Bollettino Unione Matematica Italiana

G. Besson

On the Geometrisation Conjecture

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **2** (2009), n.1, p. 245–257.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2009_9_2_1_245_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2009.

Bollettino U. M. I.
(9) **II** (2009), 245-257

On the Geometrisation Conjecture (*)

G. Besson (**)

1. – Introduction.

This is the text of a Lecture given at a joint meeting of the Italian and French mathematical societies during the summer of 2006, held in Torino. It aims at describing some of the features of the proof of the geometrization conjecture following G. Perelman and R. Hamilton and a variation of it described in [3]. Extended notes have been published by H.-D. Cao and X.-P. Zhu ([10]), B. Kleiner and J. Lott ([17]) and J. Morgan and G. Tian ([20]). The reader may also look at the following survey papers [1, 5, 23]. The following text has some overlap with [4].

2. – The conjectures.

Let us recall the famous conjectures that subtend the works presented here. The following is the Poincaré conjecture.

Conjecture 2.1 (Poincaré [24], 1904). – *If M^3 is closed and simply connected then M is homeomorphic (diffeo) to the 3-sphere S^3.*

The question was published in an issue of the Rendiconti del Circolo Matematica di Palermo ([24]). It is known that in dimension 3 the homeomorphism classes and the diffeomorphism classes are the same. The second conjecture played an important role in the understanding of the situation.

Conjecture 2.2 (Thurston [26], 1982). – *M^3 can be cut open into geometric pieces.*

The precise meaning of this statement can be checked in [5]. It means that M can be cut open along a finite family of incompressible tori so that each piece left

(*) Plenary lecture at the Joint Meeting UMI-SIMAI-SMAI-SMF (July 5, 2006).
(**) The author wishes to thank the Unione Matematica Italiana and the Société Mathématiques de France for their kind invitation.

carries one of the eight 3-dimensional geometries (see [25]). These geometries are characterized by their group of isometries. Among them are the three constant curvature geometries: spherical, flat and hyperbolic. One also finds five others among which the one given by the Heisenberg group and the Sol group (check the details in [25]). Among them four are Seifert geometries and the last one is a graphed manifold. Let us recall that, roughly speaking, a graphed manifold is a bunch of Seifert bundles glued along their boundaries (which are tori) and a Seifert bundle is a circle bundle over a 2-orbifold (a bundle with some exceptional fibers). For more precise definitions the reader is referred to [15].

Thurston's conjecture has put the Poincaré conjecture in a geometric setting, namely the purely topological statement of Poincaré is understood in geometrical terms: a simply connected 3-manifold should carry a spherical geometry. Letting the geometry enter the picture opens the Pandora box; the analysis comes with the geometry.

3. – The Ricci flow.

This is an evolution equation on the Riemannian metric g, introduced by R. Hamilton. Its expected effect is to make the curvature constant, or at least Einstein, which is the same in dimension 3. However not every 3-manifold carries a constant curvature metric (think to $S^2 \times S^1$). Rather it should look like one in the list of the geometries in 3 dimensions. This expectation is however far too optimistic and not yet proved to be achieved by this technique. Nevertheless, this "flow" turns out to be sufficiently efficient to prove both Poincaré and Thurston's conjectures. The inspiration for this beautiful idea is explained in [13] and [9]. let (M, g_0) be a Riemannian manifold, we are looking for a family of Riemannian metrics depending on a parameter $t \in \boldsymbol{R}$, such that $g(0) = g_0$ and,

$$\frac{dg}{dt} = -2\,\mathrm{Ricci}_{g(t)}\,.$$

The coefficient 2 is completely irrelevant whereas the minus sign is crucial. This could be considered as a differential equation on the space of Riemannian metrics (see [9]), it is however difficult to use this point of view for practical purposes. It is more efficient to look at it in local coordinates in order to understand the structure of this equation ([13]). This turns out to be a non-linear heat equation, which is schematically like

$$\frac{\partial}{\partial t} = \Delta_{g(t)} + Q\,.$$

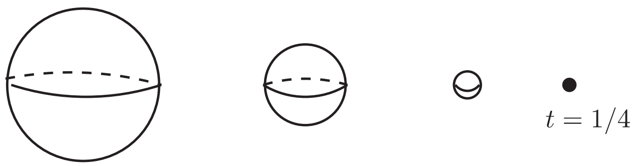
Here Δ is the Laplacian associated to the evolving Riemannian metric $g(t)$. The minus sign in the definition of the Ricci flow ensures that this heat equation is not backward and thus have solutions, at least for small time. The expression encoded in Q is quadratic in the curvatures. Such equations are called **reaction-diffusion** equations. The diffusion term is Δ; indeed if Q is equal to zero then it is an honest (time dependent) heat equation whose effect is to spread the initial temperature density. The reaction term is Q; if Δ were not in this equation then the prototype would be the ordinary differential equation,

$$f' = f^2 ,$$

for a real valued function f which blows up in finite time. From the competition between these two effects come the beauty of Hamilton and Perelman's works.

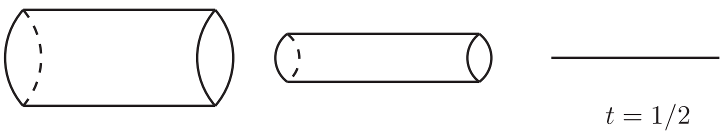
The following examples give an idea of the behaviour of the flow.

1. Flat tori, $g(t) \equiv g_0$; (it is said to be an eternal solution).
2. Round sphere $g(t) = (1 - 4t)g_0$; (ancient solution).



3. Hyperbolic space $g(t) = (1 + 4t)g_0$; (immortal solution).

4. Cylinder $g(t) = (1 - 2t)g_{S^2} \oplus g_{\boldsymbol{R}}$.



Two features deserve to be emphasized. For the round sphere the flow stops in finite positive time but has an infinite past. For the hyperbolic manifolds, on the contrary, the flow has a finite past but an infinite future. We find these aspects in the core of the proofs of the two conjectures. Indeed, for the Poincaré conjecture one is led to (although it is not strictly necessary) show that starting from any Riemannian metric on a simply connected 3-manifold the flow stops in finite time whereas for Thurston's conjecture one ought to study the long term behaviour of the evolution.

3.1 – *The first result.*

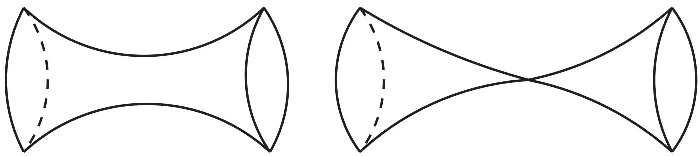
The following result is the seminal theorem at the foundation of all subsequent works.

THEOREM 3.1 ([13], R. Hamilton, 1982). – *Let M be a closed, orientable simply connected 3-dimensional manifold. If M carries a metric g_0 which has positive Ricci curvature then it is diffeomorphic to S^3.*

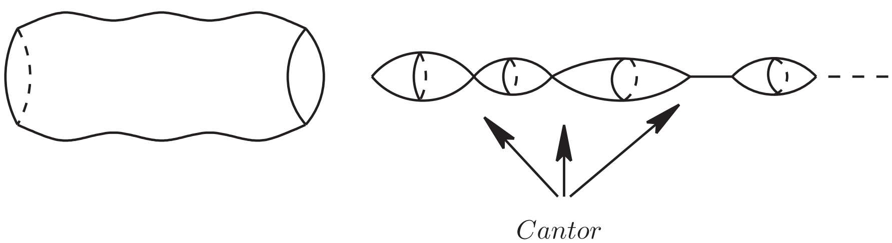
Clearly it is a step towards the proof of the Poincaré conjecture. The only restriction is important since it is not known whether a simply-connected manifold carries a metric of positive Ricci curvature. The proof is done by showing that the manifold becomes more and more round while contracting to a point.

3.2 – *The surgery.*

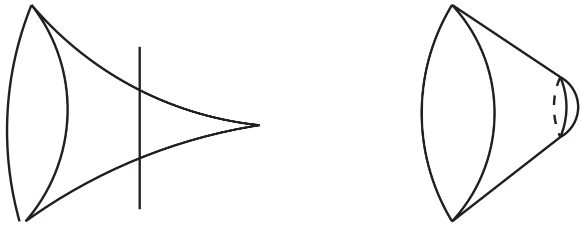
The question is now what happens if we start with a random metric g_0? It turns out that there are examples showing that the manifold may become singular, *i.e.* that the scalar curvature may become infinite on a subset of M. This is the case for the neckpinch:



which is a metric on a cylinder which develops a singularity in finite time (see [11]). But it could also be worse,



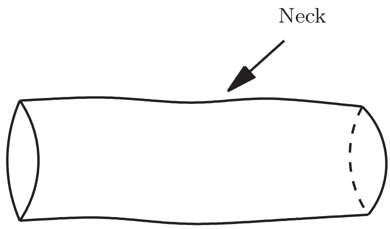
it could be that the singularities appear in a cylinder spread on a cantor subset of transversal spheres. Notice however that there are, at the moment, no explicit examples of such a behaviour. The idea introduced by R. Hamilton in [14] is to do surgery in the necks and restart the flow with a new metric on a (possibly) new manifold. It is schematically summarized by the picture below,

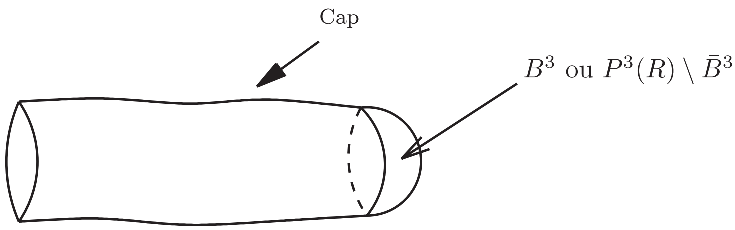


The picture on the left represents a so-called horn. The precise definitions are quite involved and the reader is referred to the original papers by G. Perelman ([21, 22]) or the monographies written on this work ([17, 20, 10]).

3.3 – *Perelman's breakthrough.*

One of the technical achievements obtained by G. Perelman is the so-called canonical neighbourhood theorem (see [21], 12.1). Roughly, it shows that there exists a universal number r_0 such that if we start with a suitably normalised metric g_0 then the points of scalar curvature larger than r_0^{-2} have a neighbourhood in which the geometry is close to a model. There is a finite list of such model geometries and a very restricted list of topologies. The neighbourhood is either a cylinder, called a neck, with a metric close to a standard round cylinder, a so-called cap which is a metric on a ball or on the complement of a ball in the projective space which looks like a cylinder out of a small set, or the manifold M is a quotient of the 3-sphere by a subgroup Γ of $O(3)$.





3.4 – *The flow-with-surgery.*

Let us now take an arbitrary 3-manifold (M, g_0). We start the flow and let it go up to the first singular time, that is up to the first time when the scalar curvature reaches $+\infty$. Two cases may occur:

i) the curvature becomes big everywhere

Just before the singular time the curvature may be big everywhere and the manifold may be entirely covered by canonical neighbourhoods. In that case we say that the manifold becomes extinct. Pasting together these neighbourhoods whose topology is known, leads to the following result

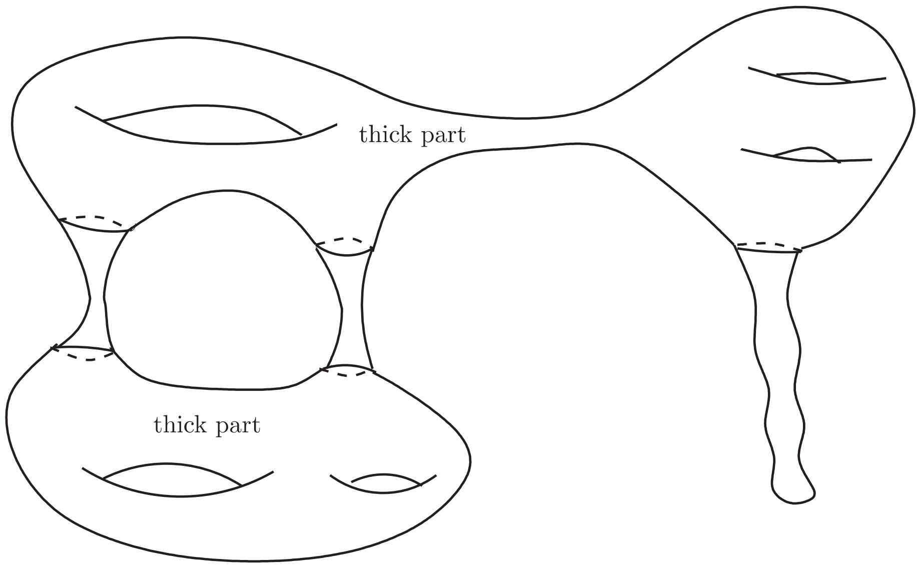
THEOREM 3.2 (Perelman, [22]). – *If the manifold becomes extinct then it is,*

i) S^3/Γ, $(\Gamma \subset SO(4))$,
ii) $S^1 \times S^2$ or $(S^1 \times S^2)/\boldsymbol{Z}^2 = \boldsymbol{P}^3(\boldsymbol{R})\#\boldsymbol{P}^3(\boldsymbol{R})$.

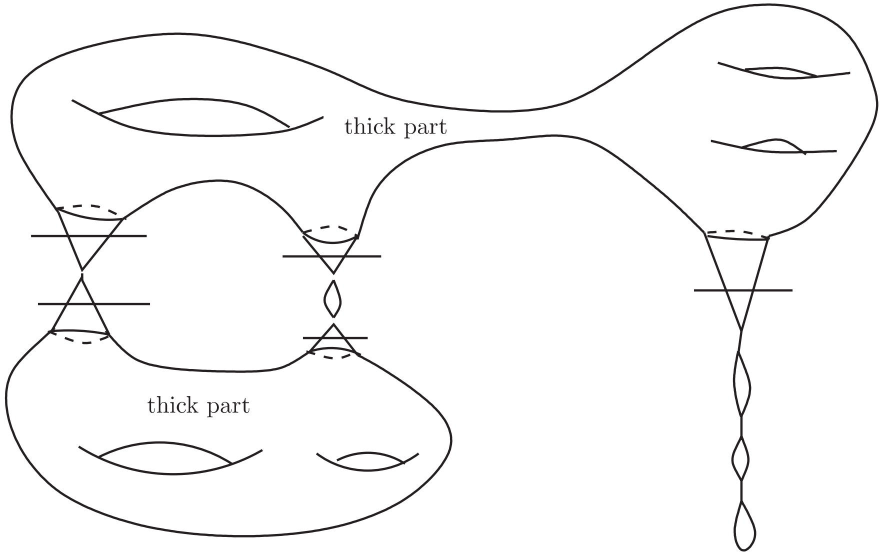
In that case we can stop the process since we have understood the topology of M. This is why it is said that the manifold becomes extinct (the curvature is high hence the manifold is small!).

ii) The manifold does not completely disappear

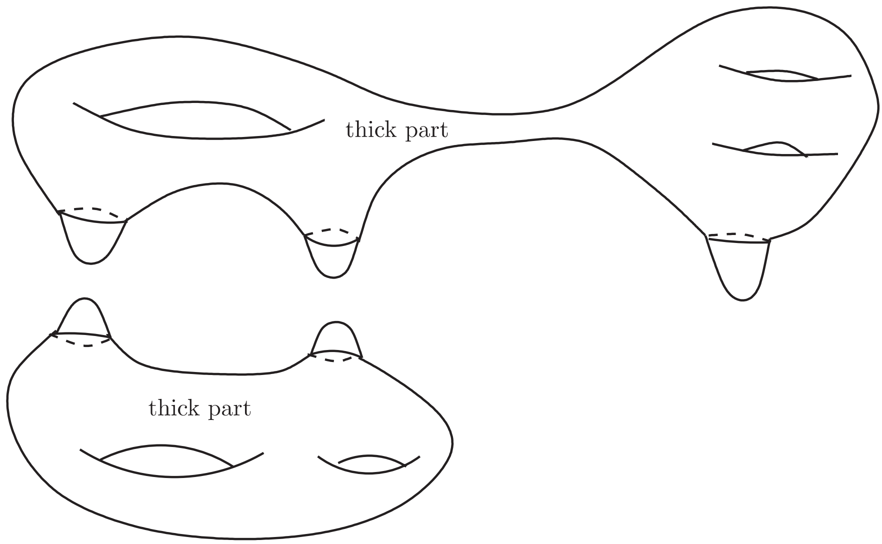
An open subset Ω is left. This is the subset of M where the scalar curvature is finite at the singular time. Just before the singular time, the manifold splits in a thick part where the scalar curvature is smaller than the scale of the canonical neighbourhoods and a thin part. So it looks like the drawing below.



At the singular time, applying surgery in the horns as shown below,



leads to a new manifold M_1, possibly not connected.



From this new Riemannian manifold one starts the Ricci flow up to the next singular time. To each connected component we apply the same dichotomy. The question is now to know whether this can be done for all time.

In [22] section 5 it is shown that this procedure leads to the Ricci flow-with-surgery, which is a non continuous version of the smooth Ricci flow (the manifold is not even fixed) defined for all time. Stating precisely the result would be too technical and beyond the scope of this note; the reader is referred to the above mentioned texts. The key step is to show that the surgeries do not accumulate, that is, on a given finite interval of time there are only finitely many of them. Globally, there may be infinitely many surgeries to perform. The question whether on an infinite interval of time we reach some special geometry will be discussed later.

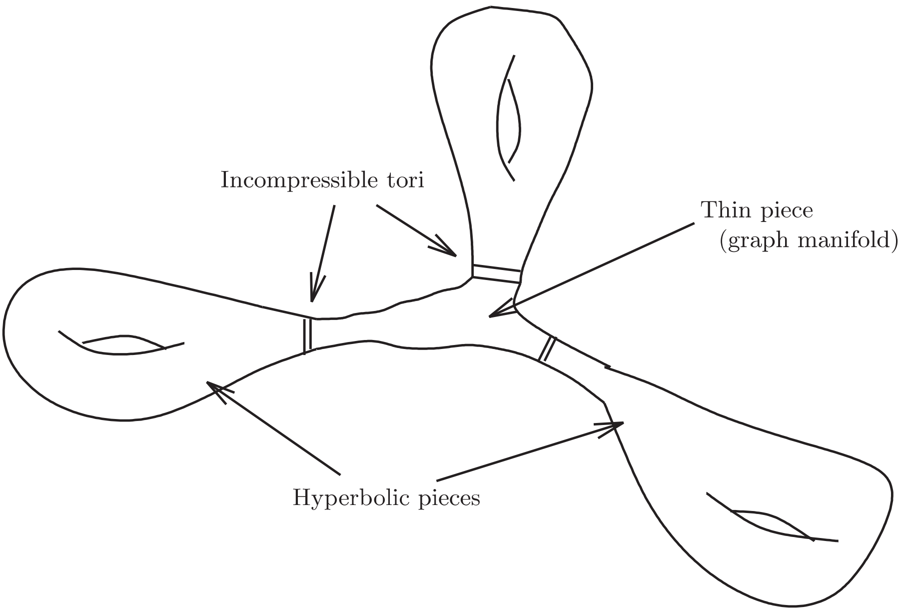
The proof of this result is quite involved. The point is that if we start with a normalised metric (see the references for a precise definition) then after the first surgery it is not any more. Thus, the surgery parameter r_0 has changed. It changes in fact after each surgery and it may go to zero in finite time which will stop the procedure and corresponds to an accumulation of surgeries. Showing that it is not the case is a *tour de force* which is a masterpiece of Riemannian geometry.

4. – The geometrisation conjecture.

In this case, as is shown in the example of the hyperbolic manifolds, we need to study the flow-with-surgery for an infinite time interval. The main result is summarised in the following rough claim.

CLAIM 4.1 (Perelman [22], sections 6-8). – *For large t,* $(M, g(t))$ *decomposes into thick and thin pieces (possibly empty).*

The picture below gives a hint of what may happen,



hyperbolic pieces emerge from the thick part and are bounded by incompressible tori, that is tori whose fundamental group injects in the one of the manifold. The thin pieces are collapsing, that is the volume of balls goes to zero while the sectional curvature is bounded below. The reader should check the precise definition of collapsing since it differs from the familiar one.

More precisely, it is the rescaled metric $\frac{1}{4t}g(t)$ which becomes thick or thin as in the case of the hyperbolic metric which behaves like $(1+4t)g_0(t)$. The assertion is that the thin part is a graphed manifold.

From this the proof of the geometrisation conjecture is described in [22], [10] and [17]. We briefly sketch below the proof given in [3].

4.1 – *Ricci flow, simplicial volume and aspherical* 3*-manifolds.*

In the forthcoming monograph [2] we shall present a (slight) variation of the construction of the flow-with-surgery which is simpler when the manifold is assumed to be irreducible. It is a step towards considering the flow-with-surgery as a generalised solution of the Ricci flow equation in the sense of analysis. The reader is referred to [18] for a more precise description and to [2] for all the details.

A manifold M is said to be *aspherical* when $\pi_k(M) = 0$ for all $k \geq 2$. Throughout the rest of this text, M is a closed, orientable and irreducible 3-manifold with infinite fundamental group. From the sphere theorem the above hypothesis imply that M is aspherical. Let us recall that a 3-manifold is said to be *Haken* if it is connected, compact, orientable, irreducible and contains an incompressible surface. Any connected, compact, orientable and irreducible 3-manifold whose boundary is not empty is Haken. Thurston has proved that Haken manifolds have geometric decompositions, *i.e.* can be decomposed in pieces carrying one the geometries described in [25]. A Haken manifold is a graphed manifold if and only if all pieces in its geometric decomposition are Seifert. See [15] for the classical topology of 3-manifolds and [8] for some post-Thurston's results.

Let g be a Riemannian metric on a manifold M and $\varepsilon > 0$ a real number. Following Perelman [22], we call *ε-thin part* of (M, g) the set $M^-(\varepsilon)$ of those points $x \in M$ for which there exists $0 < \rho \leq 1$ such that on the ball $B(x, \rho)$, the sectional curvature is not smaller than $-\rho^{-2}$ and the volume of this ball is smaller than $\varepsilon\rho^3$. Its complementary set is called *ε-thick part* and denoted by $M^+(\varepsilon)$. For a sequence of metrics g_n on M, we denote $M_n^-(\varepsilon)$ the ε-thin part of (M, g_n) and similarly $M_n^+(\varepsilon)$ its ε-thick part.

The most difficult part of the following work is to describe the thin part. Let us briefly sketch the main ideas.

DEFINITION 4.2. – *Let g_n be a sequence of Riemannian metrics on M. We say that g_n has a locally controlled curvature in the sense of Perelman if it has the following property: for all $\varepsilon > 0$ there exists $\overline{r}(\varepsilon) > 0$, $K_0(\varepsilon)$, $K_1(\varepsilon) > 0$, such that for n big enough, if $0 < r \leq \overline{r}(\varepsilon)$ and $x \in M_n$ satisfy $\mathrm{vol}\,(B(x,r))/r^3 \geq \varepsilon$ and the sectional curvature on $B(x,r)$ is $\geq -r^{-2}$, then $|\mathrm{Rm}(x)| < K_0 r^{-2}$ and $|\nabla \mathrm{Rm}(x)| < K_1 r^{-3}$.*

This is a technical condition which allows to avoid the use of Alexandrov spaces, and in particular Perelman's stability theorem (see [16]).

For any metric g_0 the Ricci flow produces a sequence of Riemannian metrics $\{g_n\}_{n\in \boldsymbol{N}}$ satisfying the following properties:

- (Boundedness of volume) There exists $C > 0$ such that, for all n, $\mathrm{vol}\,(M, g_n) \leq C$,
- (Hyperbolic limits) For all $\varepsilon > 0$ and $x_n \in M_n^+(\varepsilon)$, there exists a complete finite volume hyperbolic pointed manifold (M_∞, x_∞) such that

$$(M_n, g_n, x_n) \underset{n\to+\infty}{\longrightarrow} (M_\infty, \mathrm{hyp}, x_\infty)\,,$$

- g_n has a locally controlled curvature in the sense of Perelman.

The sequence of metrics can be taken to be $g_n = \dfrac{1}{4n}\, g(n)$ if $g(t)$ is the solution of the Ricci flow-with-surgery. Indeed, on the thick part $\dfrac{1}{4t}\, g(t)$ tends to an hyperbolic metric (see [22] chapters 6 and 7 and [2]). Let $M_n = (M, g_n)$, a simple statement of the main theorem proved in [3] is

THEOREM 4.3 (see [3]). – *With the above assumptions on M, let us assume that there exists a sequence ε_n going to 0 when n goes to infinity such that $M_n = M_n^-(\varepsilon_n)$ (that is M_n is ε_n-thin) then M_n is a graphed manifold for n large enough.*

The statement may seem strange since being a graph manifold is a topological property and the underlying manifold does not change. It really means that it is only if n is large enough that one can see the graphed structure. One could also take instead of a fixed differentiable manifold a sequence depending on n.

Some of the ideas appearing in the proof. It always starts in the same way, one uses the fact that the manifold is thin to show that locally, around any point, the geometric structure is close to a finite list of models. Then one may try to glue these local models in order to construct the global graphed structure. This is the approach developed in [22], [17] and [10]. Here we extract form the collection of balls close to the local models a finite covering of the manifold and we use two simple covering arguments to conclude.

The local models are given by the following proposition.

PROPOSITION 4.4 (see [3]). – *For all* $D > 1$ *there exists* $n_0(D)$ *such that if* $n > n_0(D)$*, then for all* $x \in M_n^-(\varepsilon_n)$ *we have the following alternative:*

(*a*) *Either* M_n *is* $\frac{1}{D}$*-close of a compact Euclidean manifold.*

(*b*) *Or there exists a radius* $\nu_n(x)$ *and a complete non compact Riemannian 3-manifold* $X_{n,x}$*, with non negative sectional curvature and soul* $S_{n,x}$*, such that* $B(x, \nu_n(x))$ *is* $\frac{1}{D}$*-close of a metric ball in* $X_{n,x}$*.*

REMARK 4.5. – The soul $S_{n,x}$ can be homeomorphic to a point, a circle, a 2-sphere, a 2-torus or a Klein bottle. Consequently, the ball $B(x, \nu_n(x))$ is homeomorphic to B^3, $S^1 \times D^2$, $S^2 \times I$, $T^2 \times I$ or to the twisted I-bundle on the Klein bottle. The case where $S_{n,x}$ is homeomorphic to the projective plane is excluded for the only closed, orientable and irreducible 3-manifold containing a projective plane is RP^3, which is not aspherical.

More technical properties of these balls close to local models are described in [3].

Now, one can show that from the collection of balls $B(x, \nu_n(x))$ a (minimal) finite covering can be extracted such that one of the sets appearing, which we shall call $\mathcal{V}$, satisfies that $\mathrm{Im}(\pi_1(\mathcal{V}) \longrightarrow \pi_1(M_n))$ is not trivial. If this is true then one can show that $M \setminus \mathcal{V}$ is irreducible with boundary hence Haken. In order to prove the existence of a nontrivial (in the above sense) set in the covering, one argues by contradiction. If it is not true, that is if all local models of the covering are trivial, then one shows that it can be deformed into a finite covering of dimension at most 2. Let us recall that the dimension of a covering is the smallest integer q such that any point in M belongs to at most $q + 1$ sets of the covering. The contradiction then comes from [19] where C. McMullen proves a theorem on the dimension of the coverings of a n-torus with $n \geq 1$ whose proof implies the following result:

THEOREM 4.6 ([19]). – *Let* N^d *be a closed, orientable and aspherical d-manifold with* $d \geq 1$*. Every locally finite covering of* N^d *by homotopically trivial open sets has dimension at least d.*

Once we have a local model $\mathcal{V}$ such that $M \setminus \mathcal{V}$ is Haken, then we know by Thurston's geometrisation of Haken manifolds that $M \setminus \mathcal{V}$ has a geometric decomposition in hyperbolic pieces and Seifert pieces. If we can show that only

Seifert pieces appear and since the local models are fibred it is then easy to show that M itself is a graphed manifold. In order to show that $M \setminus \mathcal{V}$ is graphed, we use the same argument to construct a 2-dimensional covering with sets such that the image of their fundamental group in M is virtually abelian. Indeed, the fundamental group of each local model is virtually abelian. A classical result by Gromov (see [12]) then shows that the simplicial volume of $M \setminus \mathcal{V}$ is zero. On the other hand if there were hyperbolic pieces in the geometric decomposition of $M \setminus \mathcal{V}$ it would be non-zero, a contradiction. The conclusion is that $M \setminus \mathcal{V}$ is graphed and so is M. □

4.2 – *Conclusion.*

The proof sketched above extends to the case when the manifold has finite but not trivial fundamental group (see [3]). Its strength is that it is simpler to use these two covering arguments than gluing local models in order to construct the graphed structure. The weakness is that we have to use Thurston's geometrization of Haken manifolds and that we do not "see" the graphed structure. Let us finally mention that the above technique is borrowed from [6] or [7].

REFERENCES

[1] L. Bessières, *Conjecture de Poincaré : la preuve de R. Hamilton and G. Perelman*, La gazette des mathématiciens, **106** (2005).

[2] L. Bessières - G. Besson - M. Boileau - S. Maillot - J. Porti, *Geometrization of 3-manifolds*, Monograph in preparation.

[3] L. Bessières - G. Besson - M. Boileau - S. Maillot - J. Porti, *Weak collapsing and geometrisation of aspherical 3-manifolds*, arXiv:0706.2065 January 2008.

[4] G. Besson, *The Geometrization Conjecture after R. Hamilton and G. Perelman*, To appear in Rendiconti del Seminario Matematico, Torino.

[5] G. Besson, *Preuve de la conjecture de Poincaré en déformant la métrique par la courbure de Ricci (d'après Perelman)*, In *Séminaire Bourbaki 2004-2005*, Astérisque, **307**, 309-348. Société mathématiques de France, Paris, France, 2006.

[6] M. Boileau - J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque, **272** (2001), 208. Appendix A by Michael Heusener and Porti.

[7] M. Boileau - B. Leeb - J. Porti, *Geometrization of 3-dimensional orbifolds*, Ann. of Math. (2) **162** (1) (2005), 195-290.

[8] M. Boileau - S. Maillot - J. Porti, *Three-dimensional orbifolds and their geometric structures*, volume 15 of *Panoramas and Synthèses*. Société Mathématique de France, Paris 2003.

[9] J.-P. Bourguignon, *L'équation de la chaleur associée à la courbure de Ricci*, In *Séminaire Bourbaki 1985-86*, Astérisque, 45-61. Société Mathématique de France 1987.

[10] H.-D. Cao - X.-P. Zhu, *A Complete Proof of the Poincaré and Geometrization Conjectures and application of the Hamilton-Perelman theory of the Ricci flow*, Asian Journal of Mathematics, **10** (2) (2006), 165-492.

[11] B. Chow - D. Knopf, *The Ricci flow: an introduction*, Mathematical surveys and monographs, **110** A.M.S. 2004.
[12] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math., 56:5-99 (1983), 1982.
[13] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry, **17** (1982), 255-306.
[14] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geometry, **24**, 153-179 (1986).
[15] J. Hempel, 3-*manifolds*, Annals of mathematics studies, **086**, Princeton University Press Princeton, 1976.
[16] V. Kapovitch, *Perelman's Stability Theorem*. ArXiv: math.DG/0703002, 13 mars, 2007.
[17] B. Kleiner - J. Lott, *Notes on Perelman's papers*, ArXiv: math.DG/0605667, 25 mai, 2006.
[18] S. Maillot. *Some application of the Ricci flow to* 3-*manifolds*, Séminaire de théorie spectrale et géométrie, **25**, 2006-2007.
[19] C. T. McMullen, *Minkowski's conjecture, well-rounded lattices and topological dimension*, J. Amer. Math. Soc., **18** (3) (2005), 711-734 (electronic)
[20] J. Morgan - G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay mathematics monographs, **3**, American Mathematical Society, 2007.
[21] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. ArXiv: math.DG/0211159, november, 2002.
[22] G. Perelman, *Ricci flow with surgery on three-manifolds*, ArXiv: math.DG/0303109, mars, 2003.
[23] V. Poenaru *Poincaré and l'hypersphère*, Pour la Science, Dossier hors-série n. 41 (2003), 52-57.
[24] H. Poincaré, *Cinquième complément à l'analysis situs*, Rend. Circ. Mat. Palermo, **18** (1904), 45-110.
[25] P. Scott, *The geometries of* 3-*manifolds*, Bull. London Math. Soc., **15**, (1983), 401-487.
[26] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups* and hyperbolic geometry, Bulletin of the Amer. Math. Soc, **6**, (3) (1982), 357-381.

Institut Fourier - C.N.R.S.
Université de Grenoble
e-mail: vg.besson@ujf-grenoble.fr

Received May 12, 2008