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Some Developments on Dirichlet Problems with Discontinuous Coefficients

LUCIO BOCCARDO

Tu sei lo mio maestro e il mio autore;
tu sei solo colui da cui io tolsi
lo bello stilo...
(Dante: Inferno I)

Abstract. – *This paper, dedicated to the memory of Guido Stampacchia in the thirtieth anniversary of his death, starts from his lectures on Dirichlet problems of forty years ago.*

As Sergei Prokofiev named his first symphony the "Classical", since it was written in the style that Joseph Haydn would have used if he had been alive at the time, this paper strongly follows the one by Guido Stampacchia about elliptic equations with discontinuous coefficients ([8]).

1. – Roma, Istituto Matematico, 1968-1970: an important moustache in my life.

During the academic years 1968-1969 and 1969-1970, at the University of Roma, Guido Stampacchia taught the "Istituzioni di Analisi Superiore" and "Analisi Superiore" courses.

I had the luck of being one of the students of these courses. The second part of "Analisi Superiore" was devoted to the presentation[1] of the results of the paper [8].

In particular, I had the opportunity of studying his results about the Dirichlet problem for second order elliptic equations with discontinuous coefficients.

Guido Stampacchia proved that the boundary value problem

$$(1)\qquad \begin{cases} -\operatorname{div}(M(x)\nabla u - u\,E(x)) + B(x)\nabla u + \mu u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is solvable, in weak sense, in $W_0^{1,2}(\Omega)$.

This paper, dedicated to the memory of Guido Stampacchia in the thirtieth anniversary of his death, starts from his lectures on the Dirichlet problem (1).

([1]) G. Stampacchia: Corso di Analisi Superiore; Università di Roma, 1969-1970.

2. – Introduction.

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that

$$\alpha|\xi|^2 \le M(x)\xi \cdot \xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N . \tag{2}$$

We also suppose $|B|$, $|E| \in L^N(\Omega)$, $f \in L^{\frac{2N}{N+2}}(\Omega)$ and $\mu > 0$ large enough.

Guido Stampacchia proved that the boundary value problem (1) has a unique solution u such that

- if $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;
- if $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$.

Furthermore he also proved the existence (thanks to a duality method) of a solution u even if the summability of f is less than $\frac{2N}{N+2}$:

- if $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, then $u \in W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$;
- if $f \in L^1(\Omega)$, then $u \in W_0^{1,q}(\Omega)$, $\forall\, q < \frac{N}{N-1}$.

For nonlinear problems of the same type, see [6], [3], [4]. We point out that the techniques we will use issue from those of Guido Stampacchia and of these papers.

Here we study the case $\mu = 0$: the main difficulty is due to noncoercivity of the differential operator $-\operatorname{div}(M(x)\nabla v) + \operatorname{div}(v\,E(x))$.

We assume that $E(x)$ is a vector field and $f(x)$ is a function such that

$$f \in L^m(\Omega), \quad 1 \le m < \frac{N}{2}\ , \tag{3}$$

$$|E| \in L^N(\Omega), \tag{4}$$

and we consider the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

Roughly speaking, in this paper we prove that the same existence and regularity results which hold under (3) in the case $E = 0$ hold under assumptions (3) and (4).

Remark 2.1. – Of course, if $\operatorname{div}(E) = 0$, the situation is easier and looks like the nonlinear problems studied in [2], since, roughly speaking, with the use of the test functions $|u|^r u$ in (5), the contribution of

$$\int_\Omega u E \cdot \nabla(|u|^r u)$$

is zero, because of the Divergence Theorem and the boundary condition.

In a forthcoming paper ([1]), equations with coefficients E which do not belong to $(L^N(\Omega))^N$ are studied.

3. – A nonlinear approach to a linear noncoercive problem.

Recall Stampacchia's definition of truncate:

$$T_n(s) = \begin{cases} s, & \text{if } |s| \le n, \\ n\dfrac{s}{|s|}, & \text{if } |s| > n, \end{cases}$$

let $G_k(s) = s - T_k(s)$, $f_n(x) = T_n(f(x))$ and consider the following approximate problems

$$\text{(6)} \quad u_n \in W_0^{1,2}(\Omega): \; -\text{div}(M(x)\nabla u_n) = -\text{div}\left(\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + f_n(x).$$

Note that a weak solution u_n of (6) exists thanks to Schauder fixed point theorem (see also [7]). Moreover, since for every fixed n the function

$$\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}$$

belongs to $(L^\infty(\Omega))^N$, every u_n is bounded thanks to Stampacchia's boundedness theorem (see [8]).

4. – Main estimates.

Even if in this paper we assume $|E| \in L^N(\Omega)$, in this section we will only need that $|E| \in L^2(\Omega)$.

LEMMA 4.1. – *Assume* (2), $|E| \in L^2(\Omega)$ *and* $f \in L^1(\Omega)$. *Then the solutions* u_n *of* (6) *satisfy*

$$\text{(7)} \qquad \left[\int_\Omega |\log(1+|u_n|)|^{2^*}\right]^{\frac{2}{2^*}} \le \frac{1}{\mathcal{S}^2 a^2}\int_\Omega |E|^2 + \frac{2}{\mathcal{S}^2 a}\int_\Omega |f|,$$

where $\mathcal{S}$ *is the Sobolev constant.*

PROOF. – Take $\dfrac{u_n}{1+|u_n|}$ as test function in (6). We have

$$\int_\Omega \frac{M(x)\nabla u_n\cdot\nabla u_n}{(1+|u_n|)^2} = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\cdot\frac{\nabla u_n}{(1+|u_n|)^2} + \int_\Omega \frac{f_n\, u_n}{1+|u_n|}\,.$$

Since $\dfrac{|u_n|}{1+|u_n|}\leq 1$ we have, using (2) and the fact that $|f_n|\leq|f|$,

$$a\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^2}\leq \int_\Omega \frac{E\cdot\nabla u_n}{1+|u_n|}+\int_\Omega |f|\,,$$

so that (thanks to Young inequality),

$$\frac{a}{2}\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^2}\leq \frac{1}{2a}\int_\Omega |E|^2+\int_\Omega |f|,$$

which implies

$$\frac{\mathcal{S}^2 a}{2}\left[\int_\Omega |\log(1+|u_n|)|^{2^*}\right]^{\frac{2}{2^*}}\leq \int_\Omega |\nabla\log(1+|u_n|)|^2\leq \frac{1}{2a}\int_\Omega |E|^2+\int_\Omega |f|,$$

which is (7). □

REMARK 4.2. – Remark that for any $\varepsilon>0$, it is possible to choose k_ε such that

$$\text{meas}\{x\in\Omega:|u_n(x)|>k\}^{\frac{2}{2^*}}\leq\varepsilon,\quad \forall\, k>k_\varepsilon,$$

thanks to the estimate (7), which implies

(8) $$\text{meas}\{x\in\Omega:|u_n(x)|>k\}^{\frac{2}{2^*}}\leq \frac{1}{|\log(1+k)|^2}\int_\Omega\left[\frac{1}{S^2a^2}|E|^2+\frac{2}{S^2a}|f|\right].$$

□

LEMMA 4.3. – *Assume* (2), $|E|\in L^2(\Omega)$ *and* $f\in L^1(\Omega)$. *Then, for any* $k\in\mathbb{R}^+$, *the sequence* $T_k(u_n)$ *is bounded in* $W_0^{1,2}(\Omega)$. *More precisely we have:*

(9) $$\frac{a}{2}\int_\Omega |\nabla T_k(u_n)|^2\leq \frac{k^2}{2a}\int_\Omega |E|^2+k\int_\Omega |f|$$

PROOF. – The proof follows using $T_k(u_n)$ as test function in (6) and Young inequality as in the proof of Lemma 4.1. □

5. – Finite energy solutions.

5.1 – *Unbounded solutions.*

LEMMA 5.1. – *Assume* (2), (3) *and* (4). *If* $m = \dfrac{2N}{N+2}$, *then there exists* k^* *such that the sequence* $\{G_k(u_n)\}$ *is bounded in* $W_0^{1,2}(\Omega)$, *for every* $k > k^*$.

PROOF. – Define

$$A_n(k) = \{x \in \Omega : k \le |u_n(x)|\}.$$

The use of $G_k(u_n)$ as test function in (6), and Young, Hölder and Sobolev inequalities imply that, for $\varepsilon > 0$,

$$a\int_\Omega |\nabla G_k(u_n)|^2 \le \int_\Omega |G_k(u_n)||E||\nabla G_k(u_n)| + k\int_\Omega |E||\nabla G_k(u_n)| + \int_\Omega |G_k(u_n)||f|$$

$$\le \frac{1}{\mathcal{S}}\Big(\int_{A_n(k)} |E|^N\Big)^{\frac{1}{N}}\int_\Omega |\nabla G_k(u_n)|^2 + \varepsilon\int_\Omega |\nabla G_k(u_n)|^2 + \frac{k^2}{4\varepsilon}\int_{A_n(k)} |E|^2$$

$$+\varepsilon\int_\Omega |\nabla G_k(u_n)|^2 + \frac{\mathcal{S}^2}{4\varepsilon}\Big[\int_{A_n(k)} |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}},$$

where $\mathcal{S}$ is the Sobolev constant. Thus it results

$$\Big[a - \frac{1}{\mathcal{S}}\Big(\int_{A_n(k)} |E|^N\Big)^{\frac{1}{N}} - 2\varepsilon\Big]\int_\Omega |\nabla G_k(u_n)|^2$$

$$\le \frac{k^2}{4\varepsilon}\int_{A_n(k)} |E|^2 + \frac{\mathcal{S}^2}{4\varepsilon}\Big[\int_{A_n(k)} |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}.$$

Fix ε so that $2\varepsilon = \dfrac{a}{4}$. Then Lemma 4.2 implies that there exists k^*, such that

$$\frac{1}{\mathcal{S}}\Big[\int_{A_n(k)} |E|^N\Big]^{\frac{1}{N}} \le \frac{a}{4}, \quad k \ge k^*.$$

Thus, for some $C_1 > 0$, we have, if $k \ge k^*$,

$$\int_\Omega |\nabla G_k(u_n)|^2 \le C_1 k^2 \int_{A_n(k)} |E|^2 + C_1\Big[\int_{A_n(k)} |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}. \tag{10}$$

□

COROLLARY 5.2. – *Assume* (2), (3) *and* (4). *If* $m = \dfrac{2N}{N+2}$, *then the sequence* $\{u_n\}$ *is bounded in* $W_0^{1,2}(\Omega)$.

PROOF. – The estimates (9) and (10) imply that, if $k \geq k^*$,

$$\int_\Omega |\nabla u_n|^2 \leq \frac{k^2}{2a}\int_\Omega |E|^2 + k\int_\Omega |f| + C_1 k^2 \int_\Omega |E|^2 + C_1\left[\int_\Omega |f|^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}}.$$

□

THEOREM 5.3. – *Assume* (2), (3) *and* (4). *If* $m = \dfrac{2N}{N+2}$, *then there exists* $u \in W_0^{1,2}(\Omega)$ *weak solution of* (5), *that is*

$$\int_\Omega M(x)\nabla u \nabla v = \int_\Omega u\, E(x)\nabla v + \int_\Omega f\, v, \qquad \forall\ v \in W_0^{1,2}(\Omega).$$

PROOF. – Since the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, by Corollary 5.2, then, up to subsequences, u_n converges weakly in $W_0^{1,2}(\Omega)$ to a function u. Passing to the limit as n tends to infinity yields the result thanks to the linearity of the problem: therefore u is a weak solution of (5). □

LEMMA 5.4. – *Assume* (2), (3) *and* (4). *If* $\dfrac{2N}{N+2} \leq m < \dfrac{N}{2}$, *then the sequence* $\{u_n\}$ *is bounded in* $L^{m^{**}}(\Omega)$.

PROOF. – The use of $\dfrac{|G_k(u_n)|^{2(\lambda-1)}G_k(u_n)}{2\lambda-1}$, $\lambda = \dfrac{m^{**}}{2^*}$, as test function in (6) and Young inequality imply that

$$a\int_\Omega |G_k(u_n)|^{2(\lambda-1)}|\nabla G_k(u_n)|^2$$

$$\leq \int_\Omega |u_n|(|G_k(u_n)|^{\lambda-1}|E|)(|G_k(u_n)|^{\lambda-1}|\nabla G_k(u_n)|) + \frac{1}{2\lambda-1}\int_\Omega |f||G_k(u_n)|^{2\lambda-1}$$

$$\leq \frac{1}{4\varepsilon}\int_{A_n(k)} |u_n|^2|G_k(u_n)|^{2(\lambda-1)}|E|^2 + \varepsilon\int_\Omega |G_k(u_n)|^{2(\lambda-1)}|\nabla G_k(u_n)|^2$$

$$+\frac{\|f\|_m}{2\lambda-1}\left[\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right]^{\frac{1}{m'}}.$$

Taking $2\varepsilon = a$, we have, for some constants $C_i > 0$ independent on n, thanks to Sobolev inequality,

$$
(11)\qquad \begin{cases} C_1\left[\int\limits_\Omega |G_k(u_n)|^{2^*\lambda}\right]^{\frac{2}{2^*}} \le \frac{a}{2}\int\limits_\Omega |G_k(u_n)|^{2(\lambda-1)}|\nabla G_k(u_n)|^2 \\ \le C_2 \int\limits_{A_n(k)} |G_k(u_n)|^{2\lambda}|E|^2 + C_2\, k^2 \int\limits_{A_n(k)} |G_k(u_n)|^{2(\lambda-1)}|E|^2 \\ +\frac{\|f\|_m}{2\lambda-1}\left[\int\limits_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right]^{\frac{1}{m'}}. \end{cases}
$$

Hölder inequality implies that

$$
C_1\left[\int\limits_\Omega |G_k(u_n)|^{2^*\lambda}\right]^{\frac{2}{2^*}} \le C_2\left[\int\limits_\Omega |G_k(u_n)|^{2^*\lambda}\right]^{\frac{2}{2^*}}\left[\int\limits_{A_n(k)} |E|^N\right]^{\frac{2}{N}}
$$

$$
+C_2\, k^2\, (\operatorname{meas} A_n(k))^{\frac{2}{2^*\lambda}}\left[\int\limits_\Omega |G_k(u_n)|^{2^*\lambda}\right]^{\frac{2(\lambda-1)}{2^*\lambda}}\left[\int\limits_{A_n(k)} |E|^N\right]^{\frac{2}{N}}
$$

$$
+\frac{\|f\|_m}{2\lambda-1}\left[\int\limits_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right]^{\frac{1}{m'}}
$$

Now, assumption $|E| \in L^N(\Omega)$ and (8) imply that there exist $\tilde{k}$ such that, if $k \ge \tilde{k}$ (we can suppose $\tilde{k} \ge k^*$), we have that

$$
C_2\left(\int\limits_{A_n(k)} |E|^N\right)^{\frac{2}{N}} \le \frac{C_1}{2}.
$$

Note that the choice of λ gives

$$
2^*\lambda = (2\lambda-1)m' = m^{**} \quad \text{and} \quad \frac{2}{2^*} > \frac{1}{m'} \text{ if and only if } m < \frac{N}{2}.
$$

Moreover, $\dfrac{2(\lambda-1)}{2^*\lambda} < \dfrac{2}{2^*}$. Thus, for $k > \tilde{k}$, we have

$$
\frac{C_1}{2}\left[\int\limits_\Omega |G_k(u_n)|^{m^{**}}\right]^{\frac{2}{2^*}} \le C_3\, k^2\left[\int\limits_\Omega |G_k(u_n)|^{m^{**}}\right]^{\frac{2(\lambda-1)}{2^*\lambda}}\left[\int\limits_{A_n(k)} |E|^N\right]^{\frac{2}{N}}
$$

$$
+\frac{\|f\|_m}{2\lambda-1}\left[\int\limits_\Omega |G_k(u_n)|^{m^{**}}\right]^{\frac{1}{m'}}.
$$

Then, for $k > \tilde{k}$, the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, since $\frac{2}{2^*} > \frac{1}{m'}$. The conclusion follows as in Corollary 5.2. $\square$

Moreover, we can follow the proof of Theorem 5.3 in order to state the summability result below.

THEOREM 5.5. – *Assume* (2), (3) *and* (4). *If* $\frac{2N}{N+2} \leq m < \frac{N}{2}$, *then there exists a weak solution u of* (5), *which belongs to* $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.

5.2 – *Bounded solutions.*

In this section we shall prove that $|E| \in L^m(\Omega)$, $m > N$, and $f \in L^r(\Omega)$, $r > \frac{N}{2}$, imply that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, so that the solution u belongs to $L^\infty(\Omega)$ as well.

Our proof follows Stampacchia's method ([8]) and hinges on the boundedness of $\log(1 + |u|)$.

THEOREM 5.6. – *Assume* (2). *If* $|E| \in L^m(\Omega)$, $m > N$, *and* $f \in L^r(\Omega)$, $r > \frac{N}{2}$, *then there exists a weak solution* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *of* (5).

PROOF. – Take as test function in (5)

$$\psi(u) = \begin{cases} 0, & \text{if } |u(x)| \leq k, \\ \dfrac{u}{1+u} - \dfrac{k}{1+k}, & \text{if } u(x) > k, \\ \dfrac{u}{1-u} + \dfrac{k}{1+k}, & \text{if } u(x) < -k, \end{cases}$$

and use Young inequality. Then, since $|\psi(u)| \leq 1$ we have

$$\frac{a}{2} \int_{|u|>k} \frac{|\nabla u|^2}{(1+|u|)^2} \leq \frac{1}{2a} \int_{|u|>k} |E|^2 + \int_{|u|>k} |f|,$$

which implies (with $k = e^h - 1$)

$$\frac{a}{2} \int_{\log(1+|u|)>h} |\nabla \log(1+|u|)|^2 \leq \int_{\log(1+|u|)>h} \left[\frac{1}{2a}|E|^2 + |f|\right].$$

Now, since $|E|^2 + |f|$ belongs to $L^r(\Omega)$, with $r > \frac{N}{2}$, it follows from Stampacchia's theorem (see [8]) that there exists a positive constant L such that

$$\|\log(1+|u|)\|_{L^\infty(\Omega)} \leq L,$$

and so

$$\|u\|_{L^\infty(\Omega)} \leq e^L - 1.$$

□

6. – Uniqueness of weak solutions.

THEOREM 6.1. – *Assume* (2), (3), *with* $m = \dfrac{2N}{N+2}$, *and* (4). *Then the weak solution* $u \in W_0^{1,2}(\Omega)$ *of* (5) *is unique.*

PROOF. – Let u, w be weak solutions of (5) and let $\delta \in \mathbb{R}^+$. We follow the outline of [5] and we use $T_\varepsilon(u-w)$, $0<\varepsilon<\delta$ as test function to have

$$\int_\Omega M(x)\nabla(u-w)\nabla T_\varepsilon(u-w) = \int_\Omega (u-w)E(x)\nabla T_\varepsilon(u-w).$$

Using Hölder inequality and (2), we obtain

$$a^2\int_\Omega |\nabla T_\varepsilon(u-w)|^2 \leq \varepsilon^2 \int_{0<|u(x)-w(x)|<\varepsilon} |E|^2.$$

Then

$$\int_{\delta<|u(x)-w(x)|} |T_\varepsilon(u-w)|^2 \leq \int_\Omega |T_\varepsilon(u-w)|^2 \leq \frac{\varepsilon^2}{\lambda_1 a^2} \int_{0<|u(x)-w(x)|<\varepsilon} |E|^2,$$

where (as usual) λ_1 is the first eigenvalue of the Dirichlet problem for the Laplacian operator in Ω. Thus,

$$\lambda_1\,\varepsilon^2\,\mathrm{meas}\,(\{\delta<|u(x)-w(x)|\}) \leq \frac{\varepsilon^2}{a^2} \int_{0<|u(x)-w(x)|<\varepsilon} |E|^2.$$

Since

$$\bigcap_{\varepsilon>0}\{0<|u(x)-w(x)|<\varepsilon\} = \{0<|u(x)-w(x)| \leq 0\} = \emptyset\ ,$$

the continuity of the measure with respect to intersection then implies that

$$\mathrm{meas}\,(\{0<|u(x)-w(x)|<\varepsilon\}) \to 0,\ \text{as } \varepsilon \to 0.$$

Then

$$\int\limits_{0<|u(x)-w(x)|<\varepsilon} |E|^2 \to 0,$$

and so meas$\{\delta < |u(x) - w(x)|\} = 0$ for any $\delta > 0$, that is $u(x) = w(x)$ a.e. We point out that in the proof we only used the L^2-summability of $|E|$. □

7. – Infinite energy solutions.

In this section we study the existence of solutions if $|E|$ still belongs to $L^N(\Omega)$, but f does not belong to $L^{\frac{2N}{N+2}}(\Omega)$, so that we cannot expect the existence of $W_0^{1,2}(\Omega)$ solutions.

LEMMA 7.1. – *Assume* (2), (3), $1 < m < \dfrac{2N}{N+2}$, *and* (4). *Then the sequence* $\{u_n\}$ *is bounded in* $L^{m^{**}}(\Omega)$.

PROOF. – Use $\dfrac{[1+|G_k(u_n)|]^{2\gamma-1}-1}{2\gamma-1}\operatorname{sgn}(u_n)$, $\gamma = \dfrac{m^{**}}{2^*}$, as test function in (6). Remark that $\gamma > \dfrac{1}{2}$ if and only if $m > 1$. Moreover $(2\gamma-1)m' < m^{**}$, if $m < \dfrac{N}{2}$. Then, since $\dfrac{|s|}{[1+|G_k(s)|]^{1-\gamma}} \le |G_k(s)|^\gamma + k, \ \forall\, |s| > k$,

$$a\int\limits_\Omega \frac{|\nabla G_k(u_n)|^2}{[1+|G_k(u_n)|]^{2-2\gamma}} \le \frac{1}{2\gamma-1}\int\limits_\Omega |f|[1+|G_k(u_n)|]^{2\gamma-1}$$

$$+\int\limits_\Omega |G_k(u_n)|^\gamma |E| \frac{|\nabla G_k(u_n)|}{[1+|G_k(u_n)|]^{1-\gamma}} + \int\limits_\Omega k|E|\frac{|\nabla G_k(u_n)|}{[1+|G_k(u_n)|]^{1-\gamma}}.$$

Using the inequalities of Young and Sobolev and the choice of γ, we obtain

$$C_1\left[\int\limits_\Omega |[1+|G_k(u_n)|]^\gamma - 1|^{2^*}\right]^{\frac{2}{2^*}} \le \frac{a}{3}\int\limits_\Omega \frac{|\nabla G_k(u_n)|^2}{[1+|G_k(u_n)|]^{2-2\gamma}}$$

$$\le C_2\int\limits_\Omega |f|[1+|G_k(u_n)|]^{2\gamma-1} + C_2\int\limits_\Omega |G_k(u_n)|^{2\gamma}|E|^2 + C_1\int\limits_{A_n(k)} k^2|E|^2$$

$$\le C_2\|f\|_{L^m(\Omega)}\left[\int\limits_\Omega [1+|G_k(u_n)|]^{(1-2\gamma)m'}\right]^{\frac{1}{m'}}$$

$$+C_2\Bigg[\int_{A_n(k)}|E|^2\Bigg]^{\frac{2}{N}}\Bigg[\int_\Omega |G_k(u_n)|^{m^{**}}\Bigg]^{\frac{2}{2^*}}+C_2\,k^2\int_{A_n(k)}|E|^2.$$

That is

$$\begin{aligned}(12)\qquad & C_1\Bigg[\int_\Omega \big|[1+|G_k(u_n)|]^{\frac{m^{**}}{2^*}}-1\big|^{2^*}\Bigg]^{\frac{2}{2^*}}-C_2\Bigg[\int_{A_n(k)}|E|^2\Bigg]^{\frac{2}{N}}\Bigg[\int_\Omega |G_k(u_n)|^{m^{**}}\Bigg]^{\frac{2}{2^*}}\\ & \le C_3\|f\|_{L^m(\Omega)}\Bigg[\int_\Omega [1+|G_k(u_n)|]^{m^{**}}\Bigg]^{\frac{1}{m'}}+C_3\,k^2\int_{A_n(k)}|E|^2.\end{aligned}$$

Once more we can say that, for k fixed but greater of some $\tilde{k}$, we have, since $\dfrac{2}{2^*}>\dfrac{1}{m'}$, that the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, for $k>\tilde{k}$, by a constant depending on k. The conclusion then follows as in Corollary 5.2.

Moreover, going back to the gradient, we also can say that

$$\int_\Omega \frac{|\nabla G_k(u_n)|^2}{[1+|G_k(u_n)|]^{2-2\gamma}}$$

is bounded. Then

$$\int_\Omega |\nabla G_k(u_n)|^{m^*}=\int_\Omega \frac{|\nabla G_k(u_n)|^{m^*}}{[1+|G_k(u_n)|]^{(1-\gamma)\,m^*}}\,[1+|G_k(u_n)|]^{(1-\gamma)\,m^*}$$

$$\le\Bigg[\int_\Omega \frac{|\nabla G_k(u_n)|^2}{[1+|G_k(u_n)|]^{2(1-\gamma)}}\Bigg]^{\frac{m^*}{2}}\Bigg[\int_\Omega [1+|G_k(u_n)|]^{\frac{2(1-\gamma)\,m^*}{2-m^*}}\Bigg]^{\frac{2-m^*}{2}}$$

is bounded, because of the choice of γ. □

Moreover, we can follow the proof of Theorem 5.3 (here the sequence $\{u_n\}$ is bounded in $W_0^{1,m^*}(\Omega)$) in order to state the summability result below.

THEOREM 7.2. – *Assume* (2), (3) *and* (4). *If* $1<m\le\dfrac{2N}{N+2}$, *then there exists a distributional solution* u *of* (5), *which belongs to* $W_0^{1,m^*}(\Omega)$; *that is*

$$\begin{cases} u\in W_0^{1,m^*}(\Omega):\\ \displaystyle\int_\Omega M(x)\nabla u\nabla\phi=\int_\Omega u\,E(x)\nabla\phi+\int_\Omega f\phi\\ \forall\,\phi\in\mathcal{D}(\Omega)\end{cases}$$

REMARK 7.3. – A consequence of inequality (12) is the following estimate ($k > \tilde{k}$).

$$\| u\|_{m^{**}} \leq \| G_k(u)\|_{m^{**}} + \| T_k(u)\|_{m^{**}} \leq C_0\Big(k\,|\Omega|^{\frac{1}{m^{**}}} + \|f\|_m + k^{\frac{m(N-2)}{N-2m}}\| E\|_2^2\Big).$$

REMARK 7.4. – A consequence of theorems 5.5 (where u does not belong to $W_0^{1,m^*}(\Omega)$), 7.2 (where u belongs to $W_0^{1,m^*}(\Omega)$) is that $\operatorname{div}(uE)$ always belongs to $W^{-1,m^*}(\Omega)$, as well as f. Similar situations happen for the solutions u given by theorems 5.6 and 7.5.

REMARK 7.5. – Assume (2) and (4). If $f \in L^1(\Omega)$, then there exists a distributional solution u of (5).

PROOF. – Use $\dfrac{1-[1+|G_k(u_n)|]^{1-2\sigma}}{2\sigma-1}\operatorname{sgn}(u_n)$, $2\sigma > 1$, as test function in (6). Then

$$a\int_\Omega \frac{|\nabla G_k(u_n)|^2}{[1+|G_k(u_n)|]^{2\sigma}} \leq \int_\Omega \frac{|u_n|}{[1+|G_k(u_n)|]^{\sigma}}\,|E|\,\frac{|\nabla G_k(u_n)|}{[1+|G_k(u_n)|]^{\sigma}} + \int_\Omega \frac{|f|}{2\sigma-1}.$$

Starting from this inequality, we conclude the proof as in Lemma 7.1 and Theorem 5.5, taking into account that $\sigma > \dfrac{1}{2}$. □

REMARK 7.6. – In order to prove the existence of solutions exactly in $W_0^{1,1^*}(\Omega)$ a sufficient condition is $\int_\Omega |f|\log(1+|f|) < \infty$ (related problems are studied in [8] and [4]).

REMARK 7.7. – If the right hand side is a bounded measure the existence result of Theorem 7.5 still holds with the same proof.

REMARK 7.8. – Following the duality method by Guido Stampacchia, it is possible to prove existence and summability results for the solutions of the linear (dual) boundary value problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla w) + E(x)\nabla w = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

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