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Convergence of Vanishing Viscosity Approximations of 2×2 Triangular Systems of Multi-Dimensional Conservation Laws (*)

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Abstract. – *We consider a multidimensional triangular system of conservation laws. These equations arise in models of three phase flows in porous media and include multi dimensional conservation laws with discontinuous coefficients as special cases. We study approximate solutions of these equations constructed by the vanishing viscosity method and show that the approximate solutions converge to a weak solution of the multi-dimensional triangular system.*

1. – Introduction.

In this paper, we consider the 2×2 trangular system of conservation laws of the form,

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ \partial_t v + \operatorname{div}(g(u,v)) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where u and v are the unknowns, and the flux functions are $f = (f_1, f_2, \ldots, f_N)$ and $g = (g_1, \ldots, g_N)$.

Equations of the type (1.1) arise while studying flows in porous media. In a multi-dimensional porous medium, the equations for the phase saturations are 2×2 systems of conservation laws with coefficients that are determined from an elliptic pressure equation. See [4] for details on the model. A simplified version of the fluxes in the saturation equations leads to a model where the gas saturation is independent of the other phases and results in equations of the form (1.1).

From (1.1) we observe that the evolution of u is independent of v, but the evolution of v depends on u. Writing (1.1) in quasilinear form we have

$$U_t + \sum_{i=1}^{N} A_i U_{x_i} = 0$$

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where $U = \{u, v\}$ and the directional Jacobians are given by

$$A_i = \begin{pmatrix} \partial_u f_i & 0 \\ \partial_u g_i & \partial_v g_i \end{pmatrix}.$$

The matrices A_i are lower triangular, and therefore such systems are called triangular systems. Furthermore, the eigenvalues of the matrixes A_i are real, and the system is hyperbolic. Since the eigenvalues can coincide, the system is not strictly hyperbolic. Non-strictly hyperbolic systems present many difficulties even in one space dimension. In general, it is very hard to prove rigorous results for systems of conservation laws in several space dimensions. See e.g. [2] for more detailed information.

A special case of the above system occurs when we take $f = 0$. In this case, the system reduces to a multi-dimensional scalar conservation law but with a spatially varying coefficient u which can be discontinuous. Scalar conservation laws with discontinuous coefficients arise in a wide variety of contexts including two-phase flows in heterogeneous porous media, modeling of clarifier-thickener units and in traffic flow. In one spatial dimension such equations have been studied in several papers. An incomplete list of papers includes [1, 3, 6, 7, 11, 12, 15, 16] and other references therein.

Scalar conservation laws with discontinuous coefficients in several space dimensions have not been that widely studied and the theory is not as well-developed as in the one-dimensional case. In [10], the authors considered a scalar conservation law in two space dimensions with discontinuous coefficients and obtained existence of weak solutions by showing that vanishing viscosity approximations converge. In [14], the author was able to treat a multi-dimensional scalar conservation law with discontinuous coefficients in both space and time. Existence of weak solutions was shown by proving compactness of approximations generated by smoothing the coefficients and adding vanishing viscosity. The compactness technique in [14] uses the tool of H-measures extensively, and we adapt this compactness framework to the situation herein.

In one space dimension, the triangular system (1.1) was considered in [9]. Existence of weak solutions was shown by constructing finite volume schemes and showing that the approximate solutions generated by these schemes are compact and converge to a weak solution.

In this paper, we consider a different approximation of (1.1) by studying the following parabolic system,

$$\begin{cases} \partial_t u_\varepsilon + \operatorname{div}(f(u_\varepsilon)) = \varepsilon \Delta u_\varepsilon, & x \in \mathbb{R}^N, \ t > 0, \\ \partial_t v_\varepsilon + \operatorname{div}(g(u_\varepsilon, v_\varepsilon)) = \varepsilon \Delta v_\varepsilon, & x \in \mathbb{R}^N, \ t > 0, \\ u_\varepsilon(x, 0) = u_{0,\varepsilon}(x), \ v_\varepsilon(0, x) = v_{0,\varepsilon}(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

This system represents a viscous regularization of the conservation laws. We are interested in the behaviour of the approximate solutions when $\varepsilon \to 0$ i.e., the vanishing viscosity limit. Under suitable assumptions on the fluxes and the initial data, we show that the approximate solutions generated by the viscous approximation (1.2) converge to a weak solution of (1.1). This result provides an alternative proof of existence of weak solutions of (1.1). Our main tools are the use of entropy estimates and the compactness framework developed in [14].

Furthermore, this paper is a prequel to a forthcoming paper where we will consider finite volume approximations to (1.1). Working with viscosity approximations is technically simpler than working with difference approximations, but the core techniques are similar. In this way, the present paper serves as a motivation for our work with finite volume approximations.

The rest of the paper is organized as follows, in Section 2, we outline the mathematical framework for the rest of this paper. The main convergence theorem is stated and proved in Section 3.

2. – Mathematical Framework.

We assume that the initial data in (1.1) satisfy the following assumptions.

(A.1) $f \in C^1([-M,M];\mathbb{R}^N)$, $g \in C^2([-M,M]^2;\mathbb{R}^N)$;
(A.2) $\partial_u g(\cdot,\pm M) = \partial_v g(\pm M,\cdot) = 0$;
(A.3) $\partial^2_{uu} g(\cdot,v)$ is Lipschitz continuous for each $-M \le v \le M$;
(A.4) $g(u,\cdot)$ is genuinely nonlinear for each $-M \le u \le M$, namely the map $v \in [-M,M] \mapsto \langle g(u,v),\xi\rangle$ is not affine on any nontrivial interval for every $-M \le u \le M$ and $\xi \in \mathbb{R}^N$, $|\xi| = 1$;
(A.5) $u_0 \in BV(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\|u_0\|_{L^\infty(\mathbb{R}^N)}, \|v_0\|_{L^\infty(\mathbb{R}^N)} \le M$,

for some positive constants M and $N \ge 1$.

Furthermore, the initial data for the parabolic problem (1.2) satisfy the assumptions,

$$\begin{aligned} &u_{0,\varepsilon},\ v_{0,\varepsilon} \in H^{N+1}(\mathbb{R}^N),\ \|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}, \|v_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} \le M,\ \varepsilon > 0,\\ &\sup_{\varepsilon>0}\|\nabla u_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)},\ \sup_{\varepsilon>0}\|\nabla v_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)} < \infty,\\ &u_{0,\varepsilon} \to u_0, v_{0,\varepsilon} \to v_0 \text{ in } \mathrm{L}^1(\mathbb{R}^\mathrm{N}) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.1}$$

In particular we have that

$$\sup_{\varepsilon>0}\|u_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)},\quad \sup_{\varepsilon>0}\|v_{0,\varepsilon}\|_{L^2(\mathbb{R}^N)} < \infty. \tag{2.2}$$

We will consider weak solutions of (1.1) defined below,

DEFINITION 2.1. – *Let $u, v: \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ be two functions. We say that the pair (u, v) is a weak solution of the Cauchy problem* (1.1) *if*

(D.1) $u,\ v \in L^\infty(\mathbb{R}^N \times (0,\infty))$;
(D.2) $u,\ v$ *satisfy* (1.1) *in the sense of distributions on $\mathbb{R}^N \times [0,\infty)$;*
(D.3) *for each constant $c \in \mathbb{R}$ the inequality*

$$\partial_t |u-c| + \operatorname{div}(\operatorname{sign}(u-c)(f(u)-f(c))) \leq 0$$

holds in the sense of distributions on $\mathbb{R}^N \times [0,\infty)$.

We will show existence of weak solutions defined above in the next section by proving that the viscous approximations of (1.1) converge in the vanishing viscosity limit. To this end, we need the following result of Panov ([14], Theorem 5).

LEMMA 2.1 [see [14, Theorem 5]]. – *Let u be the unique entropy solution of the single conservation law*

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.3}$$

and let $\{v_\nu\}_{\nu>0}$ be a family of functions defined on $\mathbb{R}^N \times (0,\infty)$. If $\{v_\nu\}_{\nu\in\mathbb{N}}$ lies in a bounded set of $L^1_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$ and for every constant $c \in \mathbb{R}$ the family

$$\{\partial_t |v_\nu - c| + \operatorname{div}(\operatorname{sign}(v_\nu - c)(g(u, v_\nu) - g(u, c)))\}_{\nu>0}$$

lies in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$, then there exist a sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\nu_n \to 0$, and a map $v \in L^\infty(\mathbb{R}^N \times (0,\infty))$ such that

$$v_{\nu_n} \to v \quad \text{a.e. and in } L^p_{\text{loc}}(\mathbb{R}^N \times (0,\infty)),\ 1 \leq p < \infty.$$

We also need the following technical lemma from [13],

LEMMA 2.2. – *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{\text{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$.

3. – Convergence Results.

The aim of this section is to prove that solutions $(u_\varepsilon, v_\varepsilon)$ of the parabolic system (1.2) converge (up to a subsequence) to a weak solution (u, v) of (1.1) as $\varepsilon \to 0$. The first step is to collect some standard estimates (see for instance [8]) regarding the viscous approximations of the first equation in (1.2).

LEMMA 3.1. – *For $\varepsilon > 0$, the family u_ε satisfies the following estimates:*

(i.) *(L^∞ estimate): Let $\varepsilon > 0$. We have that*

$$-M \leq u_\varepsilon(x,t) \leq M \quad \text{for each } x \in \mathbb{R}^N,\ t \geq 0.$$

(ii.) *(L^1 & BV estimates): Let $\varepsilon > 0$ and $i \in \{1, ..., N\}$. The functions*

$$t \mapsto \|u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^N)},$$
$$t \mapsto \left\|\partial_{x_i} u_\varepsilon(\cdot,t)\right\|_{L^1(\mathbb{R}^N)},$$
$$t \mapsto \|\partial_t u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^N)}$$

are non-increasing. In particular, the family $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty((0,\infty); L^1(\mathbb{R}^N)) \cap BV(\mathbb{R}^N \times (0,\infty))$.

(iii.) *(L^2 estimate): The following estimate holds*

$$\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + 2\varepsilon \int_0^t \|\nabla u_\varepsilon(\cdot,s)\|^2_{L^2(\mathbb{R}^N)} ds = \left\|u_{0,\varepsilon}\right\|^2_{L^2(\mathbb{R}^N)},$$

for each $t \geq 0$.

Furthermore, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_n \to 0$, such that

$$u_{\varepsilon_n} \to u \quad \text{a.e. and in } L^p_{\text{loc}}(\mathbb{R}^N \times (0,\infty)),\ 1 \leq p < \infty,$$

where $u \in L^\infty(\mathbb{R}^N \times (0,\infty)) \cap BV(\mathbb{R}^N \times (0,\infty))$ is the unique entropy solution of (2.3).

The next step is to prove some estimates on the approximate solutions v_ε. We start with the following estimate,

LEMMA 3.2 (L^∞ estimate). – *Let $\varepsilon > 0$. We have that*

$$-M \leq v_\varepsilon(x,t) \leq M \qquad \text{for each } x \in \mathbb{R}^N,\ t \geq 0.$$

PROOF. – Due to (**A.2**) the maps with constant values M and $-M$ solve the second equation of (1.2). Hence the claim is consequence of (2.1) and of the comparison principle for parabolic equations. □

We remark that assumption (**A.2**) is just a sufficient condition to obtain L^∞ bounds and can be relaxed. Next, we prove bounds on v_ε in L^2 below,

LEMMA 3.3. – *[L^2 estimate and Entropy Dissipation] Let (η, Q) be such that*

$$\begin{aligned} &\eta \in C^2([-M,M]), \qquad Q \in C^2([-M,M]^2;\mathbb{R}^N),\\ &\partial_v Q(u,v) = \partial_v g(u,v)\eta'(v), \qquad \eta(0) = Q(u,0) = 0. \end{aligned} \tag{3.1}$$

The following estimate holds

$$\int_{\mathbb{R}^N} \eta(v_\varepsilon(t,x))dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} \eta''(v_\varepsilon)|\nabla v_\varepsilon|^2 ds dx \le \int_{\mathbb{R}^N} \eta(v_{0,\varepsilon})dx + C(\eta,Q)t,$$

for each $\varepsilon > 0$, $t \ge 0$, where

$$C(\eta,Q) = \Big(\|\eta'\|_{L^\infty([-M,M])}\|\partial_u g\|_{L^\infty([-M,M]^2)} + \|\partial_u Q\|_{L^\infty([-M,M]^2)}\Big) \sup_{\varepsilon>0}\|\nabla u_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)}.$$

In the special case

$$\eta(u) = \frac{u^2}{2}$$

we have that

$$\|v_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + 2\varepsilon \int_0^t \|\nabla v_\varepsilon(\cdot,s)\|^2_{L^2(\mathbb{R}^N)} ds \le \|v_{0,\varepsilon}\|^2_{L^2(\mathbb{R}^N)} + C_1 t,$$

for each $\varepsilon > 0$, $t \ge 0$, where

$$C_1 = M\Big(2M\|\partial^2_{uv} g\|_{L^\infty([-M,M]^2)} + \|\partial_u g\|_{L^\infty([-M,M]^2)}\Big) \sup_{\varepsilon>0}\|\nabla u_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)}.$$

PROOF. – Fix positive ε and t. By (1.2) and (3.1)

$$\begin{aligned} &\partial_t \eta(v_\varepsilon) + \operatorname{div}(Q(u_\varepsilon, v_\varepsilon))\\ &\quad = \eta'(v_\varepsilon)\partial_t v_\varepsilon + \langle \partial_u Q(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon\rangle + \langle \partial_v Q(u_\varepsilon, v_\varepsilon), \nabla v_\varepsilon\rangle\\ &\quad = \varepsilon\eta'(v_\varepsilon)\Delta v_\varepsilon - \langle \eta'(v_\varepsilon)\partial_u g(u_\varepsilon, v_\varepsilon) - \partial_u Q(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon\rangle\\ &\quad = \varepsilon\Delta(\eta(v_\varepsilon)) - \varepsilon\eta''(v_\varepsilon)|\nabla v_\varepsilon|^2 - \langle \eta'(v_\varepsilon)\partial_u g(u_\varepsilon, v_\varepsilon) - \partial_u Q(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon\rangle. \end{aligned}$$

Integrating over $\mathbb{R}^N$ and using Lemmas 3.1 and 3.2 we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N} \eta(v_\varepsilon)dx + \varepsilon\int_{\mathbb{R}^N} \eta''(v_\varepsilon)|\nabla v_\varepsilon|^2 dx\\ &\qquad \le \big(\|\eta'\|_{L^\infty([-M,M])}\|\partial_u g\|_{L^\infty([-M,M]^2)} + \|\partial_u Q\|_{L^\infty([-M,M]^2)}\big)\|\nabla u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Finally one more integration over $(0,t)$, Lemma 3.1, and equation (2.1) proves the claim.

We are in a position to state and prove the main convergence theorem.

THEOREM 3.1. – *There exist a sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_n \to 0$, *and a weak solution* (u,v) *of* (1.1) *such that*

$$u_{\varepsilon_n} \to u,\ v_{\varepsilon_n} \to v \quad a.e. \text{ and in } L^p_{\mathrm{loc}}(\mathbb{R}^N \times (0,\infty)),\ 1 \leq p < \infty.$$

PROOF. – We want to apply Lemma 2.1. Let $c \in \mathbb{R}$ be fixed, we claim that the family

$$\{\partial_t |v_\varepsilon - c| + \operatorname{div}(\operatorname{sign}(v_\varepsilon - c)(g(u,v_\varepsilon) - g(u,c)))\}_{\varepsilon>0}$$

is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^N \times (0,\infty))$. For the sake of notational simplicity we introduce the following notations

$$\begin{aligned}
\eta_0(v) &= |v-c| - |c|,\\
Q_0(u,v) &= \operatorname{sign}(v-c)\big(g(u,v) - g(u,c)\big) - \operatorname{sign}(-c)\big(g(u,0) - g(u,c)\big),
\end{aligned}$$

and we observe that

$$\eta_0(0) = Q_0(u,0) = 0,$$

$$\begin{aligned}
\partial_t |v_\varepsilon - c| &+ \operatorname{div}(\operatorname{sign}(v_\varepsilon - c)(g(u,v_\varepsilon) - g(u,c)))\\
&= \partial_t \eta_0(v_\varepsilon) + \operatorname{div}(Q_0(u,v_\varepsilon)) + \operatorname{sign}(-c)\operatorname{div}((g(u,0) - g(u,c))).
\end{aligned} \tag{3.2}$$

Let $\{(\eta_\varepsilon, Q_\varepsilon)\}_{\varepsilon>0}$ be a family of maps such that

$$\begin{aligned}
&\eta_\varepsilon \in C^2([-M,M]), \quad Q_\varepsilon \in C^2([-M,M]^2;\mathbb{R}^N),\\
&\partial_v Q_\varepsilon(u,v) = \partial_v g(u,v)\eta'_\varepsilon(v), \quad \eta''_\varepsilon \geq 0,\\
&\|\eta_\varepsilon - \eta_0\|_{L^\infty([-M,M])} \leq \varepsilon, \quad \|\eta'_\varepsilon - \eta'_0\|_{L^1([-M,M])} \leq \varepsilon,\\
&\|\eta'_\varepsilon\|_{L^\infty([-M,M])} \leq 1, \quad \eta_\varepsilon(0) = Q_\varepsilon(u,0) = 0,
\end{aligned} \tag{3.3}$$

for each $\varepsilon > 0$. Since

$$Q_0(u,v) = \int_0^v \partial_v g(u,\xi)\eta'_0(\xi)d\xi \quad \text{and} \quad Q_\varepsilon(u,v) = \int_0^v \partial_v g(u,\xi)\eta'_\varepsilon(\xi)d\xi,$$

we also have

$$\begin{aligned}
&\|\partial_u Q_\varepsilon\|_{L^\infty([-M,M]^2)} \leq M\|\partial^2_{uv} g\|_{L^\infty([-M,M]^2)},\\
&\|Q_\varepsilon - Q_0\|_{L^\infty([-M,M]^2)} \leq \|\partial_v g\|_{L^\infty([-M,M]^2)}\,\varepsilon.
\end{aligned} \tag{3.4}$$

By (1.2)

$$\begin{aligned}
&\partial_t\eta_0(v_\varepsilon) + \mathrm{div}(Q_0(u, v_\varepsilon))\\
&\quad = \partial_t\eta_\varepsilon(v_\varepsilon) + \mathrm{div}(Q_\varepsilon(u_\varepsilon, v_\varepsilon)) + \partial_t(\eta_0(v_\varepsilon) - \eta_\varepsilon(v_\varepsilon))\\
&\qquad + \mathrm{div}(Q_0(u, v_\varepsilon) - Q_\varepsilon(u, v_\varepsilon)) + \mathrm{div}(Q_\varepsilon(u, v_\varepsilon) - Q_\varepsilon(u_\varepsilon, v_\varepsilon))\\
&\quad = \varepsilon\eta'_\varepsilon(v_\varepsilon)\Delta v_\varepsilon\\
&\qquad - \left\langle \eta'_\varepsilon(v_\varepsilon)\partial_u g(u_\varepsilon, v_\varepsilon) - \partial_u Q_\varepsilon(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon \right\rangle + \partial_t(\eta_0(v_\varepsilon) - \eta_\varepsilon(v_\varepsilon))\\
&\qquad + \mathrm{div}(Q_0(u, v_\varepsilon) - Q_\varepsilon(u, v_\varepsilon)) + \mathrm{div}(Q_\varepsilon(u, v_\varepsilon) - Q_\varepsilon(u_\varepsilon, v_\varepsilon))\\
&\quad = \varepsilon\Delta(\eta_\varepsilon(v_\varepsilon)) - \varepsilon\eta''_\varepsilon(v_\varepsilon)|\nabla v_\varepsilon|^2\\
&\qquad - \left\langle \eta'_\varepsilon(v_\varepsilon)\partial_u g(u_\varepsilon, v_\varepsilon) - \partial_u Q_\varepsilon(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon \right\rangle + \partial_t(\eta_0(v_\varepsilon) - \eta_\varepsilon(v_\varepsilon))\\
&\qquad + \mathrm{div}(Q_0(u, v_\varepsilon) - Q_\varepsilon(u, v_\varepsilon)) + \mathrm{div}(Q_\varepsilon(u, v_\varepsilon) - Q_\varepsilon(u_\varepsilon, v_\varepsilon)).
\end{aligned}$$

Therefore, thanks to (3.2)

$$\begin{aligned}
\text{(3.5)}\quad &\partial_t|v_\varepsilon - c| + \mathrm{div}(\mathrm{sign}(v_\varepsilon - c)(g(u, v_\varepsilon) - g(u, c)))\\
&\qquad = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + I_{4,\varepsilon} + I_{5,\varepsilon} + I_{6,\varepsilon} + I_7
\end{aligned}$$

where

$$\begin{aligned}
I_{1,\varepsilon} &= \varepsilon\Delta(\eta_\varepsilon(v_\varepsilon)),\\
I_{2,\varepsilon} &= -\varepsilon\eta''_\varepsilon(v_\varepsilon)|\nabla v_\varepsilon|^2,\\
I_{3,\varepsilon} &= -\left\langle \eta'_\varepsilon(v_\varepsilon)\partial_u g(u_\varepsilon, v_\varepsilon) - \partial_u Q_\varepsilon(u_\varepsilon, v_\varepsilon), \nabla u_\varepsilon \right\rangle,\\
I_{4,\varepsilon} &= \partial_t(\eta_0(v_\varepsilon) - \eta_\varepsilon(v_\varepsilon)),\\
I_{5,\varepsilon} &= \mathrm{div}(Q_0(u, v_\varepsilon) - Q_\varepsilon(u, v_\varepsilon)),\\
I_{6,\varepsilon} &= \mathrm{div}(Q_\varepsilon(u, v_\varepsilon) - Q_\varepsilon(u_\varepsilon, v_\varepsilon)),\\
I_7 &= \mathrm{sign}(-c)\mathrm{div}((g(u, 0) - g(u, c))).
\end{aligned}$$

Due to Lemmas 3.1, 3.2, 3.3 and (2.1), (3.3), (3.4) we have

$\{I_{1,\varepsilon}\}_{\varepsilon>0}$ is converging in $H^{-1}(\mathbb{R}^N \times (0, T))$ for each $T > 0$,

$\{I_{2,\varepsilon}\}_{\varepsilon>0}$ and $\{I_{3,\varepsilon}\}_{\varepsilon>0}$ are bounded in $L^1(\mathbb{R}^N \times (0, T))$ for each $T > 0$,

$\{I_{4,\varepsilon}\}_{\varepsilon>0}$ and $\{I_{5,\varepsilon}\}_{\varepsilon>0}$ are converging in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^N \times (0, \infty))$,

$\{I_{6,\varepsilon}\}_{\varepsilon>0}$ is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^N \times (0, \infty))$,

$I_7 \in \mathcal{M}_{loc}(\mathbb{R}^N \times (0, \infty))$.

Therefore the claim follows from Lemma 2.2. □

Hence, we have shown that the vanishing viscosity approximations converge to a weak solution of (1.1). The calculations in the proof of convergence can serve as a motivation for showing convergence of other approximation schemes. In a forthcoming paper [5], we propose an Engquist-Osher type numerical scheme for the triangular system (1.1) and show that the approximate solutions converge to a weak solution of the triangular system.

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