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A Gluing Theorem for the Elliptic Adjoint Operator $L^*$ in the Plane

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dedicated to the memory of Guido Stampacchia

Abstract. – A “matching” method for planar harmonic functions is exhibited, using an elliptic adjoint equation $L^*u = 0$.

1. – Introduction.

In $\mathbb{R}^2$, let

$$T := \{-\pi \leq y \leq \pi\}$$

be the 1-dimensional torus and, for $\alpha > 0$, let

$$\Omega := (-\alpha, \alpha) \times T \subset \mathbb{R} \times T$$

and

$$\Omega_\delta := (-\alpha - \delta, \alpha + \delta) \times T, \quad \delta > 0.$$ 

Denote:

$$\Gamma_k := (-1)^k \alpha \times T, \quad k = 1, 2.$$ 

All the functions we will consider will be periodic, of period $2\pi$, in the variable $y$. Let $w_1 = w_1(x, y)$ and $w_2 = w_2(x, y)$ be two harmonic functions, defined in $\Omega_\delta$. The problem we are going to deal with is the following.

Problem (*). – Given $w_1$ and $w_2$ harmonic functions as above, does it exist a function $u$ defined in $\Omega_\delta$, a matrix $(a^{ij})$, $a\lvert \lambda \rvert^2 \leq a^{ij}(x, y)\lambda_i\lambda_j \leq a^{-1}\lvert \lambda \rvert^2$ for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, $a =$ ellipticity constant $> 0$, for a.e. $(x, y) \in \Omega_\delta$, $a^{ij} = \delta^{ij}$ in $\Omega_\delta \setminus \Omega$, such that:

$$u = w_1 \quad \text{in} \quad (\Omega_\delta \setminus \Omega) \cap \{x < 0\} \times T$$

and

$$u = w_2 \quad \text{in} \quad (\Omega_\delta \setminus \Omega) \cap \{x > 0\} \times T$$

(1)

and

$$L^*u = \partial^{ij}(a^{ij}u) = 0 \quad \text{in} \quad \Omega_\delta?$$

(2)
Here and throughout the paper, it is assumed that repeated indices are summed over. As regard to the matrix $(a^{ij})$, we will look for $a^{ij}$ regular in $\Omega_{\delta} \setminus (\Gamma_1 \cup \Gamma_2)$, that can be extended as regular functions to $\overline{\Omega}$ and to $\overline{\Omega_{\delta}} \setminus \Omega$; notice that $u^{ij}$ can be discontinuous on $\Gamma_1 \cup \Gamma_2$.

We will look for solutions $u \in C^2(\Omega_{\delta})$, periodic in $y$.

Recall, by the way, that a function $u$ satisfies the adjoint equation $L^*u = 0$ if $\int \int uL\varphi \, dx \, dy = 0$ for any $\varphi$ regular, periodic in $y$, with compact support in $x$, where $L\varphi = tr(a^{ij}D^2\varphi)$.

Our gluing theorem will give an affirmative answer to the above Problem (*), by means of a “matching” functions technique. It is worthy mentioning that, in the context of elliptic equations, as far as to my knowledge, a type of matching functions procedure goes back to M. S. Safonov [1].

Section 2 of the paper is devoted to some remarks and the proof of some compatibility conditions, that must be verified. In Section 3, we succeed in finding a solution to the Problem (*) and proving the main theorems, in the case that the harmonic functions $w_1$ and $w_2$ have the same sign. See also, the Remark, there.

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I wish to thank P. Manselli for pointing out this type of question to me.

2. Remarks and compatibility conditions.

Let us start with some remarks.

Remark 1. It turns out that

\begin{equation}
\partial_1 \int_T w_k(x, y) \, dy = c_k \quad \text{for} \quad k = 1, 2,
\end{equation}

in $|x| < a + \delta$; $c_k$ a constant.

Remark 2. One has, for $k = 1, 2$:

\[ \int_T w_k(x, y) \, dy = \int_T w_k(\overline{x}, y) \, dy + (x - \overline{x}) \int_T \partial_1 w_k(\overline{x}, y) \, dy, \]

$\overline{x} \in (-a - \delta, a + \delta)$.

We may prove the following form of compatibility conditions.
LEMMA 1. – If the Problem (*) has a solution, then:

\[ (a^{11} - w_1)|_{\Gamma_1} = 0, \quad [\partial_1(a^{11}u) + 2\partial_2(a^{12}u) - \partial_1w_1]|_{\Gamma_1} = 0, \]
\[ (a^{11} - w_2)|_{\Gamma_2} = 0, \quad [\partial_1(a^{11}u) + 2\partial_2(a^{12}u) - \partial_1w_2]|_{\Gamma_2} = 0, \]

where:

\[ a^{ij}|_{\Gamma_1} = \lim_{\varepsilon \to 0^+} a^{ij}( - a + \varepsilon, \cdot), \quad a^{ij}|_{\Gamma_2} = \lim_{\varepsilon \to 0^+} a^{ij}(a - \varepsilon, \cdot), \quad \varepsilon > 0. \]

PROOF. – Recall \( a^{ij} \) are regular in \( \Omega \) and \( \overline{\Omega_0 \setminus \Omega} \). Let \( \varphi \in C_0^\infty(\Omega_0), \varphi \) periodic in \( y \). Then, if we set, for \( \varepsilon \) and \( \varepsilon_1 \) positive numbers:

\[ A_\varepsilon = [ - a - \delta, - a - \varepsilon] \times T \]
\[ B_{\varepsilon, \varepsilon_1} = [ - a + \varepsilon, a - \varepsilon_1] \times T \]
\[ C_\varepsilon = [a + \varepsilon, a + \delta] \times T \]

we have:

\[ 0 = \iint_{\Omega_0} u(a^{ij}\partial_j\varphi)dxdy = \lim_{\varepsilon \to 0} \iint_{A_\varepsilon} w_1\partial_j\varphi dxdy \]
\[ + \lim_{\varepsilon \to 0} \iint_{B_{\varepsilon, \varepsilon_1}} u(a^{ij}\partial_j\varphi)dxdy + \lim_{\varepsilon \to 0} \iint_{C_\varepsilon} w_2\partial_j\varphi dxdy. \]

Integration by parts in each term and taking the limit as \( \varepsilon \to 0 \) and \( \varepsilon_1 \to 0 \), it yields:

\[ \iint_{\Gamma_1} \{ [a^{11}( - a^+, \cdot)u - w_1]\partial_1\varphi - [\partial_1(a^{11}u) + 2\partial_2(a^{12}u) - \partial_1w_1]\varphi \} d\Gamma = 0 \]

and

\[ \iint_{\Gamma_2} \{ [a^{11}(a^-, \cdot)u - w_2]\partial_1\varphi - [\partial_1(a^{11}u) + 2\partial_2(a^{12}u) - \partial_1w_2]\varphi \} d\Gamma = 0. \]

By choosing \( \varphi \) suitably on \( \Gamma_1 \), we get the compatibility conditions (4). The proof is complete. \( \square \)

LEMMA 2. (Integral compatibility condition). – If \( u \) is a solution to Problem (*), then the following integral compatibility condition holds for \( |x| < a + \delta \):

\[ \int_T (a^{11}u)(x, y)dxdy = \frac{x}{2a} \int_T [w_2(a, y) - w_1(-a, y)] dy \]
\[ + \frac{1}{2} \int_T [w_2(a, y) - w_1(-a, y)] dy. \]
PROOF. – We have:
\[
\partial_{11}(a^{11}u) + 2\partial_{12}(a^{12}u) + \partial_{22}(a^{22}u) = 0 \quad \text{in } \Omega.
\]
By integrating on \( T \), we get
\[
\partial_{11} \int_T (a^{11}u)(x, y)dy = 0,
\]
that is:
\[
\int_T (a^{11}u)(x, y)dy = kx + k_0,
\]
\( k, k_0 \) real constants.

By Lemma 1, an easy computation yields the condition (5). The proof of the Lemma 2 is completed. \( \Box \)

3. – Solution of the Problem (*) and construction of the function \( u \).

To solve our problem, let us assume that the harmonic functions \( w_1 \) and \( w_2 \) are both positive. Moreover:
\[
\int_T w_1(x, y)dy = \int_T w_2(x, y)dy.
\]
As a consequence of REMARK 1 of Section 2, we have for \( |x| < a + \delta \):
\[
\partial_1 \int_T w_1(x, y)dy = \partial_1 \int_T w_2(x, y)dy.
\]
Now, let us solve the problem by assuming
\[
a^{11} = 1 \quad \text{and} \quad a^{12} = 0.
\]
Let \( \psi \in C^\infty, 0 \leq \psi \leq 1 \), so defined: \( \psi \equiv 1 \) near \( x = -a \) and \( \psi \equiv 0 \) near \( x = a \).

Then construct the function \( u = u(x, y) \) as follows:
\[
u = \psi w_1 + (1 - \psi)w_2 = w_2 + \psi(w_1 - w_2).
\]
Clearly, \( u > 0 \). Moreover, as it can be easily verified, \( u \) satisfies the compatibility conditions (4) and (5).

Thus, from (5) we have:
\[
\int_T \partial_{11}u(x, y)dy = 0 \quad \text{in } \Omega_\delta.
\]
Therefore, there exists a function \( v > 0, 2\pi \) - periodic in \( y \) and such that:

\[
\frac{\partial^2 v}{\partial y^2}(x, y) = -\partial_1 u(x, y) \quad \text{in} \quad \Omega_\delta.
\]

Then:

\[
\partial_1 u + \partial_{\partial_2} \left( \frac{v}{u} \cdot u \right) = 0 \quad \text{in} \quad \Omega.
\]

So, the problem is solved in \( \Omega_\delta \) with

\[
a^{11} = 1, \quad a^{22} = \frac{v}{u}, \quad a^{12} = 0 \quad \text{in} \quad \Omega
\]

and

\[
a^{11} = a^{22} = 1 \quad \text{in} \quad \Omega_\delta \setminus \Omega.
\]

**Remark.** – The same arguments could be used to get the same result in the case \( w_1 \) and \( w_2 \) both negative. Actually, it remains to be investigated the case \( w_1 \cdot w_2 < 0 \).

Therefore, we may state the following

**Theorem 1.** – Let \( w_1 \) and \( w_2 \) two harmonic functions defined in \( \Omega_\delta \).

Assume:

\[
w_1 \cdot w_2 > 0 \quad \text{and} \quad \int_T w_1 dy = \int_T w_2 dy.
\]

Then, **Problem (*)** is solvable in \( \Omega_\delta \).

**Proof.** – It is given exactly by all the above arguments of the present section. \( \square \)

A consequence of **Theorem 1** is the following

**Theorem 2.** – Let \( w_k (k = 1, 2) \) harmonic functions in a neighbourhood \( N \) of \( x = 0 \quad (y \in \mathbb{T}) \), \( w_1 \cdot w_2 > 0 \), such that:

\[
w_1(0, y) = w_2(0, y) > 0, \quad \partial_1 \int_T w_1(0, y) dy = \partial_1 \int_T w_2(0, y) dy.
\]

Then, there exists a function \( u \), defined in \( N \), such that:

\[
\begin{align*}
L^* u &= 0 \quad \text{in} \quad N \\
u &= w_1 \quad \text{for} \quad x < \tilde{x} < 0 \\
u &= w_2 \quad \text{for} \quad x > \tilde{x} > 0.
\end{align*}
\]
Proof. – In Theorem 1, let \( a, \delta > 0 \) so small that, in \( |x| < a + \delta, w_1 \geq c_1 > 0 \) and \( w_2 \geq c_2 > 0 \). Remark 1 of Section 2 implies that

\[
\partial_1 \int_T w_1 \, dy = \partial_1 \int_T w_2 \, dy = c
\]

in \( |x| < a + \delta \) and \( w_1 = w_2 \) at \( x = 0 \) and Remark 2 imply that

\[
\int_T w_1 \, dy = \int_T w_2 \, dy.
\]

Then, \( u \) can be constructed as before and, from the previous Theorem, Theorem 2 follows.

\[\square\]

References


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