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A Gluing Theorem for the Elliptic Adjoint Operator L^* in the Plane

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dedicated to the memory of Guido Stampacchia

Abstract. – *A "matching" method for planar harmonic functions is exhibited, using an elliptic adjoint equation $L^*u = 0$.*

1. – Introduction.

In $\mathbb{R}^2$, let

$$\mathbb{T} :\cong \{-\pi \le y \le \pi\}$$

be the 1-dimensional torus and, for $a > 0$, let

$$\Omega := (-a, a) \times \mathbb{T} \subset \mathbb{R} \times \mathbb{T}$$

and

$$\Omega_\delta := (-a-\delta, a+\delta) \times \mathbb{T}, \quad \delta > 0.$$

Denote:

$$\Gamma_k := (-1)^k a \times \mathbb{T}, \quad k = 1, 2.$$

All the functions we will consider will be periodic, of period 2π, in the variable y. Let $w_1 = w_1(x, y)$ and $w_2 = w_2(x, y)$ be two harmonic functions, defined in Ω_δ. The problem we are going to deal with is the following.

Problem (*). – *Given w_1 and w_2 harmonic functions as above, does it exist a function u defined in Ω_δ, a matrix (a^{ij}), $a|\lambda|^2 \le a^{ij}(x, y)\lambda_i\lambda_j \le a^{-1}|\lambda|^2$ for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, $a =$ ellipticity constant > 0, for a.e. $(x, y) \in \Omega_\delta$, $a^{ij} = \delta^{ij}$ in $\Omega_\delta \backslash \Omega$, such that:*

$$(1) \qquad \begin{aligned} u = w_1 \quad &in \quad (\Omega_\delta \backslash \Omega) \cap (\{x<0\} \times \mathbb{T}) \\ u = w_2 \quad &in \quad (\Omega_\delta \backslash \Omega) \cap (\{x>0\} \times \mathbb{T}) \end{aligned}$$

and

$$(2) \qquad L^*u = \partial_{ij}(a^{ij}u) = 0 \quad in \quad \Omega_\delta?$$

Here and throughout the paper, it is assumed that repeated indices are summed over. As regard to the matrix (a^{ij}), we will look for a^{ij} regular in $\Omega_\delta \backslash (\Gamma_1 \cup \Gamma_2)$, that can be extended as *regular* functions to $\overline{\Omega}$ and to $\overline{\Omega_\delta \backslash \Omega}$; notice that a^{ij} can be discontinuous on $\Gamma_1 \cup \Gamma_2$.

We will look for solutions $u \in C^2(\Omega_\delta)$, periodic in y.

Recall, by the way, that a function u satisfies the adjoint equation $L^* u = 0$ if $\int\int u L\varphi \, dx \, dy = 0$ for any φ regular, periodic in y, with compact support in x, where $L\varphi = tr(a^{ij} D^2 \varphi)$.

Our gluing theorem will give an affirmative answer to the above PROBLEM (*), by means of a"matching" functions technique. It is worthy mentioning that, in the context of elliptic equations, as far as to my knowledge, a type of matching functions procedure goes back to M. S. SAFONOV [1].

Section 2 of the paper is devoted to some remarks and the proof of some compatibility conditions, that must be verified. In Section 3, we succeed in finding a solution to the PROBLEM (*) and proving the main theorems, in the case that the harmonic functions w_1 and w_2 have the same sign. See also, the REMARK, there.

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I wish to thank P. Manselli for pointing out this type of question to me.

2. – Remarks and compatibility conditions.

Let us start with some remarks.

REMARK 1. – *It turns out that*

$$\partial_1 \int_{\mathrm{T}} w_k(x, y) dy = c_k \quad \textit{for} \quad k = 1, 2, \tag{3}$$

in $|x| < a + \delta$; c_k *a constant.*

REMARK 2. – *One has, for* $k = 1, 2$*:*

$$\int_{\mathrm{T}} w_k(x, y) dy = \int_{\mathrm{T}} w_k(\overline{x}, y) dy + (x - \overline{x}) \int_{\mathrm{T}} \partial_1 w_k(\overline{x}, y) dy,$$

$\overline{x} \in (-a - \delta, a + \delta)$.

We may prove the following form of compatibility conditions.

LEMMA 1. – *If the PROBLEM (*) has a solution, then:*

$$(4)\qquad \begin{aligned} &(a^{11}_- - w_1)|_{\Gamma_1} = 0, \quad \big[\partial_1(a^{11}_- u) + 2\partial_2(a^{12}_- u) - \partial_1 w_1\big]|_{\Gamma_1} = 0,\\ &(a^{11}_+ - w_2)|_{\Gamma_2} = 0, \quad \big[\partial_1(a^{11}_+ u) + 2\partial_2(a^{12}_+ u) - \partial_1 w_2\big]|_{\Gamma_2} = 0, \end{aligned}$$

where:

$$a^{ij}_-|_{\Gamma_1} = \lim_{\varepsilon\to 0+} a^{ij}(-a+\varepsilon,\cdot), \quad a^{ij}_+|_{\Gamma_2} = \lim_{\varepsilon\to 0+} a^{ij}(a-\varepsilon,\cdot), \quad \varepsilon > 0.$$

PROOF. – Recall a^{ij} are regular in $\overline{\Omega}$ and $\overline{\Omega_\delta\backslash\Omega}$. Let $\varphi \in C_0^\infty(\Omega_\delta)$, φ periodic in y. Then, if we set, for ε and ε_1 positive numbers:

$$\begin{aligned} A_\varepsilon &= [-a-\delta, -a-\varepsilon]\times\mathbb{T}\\ B_{\varepsilon,\varepsilon_1} &= [-a+\varepsilon, a-\varepsilon_1]\times\mathbb{T}\\ C_\varepsilon &= [a+\varepsilon, a+\delta]\times\mathbb{T} \end{aligned}$$

we have:

$$0 = \iint_{\Omega_\delta} u(a^{ij}\partial_{ij}\varphi)dxdy = \lim_{\varepsilon\to 0}\iint_{A_\varepsilon} w_1\partial_{ij}\varphi\, dx\, dy$$

$$+\lim_{\substack{\varepsilon\to 0\\ \varepsilon_1\to 0}}\iint_{B_{\varepsilon,\varepsilon_1}} u(a^{ij}\partial_{ij}\varphi)dxdy + \lim_{\varepsilon\to 0}\iint_{C_\varepsilon} w_2\partial_{ij}\varphi dxdy.$$

Integration by parts in each term and taking the limit as $\varepsilon\to 0$ and $\varepsilon_1 \to 0$, it yields:

$$\int_{\Gamma_1}\big\{\big[a^{11}_-(-a^+,\cdot)u - w_1\big]\partial_1\varphi - \big[\partial_1(a^{11}_- u) + 2\partial_2(a^{12}_- u) - \partial_1 w_1\big]\varphi\big\}dy = 0$$

and

$$\int_{\Gamma_2}\big\{\big[a^{11}_+(a^-,\cdot)u - w_2\big]\partial_1\varphi - \big[\partial_1(a^{11}_+ u) + 2\partial_2(a^{12}_+ u) - \partial_1 w_2\big]\varphi\big\}dy = 0.$$

By choosing φ suitably on Γ_k, we get the compatibility conditions (4). The proof is complete. □

LEMMA 2. (Integral compatibility condition). – *If u is a solution to PROBLEM (*), then the following integral compatibility condition holds for $|x| < a+\delta$:*

$$(5)\qquad \begin{aligned} \int_{\mathbb{T}} (a^{11}u)(x,y)dxdy = \frac{x}{2a}\int_{\mathbb{T}}[w_2(a,y) - w_1(-a,y)]dy\\ + \frac{1}{2}\int_{\mathbb{T}}[w_2(a,y) - w_1(-a,y)]dy\ . \end{aligned}$$

PROOF. – We have:

$$\partial_{11}(a^{11}u) + 2\partial_{12}(a^{12}u) + \partial_{22}(a^{22}u) = 0 \quad \text{in } \Omega.$$

By integrating on $\mathbb{T}$, we get

$$\partial_{11}\int_{\mathbb{T}} (a^{11}u)(x,y)dy = 0,$$

that is:

$$\int_{\mathbb{T}} (a^{11}u)(x,y)dy = kx + k_0,$$

k, k_0 real constants.

By Lemma 1, an easy computation yields the condition (5). The proof of the Lemma 2 is completed. □

3. – Solution of the Problem (*) and construction of the function u.

To solve our problem, let us assume that the harmonic functions w_1 and w_2 are both positive. Moreover:

$$\int_{\mathbb{T}} w_1(x,y)dy = \int_{\mathbb{T}} w_2(x,y)dy.$$

As a consequence of REMARK 1 of Section 2, we have for $|x| < a + \delta$:

$$\partial_1\int_{\mathbb{T}} w_1(x,y)dy = \partial_1\int_{\mathbb{T}} w_2(x,y)dy.$$

Now, let us solve the problem by assuming

$$a^{11} = 1 \quad \text{and} \quad a^{12} = 0.$$

Let $\psi \in C^\infty$, $0 \le \psi \le 1$, so defined: $\psi \equiv 1$ near $x = -a$ and $\psi \equiv 0$ near $x = a$.

Then construct the function $u = u(x,y)$ as follows:

$$u = \psi w_1 + (1-\psi)w_2 = w_2 + \psi(w_1 - w_2).$$

Clearly, $u > 0$. Moreover, as it can be easily verified, u satisfies the compatibility conditions (4) and (5).

Thus, from (5) we have:

$$\int_{\mathbb{T}} \partial_{11}u(x,y)dy = 0 \quad \text{in} \quad \Omega_\delta.$$

Therefore, there exists a function $v > 0$, 2π - periodic in y and such that:

$$\frac{\partial^2 v}{\partial y^2}(x,y) = -\partial_{11}u(x,y) \quad \text{in} \quad \Omega_\delta.$$

Then:

$$\partial_{11}u + \partial_{22}\left(\frac{v}{u}\cdot u\right) = 0 \quad \text{in} \quad \Omega.$$

So, the problem is solved in Ω_δ with

$$a^{11} = 1,\ a^{22} = \frac{v}{u},\ a^{12} = 0 \quad \text{in} \quad \Omega$$

and

$$a^{11} = a^{22} = 1 \quad \text{in} \quad \Omega_\delta \backslash \Omega.$$

REMARK. – The same arguments could be used to get the same result in the case w_1 and w_2 both negative. Actually, it remains to be investigated the case $w_1 \cdot w_2 < 0$.

Therefore, we may state the following

THEOREM 1. – *Let w_1 and w_2 two harmonic functions defined in Ω_δ. Assume:*

$$w_1 \cdot w_2 > 0 \quad \textit{and} \quad \int_{\mathbb{T}} w_1 dy = \int_{\mathbb{T}} w_2 dy.$$

Then, PROBLEM (*) is solvable in Ω_δ.

PROOF. – It is given exactly by all the above arguments of the present section. □

A consequence of THEOREM 1 is the following

THEOREM 2. – *Let w_k $(k = 1,2)$ harmonic functions in a neighbourhood $\mathcal{N}$ of $x = 0 \quad (y \in \mathbb{T}), \quad w_1 \cdot w_2 > 0$, such that:*

$$w_1(0,y) = w_2(0,y) > 0, \quad \partial_1 \int_{\mathbb{T}} w_1(0,y)dy = \partial_1 \int_{\mathbb{T}} w_2(0,y)dy.$$

Then, there exists a function u, defined in $\mathcal{N}$, such that:

$$\begin{cases} L^* u &= 0 & \textit{in} & \mathcal{N} \\ u &= w_1 & \textit{for} & x < \tilde{x} < 0 \\ u &= w_2 & \textit{for} & x > \tilde{\tilde{x}} > 0. \end{cases}$$

PROOF. – In Theorem 1, let a, $\delta > 0$ so small that, in $|x| < a + \delta$, $w_1 \geq c_1 > 0$ and $w_2 \geq c_2 > 0$. REMARK 1 of Section 2 implies that

$$\partial_1 \int_{\mathbb{T}} w_1 dy = \partial_1 \int_{\mathbb{T}} w_2 dy = c$$

in $|x| < a + \delta$ and $w_1 = w_2$ at $x = 0$ and REMARK 2 imply that

$$\int_{\mathbb{T}} w_1 dy = \int_{\mathbb{T}} w_2 dy.$$

Then, u can be constructed as before and, from the previous Theorem, THEOREM 2 follows. □

REFERENCES

[1] M. V. SAFONOV, *Unimprovability of estimates of Hölder constants for solutions of linear elliptic equations with measurable coefficients*, Matem. Sbornïk tom 132 (174) (1987), english transl. Math. USSR Sbornïk, **60**, no. 1 (1988), 269-281.

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