Bollettino Unione Matematica Italiana

F. Giannessi, G. Matroeni, X. Q. Yang

A Survey on Vector Variational Inequalities

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **2** (2009), n.1, p. 225–237.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2009_9_2_1_225_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2009.

Bollettino U. M. I.
(9) **II** (2009), 225-237

A Survey on Vector Variational Inequalities

F. Giannessi - G. Mastroeni - X. Q. Yang

Dedicated to the memory of Guido Stampacchia

Abstract. – *The paper consists in a brief overview on Vector Variational Inequalities (VVI). The connections between VVI and Vector Optimization Problems (VOP) are considered. This leads to point out that necessary optimality conditions for a VOP can be formulated by means of a VVI when the objective function is Gâteaux differentiable and the feasible set is convex. In particular, the existence of solutions and gap functions associated with VVI are analysed. Gap functions provide an equivalent formulation of a VVI, in terms of a constrained extremum problem. Finally, Vector Complementarity Problems and their relationships with VVI are considered.*

1. – Introduction.

Variational Inequalities (for short, VI) were introduced by Stampacchia in early sixties in the field of Calculus of Variations. Since then, they have extensively been studied because they have shown to be a powerful tool in many fields of optimization: from the classical optimality conditions for constrained extremum problems to the equilibrium conditions for network flow, economic and mechanical engineering equilibrium problems [15, 13, 12, 27].

The great advantage of VI models is that the equilibrium is not necessarily the extremum of a functional, like energy, so that no such a functional must be supposed to exist. In particular, the finite-dimensional formulation of a VI has become of particular interest after it has been shown its equivalence with the Wordrop equilibrium principle for traffic equilibrium and with the classic Complementarity Problem, in the case where the feasible set of the VI is a closed convex cone.

Recently, in some fields like Industrial Systems, Logistics and Management Science, there has been a strong request of mathematical models for optimizing situations with concurrent objectives, namely Vector Optimization Problems (for short, VOP), which were initially introduced by W. Pareto. Following the development of Vector Optimization, the theory of VI has been generalized to the vector case, with the aim to exploit the advantage of both variational and extremization methods.

In the present paper, we will briefly outline some of the main topics concerning Vector Variational Inequalities (VVI). In Section 2, we will introduce VVI and consider the main connections with VOP. Necessary optimality conditions for a VOP can be formulated in terms of a VVI when the objective function of the vector problem is Gateaux differentiable and the feasible set is convex. Suitable generalized convexity assumptions, namely pseudoconvexity, ensure that a weak VVI is a sufficient optimality condition for a weak vector minimum point. In such a context the Minty VVI is of particular importance since it provides a necessary and sufficient optimality condition for a Pareto solution of a differentiable convex problem.

Section 3 will be devoted to the analysis of the existence of the solutions.

In Section 4, we will present gap functions for VVI, which provide an equivalent formulation of a VVI in terms of a constrained extremum problem. Actually, a gap function $p : K \longrightarrow R$ is a non-negative function that fulfils the condition $p(x) = 0$ if and only if x is a solution of VVI on K. This definition, which originally has been given for a VI, can be extended to the vector case, in terms of a set-valued function.

In Section 5, we will introduce Vector Complementarity Problems (VCP) and analyse their relationships with VVI and VOP. Unlike the scalar case, the equivalence between VCP and VVI is, in general, no longer preserved: we will show a particular instance of such an occurrence.

2. – Vector Variational Inequalities.

Let X and Y be Hausdorff topological vector spaces. By $L(X, Y)$, we denote the set of all linear continuous functions from X into Y. For $l \in L(X, Y)$, the value of linear function l at x is denoted by $\langle l, x\rangle$. Let $C \subset Y$ be a nonempty, pointed, closed and convex cone with $intC \neq \emptyset$. For convenience, we will denote $C \setminus \{0\}$ and $int\,C$ by C_o and $\overset{\circ}{C}$ respectively. Then (Y, C) is an ordered Hausdorff topological vector space with a partial ordering defined by

$$y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C, \quad y_1, y_2 \in Y.$$

Moreover, we also define

$$y_1 \not\leq_{C_o} y_2 \Longleftrightarrow y_2 - y_1 \notin C_o;$$

$$y_1 \not\leq_{\overset{\circ}{C}} y_2 \Longleftrightarrow y_2 - y_1 \notin \overset{\circ}{C}\ .$$

These partial orderings can also be applied to sets where the ordering is understood as element-wise.

Let $T : K \to L(X, Y)$ and $K \subset X$ be a nonempty, closed and convex subset. A

VVI consists in finding $x^* \in K$, such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{C_o} 0, \quad \forall x \in K. \tag{2.1}$$

A weak *VVI* consists in finding $x^* \in K$, such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall x \in K. \tag{2.2}$$

When $Y = \mathbb{R}$ and $X = \mathbb{R}^n$, both (2.2) and (2.1) reduce to a scalar variational inequality; see [13].

Consider a Vector Optimization Problem (*VOP*):

$$\min_{x \in K}{}_C f(x), \tag{2.3}$$

where $f : X \to Y$ is a vector-valued function. The point $x^* \in K$ is said to be a weak Pareto solution of f on K, if and only if $f(K) \not\leq_{\overset{\circ}{C}} f(x^*)$ and a Pareto solution of f on K, if and only if $f(K) \not\leq_{C_o} f(x^*)$.

$f : X \to Y$ is a C-function on K, if and only if, for any $x_1, x_2 \in K, \lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_C \lambda f(x_1) + (1 - \lambda)f(x_2).$$

$f : X \to Y$ is a strict C-function on K, if and only if, for any $x_1 \neq x_2 \in K, \lambda \in (0, 1)$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_{\overset{\circ}{C}} \lambda f(x_1) + (1 - \lambda)f(x_2).$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the previous definitions collapse to the classic convexity and strict convexity, respectively.

The following proposition summarizes the relationships between (2.1), (2.2) and (2.3).

PROPOSITION 2.1. – *Assume that f is Gâteaux differentiable with Gâteaux derivative Df. Let $T = Df$. We have*

(i) *If x is a weak Pareto solution of* (2.3), *then x solves* (2.2).

(ii) *If f is C-convex and x solves* (2.2), *then x is a weak Pareto solution of* (2.3).

(iii) *If f is a C-function and x solves* (2.1), *then x is a Pareto solution of* (2.3).

(iv) *If $-f$ is a strict C-function and x^* is a Pareto solution of* (2.3), *then x solves* (2.1).

When $Y = \mathbb{R}^\ell$, the ordering $\not\leq_{\overset{\circ}{C}}$ is defined by the algebraic interior of C and T is chosen as a weak subgradient of f at x, (i) is first given in [3]. The general case of (i) and (ii) are given in [5]. When $Y = \mathbb{R}^\ell$, and $C = \mathbb{R}^\ell_+$, (iii) and (iv) are obtained in [27]. But the general cases easily follow the same proof as in [27].

Without the assumption that $-f$ is a strict C-function, this may not be true. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the problem $\min_C f(x)$, subject to $x \in [-1, 0]$ where $f(x) = (x, x^2 + 1)^\top$. It is clear that every $x \in [-1, 0]$ is a

Pareto solution of the problem. But $x = 0$ is not a solution of (2.1). The set of solutions for (2.2) and (2.1) is $[-1, 0]$ and $[-1, 0)$ respectively.

Two directional derivatives of $h : \mathbb{R}^n \to \mathbb{R}$ at x in the direction y are defined respectively as

$$h^{\top}(x; y) = \lim_{\varepsilon \downarrow 0} \limsup_{t \downarrow 0} \inf_{v \to y} \frac{h(x + tv) - h(x)}{t},$$

$$h^{+}(x; y) = \limsup_{t \downarrow 0} \frac{h(x + ty) - h(x)}{t}.$$

Two relevant VVIs are defined respectively as of finding $x^* \in K$ such that

$$(f_1^{\top}(x^*; x - x^*), \cdots, f_\ell^{\top}(x^*; x - x^*)) \not\leq_{\overset{\circ}{C}} 0, \ \forall x \in K, \tag{2.4}$$

$$(f_1^{+}(x^*; x - x^*), \cdots, f_\ell^{+}(x^*; x - x^*)) \not\leq_{\overset{\circ}{C}} 0, \ \forall x \in K. \tag{2.5}$$

THEOREM 2.1. – [23]

(i) *Assume that the functions* $f_i, i = 1, \cdots, \ell$, *are lower semicontinuous and that*

$$\sum_{i=0}^{\ell} z_i = 0, z_0 \in N(K, x^*), z_i \in \partial^{\infty} f_i(x^*), i = 1, \cdots, \ell, \Longrightarrow z_i = 0, i = 0, 1, \cdots, \ell,$$

where $N(K, x^*)$ *and* $\partial^{\infty} f_i(x^*)$ *are the normal cone to* K *at* x^* *and singular approximate subdifferential of* f *at* x^* *respectively. If* x^* *is a weak Pareto solution of* (2.3), *then* x^* *is a solution of* (2.4).

(ii) *Assume that* $f_i, i = 1, \cdots, \ell$, *are* $\top$*-pseudoconvex at* x^*, *that is,* $\forall x \in K, f_i(x) < f_i(x^*)$ *implies* $f_i^{\top}(x^*; x - x^*) < 0$. *If* x^* *is a solution of* (2.4), *then* x^* *is a weak Pareto solution of* (2.3).

(iii) *Assume that* $f_i, i = 1, \cdots, \ell$, *are* $+$ *pseudoconvex at* x^*, *that is,* $\forall x \in K, f_i(x) < f_i(x^*)$ *implies* $f_i^{+}(x^*; x - x^*) < 0$. *The point* x^* *is a solution of* (2.5) *if and only if* x^* *is a weak Pareto solution of* (2.3).

EXAMPLE 2.1. – Let

$$f_1(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } x \geq 0 \text{ or if } x = -1/i, i \text{ is a natural number}, \\ 0, & \text{otherwise} \end{cases}$$

and $S = \mathbb{R}$. Then

$$N(K,0) = \{0\},\ \partial^{\infty} f_1(0) = \{0\},\ \partial^{\infty} f_2(0) = \mathbb{R},$$

$$f_1^{\top}(0;x) = \max\{0,x\}, f_2^{\top}(0;x) = x.$$

So the assumptions of (i) Theorem 2.1 hold at $x^* = 0$ and $f_i, i = 1,2$ are $\top$ pseudoconvex at 0. Thus 0 is a weak Pareto solution of (2.3).

The Minty VVI consists in finding $x^* \in K$, such that

$$\langle T(x), x - x^* \rangle \not\leq_{C_0} 0,\ \forall x \in K. \tag{2.6}$$

The following result shows that a Pareto solution of (2.3) can be completely characterized by the Minty VVI when $C = \mathbb{R}^{\ell}_{+}$.

THEOREM 2.2. – [11] *Let* $X = \mathbb{R}^n$, $Y = \mathbb{R}^{\ell}$ *and* $C = \mathbb{R}^{\ell}_{+}$. *Let* $T(x) = \nabla f(x)$. *Let* f *be a* $\mathbb{R}^n_+$*-function and v-hemicontinuous on K. Then,* x^* *is a Pareto solution of* (2.3) *if and only if it is a solution of the Minty VVI.*

The conclusions of this theorem still hold if f is assumed to be $\mathbb{R}^n_+$-pseudoconvex, see [24]. Relations between Minty VVI and the proper efficiency of a solution is investigated in [7].

This theorem is further generalized to a weak Minty vector variational-like inequality problem with a variable ordering cone in [1].

Gap functions for Minty VVI and their differential properties are studied in [21].

In general, a solution of (2.2) is not one for (2.1). This is shown not to be true by an example in [28] even if T is strongly monotone. However under the further assumption on K, a positive answer is as follows.

PROPOSITION 2.2. – [28] *Let* $Y = \mathbb{R}^{\ell}$. *Suppose that*

1. *for any* $x \neq x' \in K$ *and* $\lambda \in (0,1)$, $(1-\lambda)x + \lambda x' \in \hat{K}$,
2. *for each* $x \in K$, $v \to \langle T(x), v \rangle$ *is surjective.*

Then a solution of (2.2) *is one for* (2.1).

3. – Existence of Solutions.

The following generalized linearization lemma and Knaster, Kuratowski and Mazurkiewicz Theorem (KKM Theorem, in short) have played a key role in the establishment of the existence of a solution for (2.2).

LEMMA 3.1. (Generalized Linearization Lemma). – [5] *Let the mapping* $T : X \to L(X, Y)$ *be monotone and v-hemicontinuous. Then the following two problems are equivalent:*

1. $x \in K,\ \langle T(x), y - x \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall y \in K;$
2. $x \in K,\ \langle T(y), y - x \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall y \in K.$

When $Y = \mathbb{R}$, this is the linearization lemma in [2, 22]. This linearization lemma is generalized to the invex type VVI in [1].

LEMMA 3.2. (KKM Theorem). – [17] *Let* K *be a subset of a topological vector space* V. *For each* $x \in K$, *let a closed and convex set* $F(x)$ *in* V *be given such that* $F(x)$ *is compact for at least one* $x \in K$. *If the convex hull of every finite subset* $\{x_1, x_2, \cdots, x_k\}$ *of* K *is contained in the corresponding union* $\cup_{i=1}^{n} F(x_i)$, *then* $\cap_{x \in K} F(x) \neq \emptyset$.

Assume that K is compact. We set

$$F_1(y) = \{x \in K : \langle T(x), y - x \rangle \not\leq_{\overset{\circ}{C}} 0\}, \quad y \in K,$$

$$F_2(y) = \{x \in K : \langle T(y), y - x \rangle \not\leq_{\overset{\circ}{C}} 0\}, \quad y \in K.$$

It can be shown that the convex hull of every finite subset $\{x_1, x_2, \cdots, x_k\}$ of K is contained in the corresponding union $\cup_{i=1}^{n} F_1(x_i)$. Since $F_1(y) \subset F_2(y)$ for all $y \in K$, this is also true for F_2. By Lemma 3.1, we have

$$\cap_{y \in K} F_1(y) = \cap_{y \in K} F_2(y).$$

We observe that for each $y \in K$, $F_2(y)$ is a (weakly) compact subset in K.

By Lemma 3.2, we have

$$\cap_{y \in K} F_1(y) = \cap_{y \in K} F_2(y) \neq \emptyset.$$

Hence, there exists an $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall x \in K.$$

Assume that K is unbounded and $T : K \to L(X, Y)$ is weakly coercive on K, that is, there exist $x_0 \in K$ and $c \in int\, C^*$ such that

$$\langle c \circ T(x) - c \circ T(x_0), x - x_0 \rangle / \|x - x_0\| \to +\infty,$$

whenever $x \in K$ and $\|x\| \to +\infty$. In a similar way, we can show that $\cap_{y \in K} F_1(y) \neq \emptyset$.

As such we have the following result, where the weak topology of X and the norm topology of Y are used.

THEOREM 3.1. – [5] *Assume that X is a reflexive Banach space and $K \subset X$ is convex. Assume that (Y, C) is an ordered Banach space with $\overset{\circ}{C} \neq \emptyset$ and $intC^* \neq \emptyset$. Let the mapping $T : K \to L(X, Y)$ be monotone, v-hemicontinuous and let, for any $y \in K$, $T(y)$ be completely continuous on X. If*

1. *K is compact, or*

2. *K is closed, T is weakly coercive on K,*

then the weak vector variational inequality (2.2) *is solvable.*

THEOREM 3.2. – [26] *Assume that X is a reflexive Banach space and $K \subset X$ is convex. Assume that (Y, C) is an ordered Banach space with $\overset{\circ}{C} \neq \emptyset$ and $int\, C^* \neq \emptyset$. Suppose that*

1. *there is a compact subset $B \subset X$ and $y_0 \in B \cap K$, such that*

$$\langle T(y_0), y_0 - x\rangle \leq_{\overset{\circ}{C}} 0, \quad \forall x \in K \setminus B,$$

2. *for any $\{x_1, \cdots, x_k\} \subset K, x = \sum_{i=1}^{k} a_i x_i, \sum_{i=1}^{k} a_i = 1, a_i \geq 0,$*

$$\sum_{i=1}^{k} a_i \langle F(x_i), x\rangle \leq \langle F(x), x\rangle \leq \sum_{i=1}^{k} a_i \langle F(x_i), x_i\rangle,$$

then the following weak vector variational inequality which consists in finding $x^* \in K$ such that

$$\langle T(x), x - x^*\rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall x \in K,$$

is solvable.

KKM Theorem cannot be applied to the establishment of the existence of solutions of (2.1) as the sets $F_1(y)$ and $F_2(y)$ where $\not\leq_{\overset{\circ}{C}}$ is replaced by $\not\leq_{C_o}$ are not closed anymore.

Only very recently, an existence result for (2.1) has been obtained by using the Browder fixed point theorem.

THEOREM 3.3. – [9] *Assume that X is a reflexive Banach space and $K \subset X$ is convex. Assume that (Y, C) is an ordered Banach space with $\overset{\circ}{C} \neq \emptyset$ and $intC^* \neq \emptyset$. Let the mapping $T : K \to L(X, Y)$. If*

1. *K is compact, and for each $y \in K$, the set $\{x \in K : \langle T(x), y - x\rangle \leq_{C_o} 0\}$ is open in K, or*

2. *K is closed, T is continuous, and weakly coercive on K,*

then the vector variational inequality (2.1) *is solvable.*

Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $F_j(j = 1, \cdots, \ell) : \mathbb{R}^n \to \mathbb{R}^n$. Let $F = (F_1, \cdots, F_\ell)$ and

$$\ll F(x), y - x \gg = \begin{bmatrix} \langle F_1(x), y - x \rangle \\ \vdots \\ \langle F_\ell(x), y - x \rangle \end{bmatrix}.$$

The quasi-vector variational inequality problem (QVVI) consists in finding $x \in K(x)$, such that

$$\ll F(x), y - x \gg \not\leq_{\overset{\circ}{C}} 0, \forall y \in K(x). \tag{3.1}$$

THEOREM 3.4. – [3] *Suppose that there is a nonempty compact and convex set* K_1 *such that*

1. $K(K_1) \subset K$;
2. $F_j(j = 1, \cdots, \ell)$ *is continuous on* K_1;
3. K *is a nonempty continuous convex-valued mapping on* K_1.

Then there exists a solution to (3.1).

The study of a vector variational inequality has also been pursued by introducing another model with a similar form and using the tool of conjugate function of a vector-valued function. Here such a model is called a primitive of a VVI.

Let $T : X \longrightarrow L(X, Y)$ be a function, and $h : X \to Y$ is a function. The (VVI$_h$) problem consists in finding $x^* \in X$, such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{C_o} h(x^*) - h(x), \ \forall x \in X.$$

Assume that T is one-to-one (injective). Define $T' : L(X, Y) \to X$ as follows:

$$T'(l) := -T^{-1}(-l), \quad \forall l \in \text{Domain}\,(T') = -\text{Range}\,(T).$$

If T is linear, then $T' = T^{-1}$.

The primitive of (VVI$_h$) problem is defined as: finding $l^* \in \text{Domain}\,(T')$, such that

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{C_o} h^*_{\leq}(l^*) - h^*_{\leq}(l), \quad \forall l \in L(X, Y), \tag{IVVI$_\text{h}$}$$

where $h^*_{\leq}(l) := \text{Max}_C\{\langle l, x \rangle - h(x) : x \in X\}$ is the vector conjugate function of h.

THEOREM 3.5. – [26] *Let X be a Hausdorff topological vector space and (Y, C) be an ordered Hausdorff topological vector space. The function T is one-to-one and $h : X \to Y$ is continuous. Assume that $h^*_{\leq}(l) \neq \varnothing, \forall l \in L(X, Y)$.*

(i) *If x^* is a solution of (VVI$_h$), then $l^* = -T(x^*)$ is a solution of (IVVI$_h$) and* *the following relation is satisfied:*

$$\langle l^*, x^* \rangle \in h(x^*) + h^*_{\leq}(l^*).$$

(ii) *If l^* is a solution of (IVVI$_h$), C is connected, i.e., $C \cup (-C) = Y$, and h is weakly subdifferentiable at x^*, where $x^* = -T'(l^*)$, then x^* is a solution of (VVI$_h$).*

This result is generalized to a set-valued VVI in [18].

4. – Gap Functions.

The concept of a gap function is well-known both in the context of convex optimization and variational inequalities. The minimization of gap functions is a viable approach for solving variational inequalities.

A set-valued function $\phi_w : K \rightrightarrows Y$ is said to be a gap function of (2.2) if and only if (i) $0 \in \phi_w(x^*)$ if and only if x^* solves (2.2); and (ii) $0 \not\geq_{\overset{\circ}{C}} \phi_w(x), \forall x \in K$. A set-valued function $\phi : K \rightrightarrows Y$ is said to be a gap function of (2.1) if and only if (i) $0 \in \phi(x^*)$ if and only if x^* solves (2.1); and (ii) $0 \not\geq_{C\setminus\{0\}} \phi(x)$, $x \in K$.

PROPOSITION 4.1. – *Let C be a pointed and convex cone in Y. We have*

(i) *The set-valued function $\phi_w(x) := \mathrm{Max}_{\overset{\circ}{C}} \langle T(x), x - K \rangle$ is a gap function for* (2.2).

(ii) *The set-valued function $\phi(x) := \mathrm{Max}_C \langle T(x), x - K \rangle$ is a gap function for* (2.1).

The above gap functions are of set-valued nature. Special single-valued gap functions can be constructed in terms of nonlinear scalarization functions. Given a fixed $e \in \overset{\circ}{C}$ and $a \in Y$, the nonlinear scalarization function is defined by:

$$\xi_{ea}(y) = \min\{t \in \mathbb{R} : y \in a + te - C\}, \quad y \in Y.$$

PROPOSITION 4.2. – *Let $e \in \overset{\circ}{C}$. Then $x^* \in K$ solves* (2.2) *if and only if the nonpositive function $g(x) = \min_{y \in K} \xi_{e0}(\langle T(x), y - x \rangle)$ has a zero at x^*.*

In the special case where $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and $T(x) = [T_1(x), \cdots, T_\ell(x)]^\top$, the nonlinear scalarization function may be expressed in the following equivalent form:

$$\xi_{ea}(y) = \max_{1 \leq i \leq \ell} \frac{y_i - a_i}{e_i}.$$

Thus $g(x) = \min_{y \in K} \max_{1 \le i \le \ell}\{\langle T_i(x), y - x\rangle\}, x \in K$. The value of each $g(x)$ amounts to solving a linear minimax optimization problem.

Next we construct a gap function for a set-valued WVVI.

Let $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and $K \subset X$ a compact subset. Assume that $T : K \rightrightarrows L(X, \mathbb{R}^\ell)$ is a set-valued mapping with a compact set $T(x)$ for each x.

Consider the set-valued WVVI with the set-valued mapping T [16], which consists in finding $x^* \in K$, and $\bar{t} \in T(x^*)$ such that

$$\langle \bar{t}, x - x^*\rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall x \in K. \tag{4.1}$$

Let $x, y \in K$ and $t \in T(x)$. Denote

$$\langle t, y\rangle = ((\langle t, y\rangle)_1, \cdots, (\langle t, y\rangle)_\ell),$$

i.e., $(\langle t, y\rangle)_i$ is the i-th component of $\langle t, y\rangle$, $i = 1, \cdots, \ell$. We define two mappings $\phi_1 : K \times L(X, \mathbb{R}^\ell) \to R$ and $\phi : K \to \mathbb{R}$ as follows

$$\phi_1(x, t) = \min_{y \in K} \max_{1 \le i \le \ell} (\langle t, y - x\rangle)_i \tag{4.2}$$

and

$$\phi(x) = \max\{\phi_1(x, t) | t \in T(x)\}. \tag{4.3}$$

Since K is compact, $\phi_1(x, t)$ is well-defined. If X is a Hausdorff topological vector space, then $g_1(x, t)$ is a lower semi-continuous function in x. Since $T(x)$ is a compact set, $\phi(x)$ is well-defined.

THEOREM 4.1. – *$\phi(x)$ defined by* (4.3) *is a gap function of the set-valued WVVI.*

By Theorem 4.1, solving the set-valued WVVI is equivalent to finding a global solution x^* to the following generalized semi-infinite programming problem

$$\begin{aligned} \max_{x,s} \quad & s \\ \text{s.t.} \quad & \phi_1(x, t) \le s, \quad \forall t \in T(x), \\ & \phi_1(x, t_1) = s, \quad \exists t_1 \in T(x), \\ & x \in K. \end{aligned}$$

5. – Vector Complementarity Problems.

The concept of vector complementarity problems was introduced in [5, 25]. The weak vector complementarity problem (WVCP) consists in finding $x^* \in K$

such that

$$0 \not\leq_{\overset{\circ}{C}} \langle T(x^*), x^* \rangle \not\leq_{\overset{\circ}{C}} 0, \tag{5.1}$$

$$\langle T(x^*), y \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall y \in D. \tag{5.2}$$

Let the weak C-dual cone D_C^{w+} of D be defined by

$$D_C^{w+} = \{g \in L(X,Y) : \langle g, x \rangle \not\leq_{\overset{\circ}{C}} 0, \quad \forall x \in D\}.$$

Then (WVCP) can be rewritten as a problem of finding $x^* \in D$, such that

$$\langle T(x^*), x^* \rangle \not\geq_{\overset{\circ}{C}} 0, \quad T(x^*) \in D_C^{w+}.$$

Thus a solution of (2.2) is one for (WVCP), but the fact that the inverse implication is in general not true can be shown by some simple example. Nevertheless, the inverse implication can be guaranteed by the usual positiveness property on T. Indeed, let the strong C-dual cone D_C^{s+} of D be defined by

$$D_C^{s+} = \{g \in L(X,Y) : \langle g, x \rangle \geq_C 0, \quad \forall x \in D\}.$$

The positive vector complementarity problem (PVCP) consists in finding an $x^* \in D$ such that

$$\langle T(x^*), x^* \rangle \not\geq_{\overset{\circ}{C}} 0, \quad T(x^*) \in D_C^{s+}.$$

Let $\mathcal{F} = \{x \in C | T(x) \in D_C^{w+}\}$.

THEOREM 5.1. – [5] *If $H \cap (Y \setminus \overset{\circ}{C}) \neq \emptyset$, then (WVCP) has a solution, where $H = f(E)$, $f(x) = \langle T(x), x \rangle$ and E is the set of all weak Pareto solutions for* (2.3) *with $K = \mathcal{F}$.*

For a given $l \in L(X,Y)$, the nonlinear VOP consists in finding an $x \in \mathcal{F}$ such that x is a solution to ((2.3) with $f(x) = \langle l, x \rangle$ and $K = \mathcal{F}$. The weak minimal element problem (WMEP) consists in finding an $x \in \mathcal{F}$ such that $x \not\geq_{\overset{\circ}{C}} \mathcal{F}$. Let $f : X \to Y$. The vector unilateral minimization problem (VUMP) consists in finding an $x \in C$ such that x is a solution of (2.3) with $K = C$.

The equivalences among these problem are presented as follows.

THEOREM 5.2. – [5] *Assume that*

(i) *$T = Df$ is the Frechet derivative of f;*

(ii) *l is a weak positive linear operator, i.e., $x \not\geq_{\overset{\circ}{C}} 0 \Longrightarrow \langle l, x \rangle \not\geq_{\overset{\circ}{C}} 0$;*

(iii) *there exists $x \in \mathcal{F}$ such that $T(x)$ is one-to-one and completely continuous;*

(iv) *X is a topological dual space of a real normed space and the norm $\|\cdot\|$ in X is strictly monotonically increasing on C,* *If the nonlinear VOP is solvable, then, WMEP, VCP and VUMP have a solution, respectively.*

These equivalences are generalized in [8, 14].

REFERENCES

[1] Q. H. ANSARI - J. C. YAO, *On nondifferentiable and nonconvex Vector Optimization Problems*. J. Optim. Theory Appl., **106**, no. 3 (2000), 475-488.
[2] E. BROWDER FELIX, *Existence and approximation of solutions of nonlinear variational inequalities*. Proc. Nat. Acad. Sci. U.S.A., **56** (1966), 1080-1086.
[3] G. Y. CHEN - G. M. CHENG, *Vector Variational Inequality and Vector Optimization*, In "Lecture Notes in Econ. and Mathem. Systems", **285** (Springer-Verlag, New York-Berlin, 1987), 408-416.
[4] G. Y. CHEN, - X. X. HUANG - X. Q. YANG, *Vector optimization. Set-valued and variational analysis*. Lecture Notes in Economics and Mathematical Systems, **541** (Springer-Verlag, Berlin, 2005).
[5] G. Y. CHEN - X. Q. YANG, *The vector complementary problem and its equivalences with vector minimal element in ordered spaces*. J. Math. Anal. Appl., **153** (1990), 136-158.
[6] G. Y. CHEN - N. D. YEN, *On the variational inequality model for network equilibrium*. Internal Report, Department of Mathematics, University of Pisa, no. 3 (1993), 196 (724).
[7] G. P. CRESPI - I. GINCHEV and M. ROCCA, *Minty vector variational inequality, efficiency and proper efficiency*. (English summary) Vietnam J. Math., **32**, no. 1 (2004), 95-107.
[8] Y. P. FANG - N. J. HUANG, *Least element problems of feasible sets for vector F-complementarity problems with pseudomonotonicity*. (Chinese) Acta Math. Sinica, **48**, no. 3 (2005), 499-508.
[9] Y. P. FANG - N. J. HUANG, *Strong vector variational inequalities in Banach spaces*. Applied Mathematics Letters, **19** (2006), 362-368.
[10] F. GIANNESSI, *Theorems of alternative, quadratic programs and complementary problems*. In Cottle R. W. and Giannessi F. and Lions J. L. (eds.), Variational Inequality and Complementary Problems (Wiley, New York 1980).
[11] F. GIANNESSI, *On Minty variational principle*. In New Trends in Mathematical Programming (Kluwer Academic Publishers, 1998), 93-99.
[12] F. GIANNESSI (ed.), *Vector Variational Inequalities and Vector Equilibria*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
[13] P. T. HARKER - J. S. PANG, *Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications*. Math. Programming (Ser. B), **48**, no. 2 (1990), 161-220.
[14] N. J. HUANG - Y. P. FANG, *Strong vector F-complementary problem and least element problem of feasible set*. Nonlinear Anal., **61**, no. 6 (2005), 901-918.
[15] D. KINDERLEHRER - G. STAMPACCHIA, *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.
[16] I. V. KONNOV - J. C. YAO, *On the generalized vector variational inequality problem*. J. Math. Anal. Appl., **206**, no. 1 (1997), 42-58.
[17] MARC LASSONDE, *On the use of KKM multifunctions in fixed point theory and related topics*. J. Math. Anal. Appl., **97**, no. 1 (1983), 151-201.

[18] G. M. LEE - D. S. KIM - B. S. LEE - G. Y. CHEN, *Generalized Vector Variational Inequality and its duality for set-valued maps*. Appl. Mathem. Lett., **11** (1998), 21-26.

[19] G. M. LEE - D. S. KIM - B. S. LEE - G. Y. CHEN, *Vector variational inequality as a tool for studying vector optimization problems*. Nonlinear Anal., **34**, no. 5 (1998), 745-765.

[20] G. MASTROENI, *On Minty Vector Variational Inequality*. In [12], 351-361.

[21] K. W. MENG - S. J. LI, *Differential and sensitivity properties of gap functions for Minty vector variational inequalities*. J. Math. Anal. Appl., **337**, no. 1 (2008), 386-398.

[22] U. MOSCO, *Implicit variational problems and quasi variational inequalities. Nonlinear operators and the calculus of variations* (Summer School, Univ. Libre Bruxelles, Brussels, 1975), Lecture Notes in Math., **543** (Springer, Berlin, 1976), 83-156.

[23] D. E. WARD - G. M. LEE, *On relations between Vector Optimization Problems and Vector Variational Inequalities*. J. Optim. Theory Appl., **113**, no. 3 (2002), 583-596.

[24] X. M. YANG - X. Q. YANG - K. L. TEO, *Some Remarks On Minty Vector Variational Inequality*. J. Optim. Theory Appl., **121**, no. 1 (2004), 193-201.

[25] X. Q. YANG, *Vector complementarity and minimal element problems*. J. Optim. Theory Appl., **77**, no. 3 (1993), 483-495.

[26] X. Q. YANG, *Vector variational inequalities and its duality*. Nonl. Anal., TMA, **21** (1993), 867-877.

[27] X. Q. YANG - C. J. GOH, *On vector variational inequalities: application to vector equilibria*. J. Optim. Theory Appl., **95** (1997), 431-443.

[28] N. D. YEN - G. M. LEE, *On monotone and strongly monotone Vector Variational Inequalities*. In [12], 467-478.

F. Giannessi - G. Mastroeni: Dipartimento di Matematica, Università di Pisa - Italy
e-mail: gianness@gauss.dm.unipi.it gianness@dm.unipi.it

X.Q. Yang, Department of Applied Mathematics
The Hong Kong Polytechnic University - China

Received April 20, 2008 and in revised form September 17, 2008