \in \Sigma_n :\, L(\sigma) \leq s} \chi_\lambda(\sigma) \cdot t_\sigma \cdot f_{\gamma_1} = 0,$$

where $f_{\gamma_1} := f_{\gamma_1^{l_1-1}(1)} \circ \cdots \circ f_{\gamma_1(1)} \circ f_1$, $t_\sigma := \prod_{j=2}^q t_{\sigma,j}$, and $t_{\sigma,j} := \mathrm{tr}(f_{\gamma_j^{l_j-1}(k_j)} \circ \cdots \circ f_{\gamma_j(k_j)} \circ f_{k_j})$ with k_j any element in the support of γ_j (if $l_1 = n$, i.e. $q = 1$, then $t_\sigma = 1$).

Set $f_1 := \mathrm{Id}_A$ and $f_2 = \ldots = f_s := f$ (still no restrictions on $f_{s+1}, \ldots, f_n$). If $\mathrm{Supp}(\gamma_1) \subsetneq \{1, \ldots, s\}$, not all of the f's are in the composition f_{γ_1}, hence at least one of them must appear in a trace $\mathrm{tr}(f_{\gamma_j^{l_j-1}(k_j)} \circ \cdots \circ f_{\gamma_j(k_j)} \circ f_{k_j})$. But f is numerically trivial, so $t_\sigma = 0$ for any such σ, and

$$0 = \sum_{\sigma \in \Sigma_n :\, \mathrm{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \cdot f_{\gamma_1} =$$

$$\left(\sum_{\sigma \in \Sigma_n :\, \mathrm{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \right) f^{\circ(s-1)} = x \cdot f^{\circ(s-1)},$$

where $x := \sum_{\sigma \in \Sigma_n :\, \mathrm{Supp}(\gamma_1) = \{1, \ldots, s\}} \chi_\lambda(\sigma) \cdot t_\sigma \in F$. It is enough to show $x \neq 0$ for some choice of the f_i's.

If $r = 0$ then $\lambda = \nu = (n-j, 1^j)$ is itself a hook, $t_\sigma = 1$ for any σ with $l_1 = n$ and by [FH91, Exercise 4.16] x is just $(n-1)!(-1)^j \neq 0$, hence $\mathcal{N}(A)$ is nilpotent.

If λ is not a hook let $\delta := \lambda \setminus \nu$. The element $x \in F$ is a sum over $\sigma = \gamma_1 \circ \sigma'$ such that γ_1 is an s-cycle of $\{1, \ldots, s\}$ and σ' is a permutation of $\{s+1, \ldots, n\}$, so by Murnaghan-Nakayama $\chi_\lambda(\sigma) = \chi_{\lambda \setminus \nu}(\sigma')$, and $x = (-1)^{a_\delta - 1} \left| \{s - \text{cycles of } \Sigma_n\} \right| \sum_{\sigma' \in \Sigma_r} \chi_\delta(\sigma') \cdot t_\sigma$. Thus we are reduced to study elements of the form

$$y(\delta; g_1, \ldots, g_r) := \sum_{\sigma \in \Sigma_r} \chi_\delta(\sigma) \cdot \prod_{j=1}^q t_{\sigma,j},$$

where we can choose freely $g_1, \ldots, g_r \in \mathrm{End}_{\mathcal{A}}(A)$.

Since A has the sign property then there exist two endomorphisms π_0 and π_1 such that $t_i := \mathsf{Tr}(H(\pi_i)) = (-1)^i d_i$ $(i = 0, 1)$, and we have the following trace identities:

(TI1) $t_i = \mathsf{Tr}(\pi_i) = \mathsf{Tr}(\pi_i^l) \in \mathbb{Z}$ for all $l > 0$, and

(TI2) $\mathsf{Tr}(\pi_{j_1} \circ \cdots \circ \pi_{j_k}) = 0$ for any $k > 1$ and any non-constant map $j\colon \{1, \ldots, k\} \longrightarrow \{0, 1\}$.

Let us choose $g_1 = \ldots = g_r = g := a_0\pi_0 + a_1\pi_1$, then it is an immediate consequence of the properties (**TI1**) and (**TI2**) that y becomes a polynomial in a_0, a_1, t_0, t_1

$$y(\delta; g) := y(\delta; g, \ldots, g) = \frac{\dim V_\delta}{|\delta|!} \sum_{\sigma \in \Sigma_{|\delta|}} \chi_\delta(\sigma) \prod_{j=1}^{q} ({a_0}^{l_j} t_0 + {a_1}^{l_j} t_1)$$

Since H is a tensor functor, $S_\lambda(H(A)) = H(S_\lambda(A))$. But $S_\lambda(H(A)) = 0$ if and only if $(d_0 + 1, d_1 + 1) \in \lambda$ (see [Del02, 1.9]) and therefore $S_\lambda(A) = 0$ implies that $(d_0 + 1, d_1 + 1) \in \lambda$, and, in particular, $(d_0, d_1) \in \delta$. But by hypothesis, we have that $(d_0 + 2, d_1 + 2) \notin \lambda$ and then $(d_0 + 1, d_1 + 1) \notin \delta$. So (d_0, d_1) is in the maximal skew hook of δ.

From [DPM, Cor 5.3], it follows that $y(\delta; g)$ is the polynomial $P(\delta; a_0, a_1; t_0, t_1)$ in $\mathbb{Q}[a_0, a_1, t_0, t_1]$ which, when computed for $t_0 = d_0$ and $t_1 = -d_1$, is a non-zero polynomial in a_0 and a_1. Since the coefficients of this polynomial are in a field of characteristic zero, this proves the theorem. □

REMARK 3.2. – Proving that the function $y(\lambda \setminus \nu; -, \ldots, -)$ is not zero is a combinatorial problem because it does not depend on the choice of the category. In particular, it can be calculated on a super-vector space.

REMARK 3.3. – If $S_\lambda(A) = 0$ and its cohomology is of super dimension $d_0|d_1$, then $\lambda \supseteq ((d_1 + 1)^{d_0+1})$, and there exist $f_1, \ldots, f_r$ such that $y(\lambda \setminus \nu; f_1, \ldots, f_r) \neq 0$ only if $\lambda \not\supseteq ((d_1 + 2)^{d_0+2})$ (see the statement of Berele's result in [Ber88, 3.1]).

4. – Motives and nilpotency.

For a general reference, see [And04, Ch. 4]. For any admissible equivalence $\sim$ on algebraic cycles, motives of smooth projective varieties over a field k with coefficients in F form a pseudo-abelian $\otimes$-category $\mathcal{M}_\sim(k)_F$. If X is a variety, we write $\mathfrak{h}(X)$ for its motive. For any $f \in \mathrm{End}_{\mathcal{A}}(\mathfrak{h}(X))$, $\mathrm{tr}(f) = \deg(\Gamma_f \cdot \Delta_X)$ and therefore $\mathcal{N}(\mathfrak{h}(X)) = \mathcal{Z}_\sim^{\dim(X)}(X \times X)_{F,\mathrm{num}}$ (numerically trivial correspondences of degree zero).

In the special case of Chow motives, the André-Kahn result [AK02b, 9.1.14] generalizes a previous result by Kimura ([Kim05, 7.5]) who also conjectured in *loc. cit.* that all Chow motives are finite-dimensional, and hence all $\mathcal{N}(\mathfrak{h}(X))$ are nilpotent. Moreover, the conjectures of Bloch-Beilinson-Murre (together with the $\mathcal{N}(\mathfrak{h}(X))$ standard conjecture) imply the nilpotency of all endomorphism algebras and this implies the finite-dimensionality of each object.

The category of motives is of homological type for any Weil cohomology H when the adequate equivalence relation is finer than the homological equivalence generated by H. In this case, $\mathrm{tr}(f) = \sum_j (-1)^j \mathrm{Tr}(f|H^j(X))$ by the Lefschetz formula. The sign property in this case is known as the sign conjecture ([And04, 5.1.3] and [Jan07, p. 426]): we say that X satisfies the conjecture $C^+(X)$ if the projections on the even and the odd part of the cohomology are algebraic. This is a part of the conjecture on the algebraicity of the Chow-Künneth decomposition of the diagonal. The main difference is that we do not require the lifts to be idempotents or orthogonal, although they are so in cohomology. It can be shown that $C^+(X)$ is equivalent to the finite-dimensionality of the motive of X modulo homological equivalence.

Let H be any Weil cohomology, and let X be a smooth projective variety. The cohomology $H(X)$ is a super vector space of dimension (d_{ev}, d_{odd}), and we set $\lambda_{H(X)} := ((d_{odd}+1)^{d_{ev}+1})$ (the rectangle with $d_{odd}+1$ columns and $d_{ev}+1$ rows). By [Del02, 1.9], $S_\lambda(H(X)) \neq 0$ if and only if $\lambda \not\supset \lambda_{H(X)}$. Hence, $S_\lambda(\mathfrak{h}(X)) \neq 0$ if $\lambda \not\supset \lambda_{H(X)}$. So $S_\lambda(\mathfrak{h}(X)) = 0$ implies that $\lambda \supset \lambda_{H(X)}$.

PROPOSITION 4.1. – *Let X be a smooth projective variety, and let λ be a partition such that $\lambda \not\supset (d_{odd}(X)+2)^{d_{ev}(X)+2}$. If $S_\lambda(\mathfrak{h}(X)) = 0$ (and hence $\lambda \supset \lambda_{H(X)}$) and $C^+(X)$ holds, then any f in $\mathcal{N}(\mathfrak{h}(X))$ is nilpotent. Moreover, if X is a surface with $p_g = 0$, Bloch's conjecture holds for X.*

PROOF. – The motive $\mathfrak{h}(X)$ satisfies the conditions of Theorem 3.1. Bloch's conjecture for surfaces with $p_g = 0$ is now a formal consequence of [Kim05, 7.6 and 7.7]. □

REMARK 4.2. – By 3.1, we see that the index of nilpotency of an $f \in \mathcal{N}(A)$ depends on the (minimal) partition whose Schur-functor kills A, which in turn depends on the super-dimension of $H(A)$ as a $\mathbb{Q}$-vector space. However, for any pure motive $\mathfrak{h}(X)$, the global nilpotency bound for $\mathcal{N}(\mathfrak{h}(X))$ should be $\dim(X)+1$ by Bloch-Beilinson-Murre's conjectures ([Jan94, Conjecture 2.1 (strong e)]). As a partial evidence, Morihiko Saito proved in [Sai] that for smooth complex surfaces S with $p_g = 0$, Bloch's conjecture ([Jan94, Conjecture 1.8]) is equivalent to $(CH^2(S \times S)_{hom})^3 = 0$.

THEOREM 4.3. – *Let X be a smooth projective variety. Under $C^+(X)$ the following are equivalent:*

(1) $\mathfrak{h}(X)$ *is Kimura-finite;*

(2) $S_{\lambda_H}(\mathfrak{h}(X)) = 0$;
(3) $\mathcal{N}(\mathfrak{h}(X^n))$ *is nilpotent for every* $n \geq 1$.

PROOF. – It is easy to show that $1 \Rightarrow 2$. For $3 \Rightarrow 1$ we proceed as follows. As $C^+(X)$ holds and $\mathcal{N}(\mathfrak{h}(X))$ is nilpotent, then there exist two motives X_+ and X_- whose cohomologies are exactly the even and the odd part of $H(X)$. It is now easy to prove that $\mathfrak{h}(X) = M_+ \oplus M_-$ with M_+ even and M_- odd because it will be enough to check it in cohomology. We need to verify $2 \Rightarrow 3$. Assume that $S_{\lambda_{H(X)}}(\mathfrak{h}(X)) = 0$. From the proof of [Del02, Cor. 1.13], we find that $S_{\lambda_{H(X^n)}}(\mathfrak{h}(X^n)) = S_{\lambda_{H(X^n)}}(\mathfrak{h}(X)^{\otimes n}) = 0$. Since $C^+(X^n)$ holds true, Proposition 4.1 gives that $\mathcal{N}(\mathfrak{h}(X^n))$ is nilpotent. □

REFERENCES

[AK02] YVES ANDRÉ - BRUNO KAHN, *Nilpotence, radicaux et structures monoïdales*, Rend. Sem. Mat. Univ. Padova, **108** (2002), 107-291.

[And04] YVES ANDRÉ, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], **17**, Société Mathématique de France (Paris, 2004).

[Ber88] ALLAN BERELE, *Trace identities and* $\mathbf{Z}/2\mathbf{Z}$*-graded invariants*, Trans. Amer. Math. Soc., **309**, no. 2 (1988), 581-589.

[Del02] PIERRE DELIGNE, *Catégories tensorielles*, Mosc. Math. J., **2**, no. 2 (2002), 227-248.

[DP06] ALESSIO DEL PADRONE, *Schur functors, nilpotency and motives*, Phd thesis, Università degli Studi di Genova, Dipartimento di Matematica, 2006 (2006).

[DPM] ALESSIO DEL PADRONE - CARLO MAZZA, *Schur finiteness and nd endomorphisms universally of trace zero via certain trace relations*, To appear in Comm. in Alg.

[DPM05] ALESSIO DEL PADRONE - CARLO MAZZA, *Schur finiteness and nilpotency*, C. R. Math. Acad. Sci. Paris, **341**, no. 5 (2005), 283-286.

[FH91] WILLIAM FULTON - JOE HARRIS, *Representation theory*, Graduate Texts in Mathematics, **129** (1991), xvi+551.

[GP03] VLADIMIR GULETSKIĬ - CLAUDIO PEDRINI, *Finite-dimensional motives and the conjectures of Beilinson and Murre*, *K*-Theory, **30**, no. 3 (2003), 243-263.

[Jan94] UWE JANNSEN, *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., **55**, (1994), 245-302.

[Jan07] UWE JANNSEN, *On finite-dimensional motives and Murre's conjecture*, Algebraic Cycles and Motives, London Math. Soc. Lecture Note Ser., **344** (2007), 420-450.

[JSV96] ANDRÉ JOYAL - ROSS STREET - DOMINIC VERITY, *Traced monoidal categories*, Math. Proc. Cambridge Philos. Soc., **119**, n0. 3 (1996), 447-468.

[Kah] BRUNO KAHN, *On the multiplicities of a motive*, Preprint, February 5, 2007, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0818/ .

[Kim05] SHUN-ICHI KIMURA, *Chow groups are finite dimensional, in some sense*, Math. Ann., **331**, no. 1 (2005), 173-201.

[Maz04] CARLO MAZZA, *Schur functors and motives*, K-Theory, **33**, no. 2 (2004), 89-106.
[Sai] MORIHIKO SAITO, *Bloch's Conjecture and Chow Motives*, Preprint, February 7, 2000, arXiv, http://arxiv.org/abs/math/0002009v2.

Alessio Del Padrone - Carlo Mazza: Dipartimento di Matematica, Università di Genova
Via Dodecaneso, 35, 16146 Genova, Italy
E-mail: delpadro@dima.unige.it E-mail: mazza@dima.unige.it

Received January 23, 2008 and in revised form May 6, 2008