Bollettino Unione Matematica Italiana

Simon Salamon

Complex Structures and Conformal Geometry

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **2** (2009), n.1, p. 199–224.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2009_9_2_1_199_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2009.

Bollettino U. M. I.
(9) **II** (2009), 199-224

Complex Structures and Conformal Geometry (*)

Simon Salamon

Abstract. – *A characterization of certain complex structures on conformally-flat domains in real dimension* 4 *is carried out in the context of Hermitian geometry and twistor spaces. The presentation is motivated by some classical surface theory, whilst the problem itself leads to a refined classification of quadrics in complex projective* 3*-space. The main results are sandwiched between general facts in real dimension* $2n$ *and some concluding examples in real dimension* 6.

Introduction.

The title above is that of the author's talk given at the 18th Congress of the Italian Mathematical Society in Bari in September 2007. As explained at that event, a better title might be "Conformally flat Hermitian geometry", or even "Hermitian geometry without too many tensors". The article itself is based both on the Bari lecture, and a subsequent talk entitled "Advances in twistor theory", given at the CIRM in Luminy in November 2007 for the 20th anniversary of the publishing of Besse's *Einstein Manifolds* [8]. Whilst Einstein manifolds and twistor theory have a lot in common, the subject of this article is somewhat complementary to that intersection.

We shall investigate the existence and (in a suitable sense) the uniqueness of complex structures that are orthogonal with respect to a given Riemannian metric g on a manifold of even dimension $2n$. The associated endomorphism J (with $J^2 = -1$) is called an "orthogonal complex structure", abbreviated "OCS". If one starts with a complex manifold, it is always possible to choose a Hermitian metric g so that the endomorphism J is an orthogonal transformation relative to g. But in general, if we start with a metric g and $n > 1$, one can identify part of the curvature of g that provides a local obstruction to the existence of an OCS J. We shall explain how this works in Section 3.

It is actually conformal, rather than Riemannian, geometry that underlies the

(*) Conferenza Generale tenuta a Bari il 24 settembre 2007 in occasione del XVIII Congresso dell'Unione Matematica Italiana.

problem above. This is because if J is orthogonal with respect to g, it will be orthogonal with respect to any conformally related metric $e^f g$. Our focus will be on the "flat" case in four dimensions, that is the construction of complex structures on open sets of $\mathbb{R}^4$ that are orthogonal with respect to the Euclidean metric. The associated conformal structure extends to that of the one-point compactification S^4, since the "round" metric on the sphere is also conformally flat. Sections 4 and 5 incorporate characterizations of such OCSes, discovered jointly with J. Viaclovsky in an attempt to clarify some imprecisions in an earlier survey article [37].

The main technique for the investigation of OCSes is the construction of a twistor space with which to parametrize the structures pointwise. This approach has its origins in work of R. Penrose and E. Calabi in the 1970's, but took off in the 4-dimensional Riemannian setting with the papers of Atiyah-Hitchin-Singer [4] and Atiyah-Ward [5], which heralded striking applications. It led in particular to a focus of attention on that class of oriented 4-dimensional manifolds with self-dual conformal structure, meaning that half the Weyl tensor (namely W_+ or W_-) vanishes. Non-trivial examples of such manifolds were constructed by Poon [35], and subsequently LeBrun [29], and they were interpreted as quaternionic quotients by Joyce [26]. Later Pontecorvo [34, 20] developed the 4-dimensional theory in a direction closer to that of this article.

The use of twistor theory in higher dimensions was pioneered by (amongst others) Bérard Bergery-Ochiai [7], O'Brian-Rawnsley [33] and the author [8]. It was exploited in connection with harmonic maps or minimal surfaces by Bryant [12], Burstall-Rawnsley [13, 18]. Significant progress in 6 dimensions was made by Slupinski [40], and we shall touch on this aspect in Section 6. More recent applications of twistor theory in different contexts can be found in, for example, [6, 17].

The article is divided into six sections, and roughly two parts. The first part (Sections 1, 2, 3) surveys known theory, whilst the second (Sections 4, 5, 6) presents the newer work. Section 1 provides motivation from the classical theory of surfaces. The next two sections address the algebraic parmetrization of OCSes and questions of integrability. Sections 4 and 5 deal with the real 4-dimensional theory, and a final section draws together some observations mainly relevant for the 6-dimensional case.

1. – The classical case.

Let $M = M^2$ be a real surface embedded in $\mathbb{R}^3$. The standard inner (dot) product induces on M a Riemannian metric by restriction:

$$g(X, Y) = X \cdot Y, \qquad X, Y \in T_m M.$$

In local coordinates on M,

$$g = \sum_{i,j=1}^{2} g_{ij} dx_i \otimes dx_j, \tag{1}$$

where (g_{ij}) is a 2×2 postive-definite *symmetric* matrix. The same equation is written more traditionally as the *first fundamental form*

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2. \tag{2}$$

If M is the 3×2 matrix of partial derivatives determined by the parametrization, then $EG - F^2 = \det(M^\top M) > 0$ and $\sqrt{EG - F^2}\, dx \wedge dy$ is the *skew-symmetric* area form at each point. The interplay between symmetric and skew forms is an important feature of the theory of surfaces.

The vector cross product of $\mathbb{R}^3$ induces extra structure on $T_m M$. Assuming that M is orientable, one may choose a continuously-varying unit normal vector N. This allows one to define an endomorphism J of each tangent space by setting

$$JX = N \times X, \qquad X \in T_m M. \tag{3}$$

Since $J^2 X = N \times (N \times X) = -X$, we conclude that $J^2 = -1$, so that J is an *almost-complex structure* (the linear version of a complex structure on a vector space). An analogous construction with Cayley numbers allows one to define an almost-complex structure on any hypersurface of $\mathbb{R}^7$ [14, 11]; the one on S^6 will be mentioned at the end of the paper.

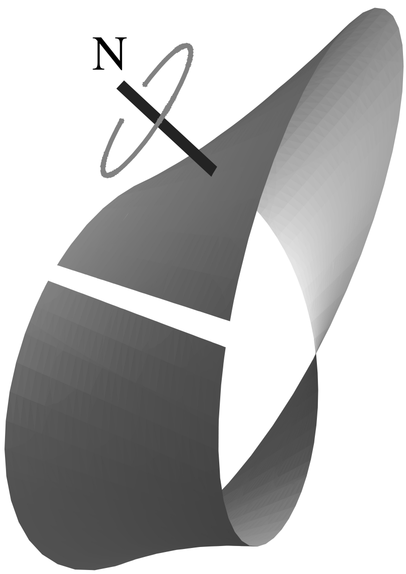

Fig. 1. – An oriented surface with unit normal vector.

In order to make a viable analytic statement, one must suppose that g is at least of class C^2 in local coordinates, though in practice it will be C^∞. The following result asserts the existence of so-called *isothermal coordinates* on such a surface M, and an effective proof can be found in [16].

THEOREM. – *On a neighourhood of any given point of M, there exist coordinates x, y for which $E = G$ and $F = 0$ in* (2) *so that $ds^2 = G(dx^2 + dy^2)$, and $g = (g_{ij})$ is a scalar function times the identity matrix.*

Given the conclusion, we can re-write (1) as

$$g = G(dx \otimes dx + dy \otimes dy),$$

and it follows that, up to sign,

$$J = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}. \tag{4}$$

Observe that the function G does not feature in the second equation. This absence reflects the conformal invariance of J, and means that the coordinates x, y render the coefficients of J *constant*. In other words, the geometrical structure defined by the tensor J is *flat* or *integrable*.

In the current situation, we can define a complex coordinate

$$z = x + iy,$$

and assert that any other $\tilde{z}$ (defined by an alternative choice $\tilde{x}, \tilde{y}$ satisfying (4)) will equal $f(z)$ where f is a *holomorphic* function (or more precisely, one satisfying the Cauchy-Riemann equations). This reflects the fact that any oriented conformal mapping between open sets of $\mathbb{C}$ is necessarily holomorphic. In this way, M has become a *complex manifold* of complex dimension one. In conclusion: in real dimension 2 an oriented conformal structure is equivalent to a complex structure.

There are many advantages in using isothermal coordinates x, y. One is that the Gaussian curvature K can be computed using the associated Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{4}\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Indeed, it follows (see e.g. [22]) that

PROPOSITION. – We have $K = -\dfrac{1}{2G}\Delta \log G$.

We next present Mercator's Projection as an example of this theory. Consider first the standard metric

$$ds^2 = d\theta^2 + \cos^2\theta \, d\phi^2$$

on the unit sphere, written in terms of latitude θ and longitude ϕ. The fact that there are no mixed terms $d\theta\, d\phi$ means that curves of constant latitude (parallels) and those of constant longitude (meridians) are everywhere mutually orthogonal. Whilst the angle θ faithfully measures distance along a meridian (note that the "earth" here has radius 1), distance along a parallel is $(\cos\theta)\phi$, a quantity which of course vanishes at the poles.

From the isothermal point of view, the combination of θ and ϕ in the second term on the right is bad, but we can improve matters by extracting $\cos^2\theta$ as a common factor:

$$ds^2 = \cos^2\theta\,(\sec^2\theta\, d\theta^2 + d\phi^2).$$

The next step is to integrate secant; setting $y = \log(\tan\theta + \sec\theta)$, we obtain $dy = \sec\theta\, d\theta$, and

$$(e^y\cos\theta - 1)^2 = \sin^2\theta = 1 - \cos^2\theta,$$

whence

$$\cos\theta = \frac{2}{e^y + e^{-y}} = \operatorname{sech} y.$$

Setting $\phi = x$ gives Mercator's parametrization

$$(5)\qquad \big(\cos x\operatorname{sech} y,\ \sin x\operatorname{sech} y,\ \tanh y\big)$$

of the sphere, with first fundamental form

$$ds^2 = \operatorname{sech}^2 y\,(dx^2 + dy^2),$$

emphasizing its conformal nature. In theory y ranges over $\mathbb{R}$, but in practice the domain of the parametrization fits into the page of an atlas because $y = 8$ reaches latitude $\theta > 1.57$.

Observations. – (i) Students discover (5) in differential geometry courses when they first compute the Gauss map of the catenoid

$$(6)\qquad \boldsymbol{x}(x,y) = (\cos x\cosh y,\ \sin x\cosh y,\ y).$$

Indeed, the unit normal vector

$$(7)\qquad N = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}$$

coincides with (5). The same is true if we replace (6) by the parametrization

$$(8)\qquad \boldsymbol{x}(x,s) = (s\cos x,\ s\sin x,\ s),$$

of the helicoid, and substitute $s = \sinh y$. It is well known that the Gauss map of any minimal surface is conformal, though the fact that (6) and (8) have the same unit normal is indicative of the more intimate relation between these surfaces [22].

(ii) The proposition tells us that

$$K = \cosh^2 y \, \frac{d^2}{dy^2} \log \cosh y \equiv 1,$$

confirming that the sphere has constant Gaussian curvature. A solution of the form $G = G(y)$ with $K \equiv -1$ is given by $G = 1/y^2$; the associated metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2), \tag{9}$$

is complete in the upper half plane H^2 and gives rise to one model of hyperbolic geometry.

(iii) According to the theory, any holomorphic function of $z = x + iy$ will give a valid conformal chart. For example, we see from (5), that

$$e^{iz} = e^{-y}(\cos x + i \sin x) = \frac{\cos x \operatorname{sech} y + i \sin x \operatorname{sech} y}{1 + \tanh x}$$

is precisely the stereographic projection (from the south pole) of the point (5) to the equatorial plane.

2. – Parametrizing orthogonal structures.

Let M^{2n} be an oriented manifold with a fixed conformal structure $[g]$. The latter denotes the set of all Riemannian metrics $e^f g$ in the same conformal class as a fixed one g.

PROBLEM. – Find, on an open subset $\Omega \subset M^{2n}$, an *orthogonal complex structure* (OCS), meaning an almost-complex structure J such that

$$g(JX, JY) = g(X, Y) \tag{10}$$

and

$$J[JX, Y] + J[X, JY] + [X, Y] - [JX, JY] = 0, \tag{11}$$

for all vector fields X, Y.

The compatibility condition (10) depends only on $[g]$. It means that JX is a vector orthogonal to X but of the same norm, and generalizes the concept of a 90° rotation in $\mathbb{R}^2$. Indeed, when $n = 1$, a conformal structure and orientation determine a unique J for which (11) is automatic. The left-hand side of (11) is the so-called *Nijenhuis tensor*, whose vanishing guarantees the existence of holomorphic coordinates [32] generalizing the theorem in Section 1.

Ignoring the integrability condition (11) for the moment, let us consider the pointwise choices constrained by (10). We shall generally suppose in addition that J is compatible with the orientation of M (so if n is odd then $-J$ is excluded).

Such a J determines a subgroup

$$U(n) \subset SO(2n)$$

(consisting of transformations that commute with J), that is a point of the Hermitian symmetric space

$$Z_n = \frac{SO(2n)}{U(n)}. \tag{12}$$

The uniqueness statement above for $n = 1$ merely reflects the fact that $SO(2) = U(1)$ and Z_1 is a point. In slightly higher dimensions, one has the following well-known isomorphisms:

$$Z_2 = \frac{SO(4)}{U(2)} = \frac{SU(2)\times SU(2)}{U(1) \times SU(2)} = \mathbb{CP}^1$$

$$Z_3 = \frac{SO(6)}{U(3)} = \frac{SU(4)/\mathbb{Z}_4}{U(3)} = \mathbb{CP}^3$$

$$Z_4 = \frac{SO(8)}{U(4)} = \frac{SO(8)}{SO(6)\times SO(2)} = \mathcal{Q}^6,$$

in which $\mathcal{Q}^6$ denotes a complex 6-dimensional quadric hypersurface of $\mathbb{CP}^7$.

An almost-complex structure J on $\mathbb{R}^{2n}$ (i.e. an endomorphism with $J^2 = -1$) is defined by specifying its $+i$ eigenspace $T^{1,0}$. The condition (10) is equivalent to asserting that $T^{1,0}$ is isotropic, so that

$$g(u, v) = 0, \qquad \forall u, v \in T^{1,0}. \tag{13}$$

In this way, we see that Z_n parametrizes maximal isotropic subspaces of $\mathbb{C}^{2n}$ compatible with the chosen orientation.

The identification above has an interpretation in terms of Clifford algebras. Let $\Delta = \Delta_+ \oplus \Delta_-$ denote the total spin representation of the double cover $Spin(2n)$ of $SO(2n)$, so that $\dim_{\mathbb{C}} \Delta_\pm = 2^{n-1}$. Clifford multiplication determines an equivariant linear mapping

$$m_+ \colon \mathbb{C}^{2n} \otimes \Delta_+ \to \Delta_-,$$

and an element $\xi \in \Delta_+$ is a *pure spinor* if its annihilator

$$T^{1,0} = \{v \in \mathbb{C}^{2n} : m_+(v \otimes \xi) = 0\}$$

is a *maximal* isotropic subspace, so that $T^{1,0} \oplus \overline{T^{1,0}} = \mathbb{C}^{2n}$. This construction identifies Z_n with the variety of such pure spinor classes $[v]$ inside the complex projective space $\mathbb{P}(\Delta_+)$.

An alternative characterization of a pure spinor can be given in terms of the

$SO(2n)$-equivariant isomorphism

$$\Delta_+ \otimes \Delta_+ \cong \begin{cases} \Lambda^n_+ \oplus \Lambda^{n-2} \oplus \cdots \oplus \Lambda^0, & n \text{ even} \\ \Lambda^n_+ \oplus \Lambda^{n-2} \oplus \cdots \oplus \Lambda^1, & n \text{ odd.} \end{cases}$$

A spinor $\xi \in \Delta_+$ is pure if and only if it has the property that $\xi \otimes \xi$ belongs to the highest weight summand Λ^n_+ [15]. Analogous conditions arise in the characterization of nilpotent elements of a Lie algebra.

A third characterization arises from the choice of a reduction of $SO(2n)$ to $U(n)$ (i.e. an "origin" in Z_n). Relative to this choice,

$$\Delta_+ \cong \lambda^0 \oplus \lambda^2 \oplus \lambda^4 \oplus \cdots \oplus \lambda^{2[n/2]}, \tag{14}$$

where the "small" exterior powers are $\lambda^k = \bigwedge^k \mathbb{C}^n$. If ξ has non-zero component in λ^0, then it must be proportional to an element of the form

$$e^\omega = 1 + \omega + \tfrac{1}{2}\omega \wedge \omega + \cdots \tag{15}$$

for some $\omega \in \lambda^2$. This fact is well known in (and explained by) the theory of generalized complex structures [23].

The next result shows how to construct the manifolds (12) by induction.

PROPOSITION. – *Z_{n+1} is the total space of a fibre bundle over the sphere S^{2n} as base, with fibre Z_n.*

PROOF. – Regard S^{2n} as the symmetric space $SO(2n+1)/SO(2n)$, and consider the homogeneous (non-symmetric) space

$$Z'_{n+1} = \frac{SO(2n+1)}{U(n)}. \tag{16}$$

An element of the latter is a coset $gU(n)$ with $g \in SO(2n+1)$, and maps in a consistent way to $gSO(2n) \in S^{2n}$. The fibre of this projection (formally the quotient of two fractions!) is $SO(2n)/U(n) = Z_n$. The point now is that Z'_{n+1} can be identified with the set of maximal isotropic subspaces (of dimension n) in $\mathbb{C}^{2n+1}$. Any such subspace extends uniquely to an oriented isotropic subspace of dimension $n+1$ in $\mathbb{C}^{2n+2}$, and we obtain an isomorphism $Z'_{n+1} \cong Z_{n+1}$. Indeed, $SO(2n+1)$ acts transitively on Z_{n+1} with stabilizer $U(n)$ and there is a well-defined map sending $gU(n) \in Z'_{n+1}$ to $gU(n+1) \in Z_{n+1}$. □

A celebrated example of this construction occurs when $n = 2$. One can identify S^4 with the quaternionic projective line $\mathbb{HP}^1$, so as to obtain the fibration

$$\pi\colon \mathbb{CP}^3 = \mathbb{P}_{\mathbb{C}}(\mathbb{H}^2) \longrightarrow \mathbb{P}_{\mathbb{H}}(\mathbb{H}^2) = \mathbb{HP}^1 = S^4 \tag{17}$$

that merely maps a complex line to the quaternionic one it spans. A study of the relevant group actions reveals that this coincides with the map of cosets we have already considered. A similar interpretation using Cayley numbers can be used to construct the fibration $Z_4 \to S^6$.

The proposition is generalized to define the twistor space $\mathscr{Z}$ of an arbitrary oriented even-dimensional Riemannian manifold $M = M^{2n}$. Let P denote the principal $SO(2n)$-bundle of oriented orthonormal frames at points of M.

DEFINITION. – *The* twistor space *of* M^{2n} *is the total space* $\mathscr{Z}$ *of the associated bundle* $P \times_{SO(2n)} Z_n = P/U(n)$.

Here again, $U(n)$ is the stabilizer of the standard complex structure J_0 on $\mathbb{R}^{2n}$.

By construction, the twistor space $\mathscr{Z}$ of S^{2n} is none other than Z_{n+1}. Its complex structure can be constructed in a more abstract fashion that extends (at least in spirit) to the general case. We shall explain this at the start of the next section.

3. – Questions of integrability.

An almost-complex structure J satisfying (10) on an open set Ω of M determines in tautological fashion a section $\tilde{J}: \Omega \to \mathscr{Z}$. We refer to this section as the *graph* of J, and its definition and properties underlie all our subsequent results.

It turns out that, if we restrict to orthogonal *complex* structures, then the endomorphisms J induced on each tangent space $\tilde{J}_*(T_m\Omega)$ are the restrictions of a unique almost-complex structure $\mathbb{J}$ on $\mathscr{Z}$. Moreover, the fibres are complex submanifolds of $(\mathscr{Z}, \mathbb{J})$, and acquire their standard complex structure in this way. It follows that, for a general smooth section $\tilde{J}: \Omega \to \mathscr{Z}$,

$$(18) \qquad J \text{ satisfies (11) on } \Omega \quad \Leftrightarrow \quad \tilde{J} \text{ is a holomorphic map},$$

where "holomorphic" is taken in the pseudo sense that $\tilde{J}_* \circ J = \mathbb{J} \circ \tilde{J}_*$.

It is a fact that the above construction, when applied to $M = S^{2n}$, yields the standard complex structure of Z_{n+1}, so the latter coincides with $\mathbb{J}$. Over a general base, the integrability of $\mathbb{J}$ is a function of the Weyl tensor W of M that (if defined carefully) depends only on the conformal class $[g]$. When $n = 2$, there is an additional splitting $W = W_+ + W_-$ where $W_\pm$ is a self-adjoint endomorphism of $\Lambda^2_\pm$. It is W_+ that is more relevant here, since (with our choice of orientation) $\mathscr{Z}$ can be identified with the 2-sphere bundle of elements of any fixed norm in Λ^2_+.

THEOREM [7, 33, 4]. – *If* $W \equiv 0$ *then* $(M, \mathbb{J})$ *is a complex manifold. The converse is true if* $n \geq 3$*, though if* $n = 2$ *a sufficient condition is* $W_+ \equiv 0$.

We omit any mention of the proofs, except to remark that the statements are closely related to the discussion of the Weyl tensor below.

EXAMPLE. – If we take M^4 to be the complex projective plane $\mathbb{CP}^2$, then we can identify $\mathscr{Z}$ with the projectivization

$$\mathbb{P}(1 \oplus \mathcal{K}) = \mathbb{P}(\mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}),$$

where 1 denotes a trivial line bundle and $\mathcal{K} = \Lambda^{2,0}\mathbb{CP}^2$ the canonical bundle. This bundle has global sections corresponding to $\pm J$ where J is the standard complex structure on $\mathbb{CP}^2$, but the fact that $\mathscr{Z} \to M$ is a holomorphic bundle is not relevant here, since $(\mathscr{Z}, \mathbb{J})$ is not integrable. With its standard orientation, $\mathbb{CP}^2$ has $W_+ \neq 0$ but $W_- = 0$. To get something integrable, we need to take the twistor space of $\overline{\mathbb{CP}}^2$ obtained by reversing orientation on the base. This time, $\mathscr{Z}$ is diffeomorphic to the flag manifold $SU(3)/T^2$, and $(\mathscr{Z}, \mathbb{J})$ is biholomorphic to the projective tangent bundle $\mathbb{P}(T^{1,0}\mathbb{CP}^2)$ relative to a different projection to $\mathbb{CP}^2$.

Much of the remainder of this section is lifted almost verbatim from [37]. This is justified on the basis that many of the problems in dimensions $2n \geq 6$ remain open. However, during the past ten years, considerable progress *has* been made in 4 dimensions with the theory of bihermitian metrics, initiated in [27] and advanced in [1, 3, 25].

Given the fixed Riemannian metric g, we have its associated Levi Civita connection ∇. Let $\mathfrak{X}^{1,0}$ denote the space of $(1,0)$-vector fields on M. Consider the condition

$$X, Y \in \mathfrak{X}^{1,0} \;\Rightarrow\; \nabla_X Y \in \mathfrak{X}^{1,0}. \tag{19}$$

Since ∇ is torsion-free, we have

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

so (19) certainly implies that

$$X, Y \in \mathfrak{X}^{1,0} \;\Rightarrow\; [X, Y] \in \mathfrak{X}^{1,0}, \tag{20}$$

which is directly equivalent to (11). However, it is known that (19) and (20) (and so (11)) are in fact equivalent. It is then condition (19) that underlies the existence of the almost-complex structure $\mathbb{J}$ on $\mathscr{Z}$.

These observations have an important consequence for the Riemann curvature tensor R, computed as

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Indeed, (19) and (20) yield immediately

$$X, Y, Z \in \mathfrak{X}^{1,0} \;\Rightarrow\; R_{XY}Z \in \mathfrak{X}^{1,0},$$

or equivalently

$$X, Y, Z, W \in \mathfrak{X}^{1,0} \;\Rightarrow\; g(R_{XY}Z,\, W) = 0.$$

This condition is equivalent to the vanishing of the component W_J of W lying in a certain real subspace $\mathcal{W}_J$ of the space $\mathcal{W}$ of Weyl tensors, identified by Tricerri-Vanhecke [41]. The tensor W_J is determined by the restriction of R to the complex subspace $\Lambda^{2,0} \otimes \Lambda^{2,0}$, and evidently has no Ricci contraction.

LEMMA [41]. – *If J is an OCS on an even-dimensional Riemannian manifold then $W_J \equiv 0$.*

The full space of curvature tensors on a Riemannian manifold of dimension n (half what we have) equals

$$r_n = \tfrac{1}{12} n^2(n^2 - 1).$$

The space $\mathcal{W}$ has dimension

$$w_n = r_n - \tfrac{1}{2} n(n+1) = \tfrac{1}{12} n(n^3 - 7n - 6).$$

The real subspace $\mathcal{W}_J$ underlies the kernel of the mapping $S^2\Lambda^{2,0} \to \Lambda^{4,0}$ arising from the first Bianchi identity (just like R itself), and so its real dimension equals $2r_n$. Hence,

$$8\,\frac{\dim \mathcal{W}_J}{\dim \mathcal{W}} = 8\,\frac{2r_2}{w_{2n}} = 1 + \varepsilon_n,$$

where $\varepsilon_n > 0$ for all n and $\varepsilon_n \to 0$. This striking "8-fold" asymptotic behaviour is illustrated in the following table.

$2n$	$2r_n$	w_{2n}	ε
4	2	10	0.6
6	12	84	0.142
8	40	300	0.067
10	100	770	0.039
12	210	1638	0.026
14	392	3080	0.018
16	672	5304	0.014
18	1080	8550	0.011
20	1650	13090	0.008

Given two OCSes J, J' on an even-dimensional Riemannian manifold M, we shall say that J, J' are *curvature-independent* if $\mathcal{W}_J \cap \mathcal{W}_{J'} = \{0\}$. This definition is tailor made so as to record the

COROLLARY. – *If M admits eight curvature-independent OCSes then it is conformally flat.*

The theory of generalized twistor bundles provides many instances in which a non-conformally flat manifold admits (at least locally) infinite families on OCSes. Such situations typically arise by investigating the zero set of the Nijenhuis tensor of $\mathbb{J}$. They have been completely classified when M is an inner symmetric space [13], but we can only conclude that the associated subspaces $\mathcal{W}_J$ are heavily constrained.

PROBLEM. – Find and classify Riemannian metrics which are not conformally flat, but which admit two or more curvature-independent OCSes.

Some examples along these lines can be found in [2].

One might imagine that knowledge of J is necessary to identify the subspace $\mathcal{W}_J$. This is not in fact the case, at least in real dimension $2n=4$. In this situation, each of the two $SO(4)$ (indeed, $SO(3)$) components W_+, W_- can be viewed as a quartic polynomial in a spinor variable parametrizing orthogonal almost-complex structures. A generic tensor W_+ will have four "roots" J, $-J$, J', $-J'$ at each point with J, J' curvature-independent (and therefore exhausting 4 of the 5 dimensions of $\mathcal{W}_+$). The point is that any positively-oriented OCS *must* be one of these four [38].

4. – Liouville theorems.

We now turn to the parametrization of complex structures on $\mathbb{R}^4$, basing our approach on the concept of deformation.

The standard complex structure J_0 on $\mathbb{C}^2$ can be characterized by means of its space of $(1,0)$-forms:

$$\Lambda^{1,0}(J_0) = \langle dz_1,\, dz_2\rangle. \tag{21}$$

Modify this structure by setting $\Lambda^{1,0}(J) = \langle \Xi_1,\, \Xi_2\rangle$, where

$$\begin{cases} \Xi_1 = dz_1 + \xi_{11}d\bar{z}_1 + \xi_{12}d\bar{z}_2, \\ \Xi_2 = dz_2 + \xi_{21}d\bar{z}_1 + \xi_{22}d\bar{z}_2. \end{cases} \tag{22}$$

It helps to think of the functions $\xi_{ij} = \xi_{ij}(z_1, z_2)$ as "small", but (22) is in fact a valid description of a generic almost-complex structure J on $\mathbb{R}^4$, subject only to the proviso that

$$\Lambda^{1,0}(J) \cap \Lambda^{0,1}(J_0) = \{0\}. \tag{23}$$

There are no other dz_k terms needed in (22) because we are effectively representing J by the echelon form

$$\left(\begin{array}{cc|cc} 1 & 0 & \xi_{11} & \xi_{12} \\ 0 & 1 & \xi_{21} & \xi_{22} \end{array}\right)$$

that defines a mapping from (21) to its conjugate space at each point. In more sophisticated notation,

$$(\xi_{ij}) \in \Gamma(\mathbb{C}^2, \Theta \otimes \Lambda^{0,1}),$$

where $\Theta = T^{1,0}(J_0)$ denotes the holomorphic tangent space relative to J_0.

In order that J be orthogonal, we need to take $\xi_{11} = \xi_{22} = 0$ and $\xi_{21} = -\xi_{12}$. This skew-symmetry reflects the fact that the cotangent space to Z_2 is isomorphic to the 1-dimensional representation $\Lambda^{2,0}$ (on which $A \in U(2)$ acts as $(\det A)^{-1}$). The remaining function is effectively a projective parameter and defines a mapping

$$\xi = \xi_{21} : \mathbb{C}^2 \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = Z_2.$$

We can easily identify J explicitly as an element in $SO(4) \cap \mathfrak{so}(4)$; it works out as the skew-symmetric matrix

$$J = \frac{-1}{1+|\xi|^2}\begin{pmatrix} 0 & 1-|\xi|^2 & 2b & 2a \\ * & 0 & 2a & -2b \\ * & * & 0 & 1-|\xi|^2 \\ * & * & * & 0 \end{pmatrix},$$

where $\xi = a + ib$ and an asterisk stands for minus the corresponding entry above the diagonal. Whilst $\xi = 0$ gives J_0, in the limit $\xi \to \infty$ the matrix tends to $-J_0$ (in the compact group $SO(4)$).

Next set

$$\begin{cases} W_1 = z_1 - \xi\overline{z}_2, \\ W_2 = z_2 + \xi\overline{z}_1, \end{cases} \tag{24}$$

and observe that

$$\Lambda^{1,0}(J) = \langle dW_1 + \overline{z}_2 d\xi,\ dW_2 - \overline{z}_1 d\xi \rangle. \tag{25}$$

The integrability condition (11) amounts to asserting that this a differential ideal, which will be the case if and only if $d\xi$ itself belongs to $\Lambda^{1,0}(J)$, so that ξ is "self-holomorphic". More precisely,

$$\begin{cases} \dfrac{\partial\xi}{\partial\overline{z}_1} - \xi\dfrac{\partial\xi}{\partial z_2} = 0, \\[2ex] \dfrac{\partial\xi}{\partial\overline{z}_2} + \xi\dfrac{\partial\xi}{\partial z_1} = 0, \end{cases}$$

but (from (25)) these equations will be automatically satisfied if we succeed in solving (24) so as to express $\xi = \xi(W_1, W_2)$ as a *holomorphic* function of W_1, W_2. Here are some examples:

0) $\xi = 0$, giving J_0 defined on $\mathbb{R}^4 = \mathbb{C}^2$. Taking ξ to be a non-zero constant would merely result in $P^{-1}J_0P$ for some constant orthogonal matrix P.

1) $\xi = W_1$, so that $\xi = z_1/(1+\overline{z}_2)$. This yields an OCS J_1 for which

$$\Lambda^{1,0}(J_1) = \langle (1+\overline{z}_2)dz_1 - z_1 d\overline{z}_2,\ (1+\overline{z}_2)dz_2 + z_1 d\overline{z}_1 \rangle.$$

In this way, we see that J_1 is defined on $\mathbb{C}^2 \setminus \{(0,-1)\}$.

2) $\xi = W_2/W_1$, which yields the quadratic equation

$$\xi^2 \overline{z}_2 + \xi(\overline{z}_1 - z_1) + z_2 = 0,$$

and two distinct roots for ξ unless $z_1 = \overline{z}_1$ and $z_2 = 0$. We can consistently choose one root so as to obtain an OCS J_2, this time defined on $\mathbb{C}^2 \setminus L$, where $L = \{(z_1, z_2) : \operatorname{Re} z_1 = 0,\ z_2 = 0\}$ is a straight line.

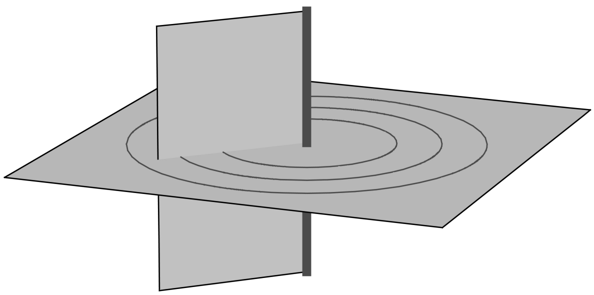

Fig. 2. The domain $\mathbb{R}^4 \setminus L \cong S^2 \times H^2$.

Both the complex structures J_0 and J_1 are defined on all of S^4 minus a single point. They are in fact conformally equivalent; this is best shown using the quaternionic formalism introduced in the next section. On the other hand, J_2 is defined on the complement

$$\Omega = \mathbb{R}^4 \setminus L \cong (\mathbb{R}^3)^* \times \mathbb{R} \cong S^2 \times H^2, \tag{26}$$

where we have identified $\mathbb{R}^* \times \mathbb{R}$ with the upper half-plane H^2. The S^2's lie in planes in $\mathbb{R}^3$ orthogonal to L, and are shown as circles in Figure 2. Just as $\mathbb{R}^3$ minus a line admits a natural vector field (representing the magnetic field of a current running along the line), so (26) admits a natural almost-complex structure preserving the transversal 2-dimensional leaves on which it induces the standard complex structures. The resulting OCS is precisely J_2, and is Hermitian relative to the product of the metrics with constant but opposite

Gaussian curvatures on S^2 and H^2 (see (9)). This makes (Ω, J_2) a complete Kähler (but conformally flat) manifold.

These simple examples led us to the

PROBLEM. – On what other domains Ω of $\mathbb{R}^4$ can we find OCSes that do not extend to $\mathbb{R}^4 \setminus \Omega$?

To tackle this problem, we consider the graph of an orthogonal complex structure J. Given our choice of inhomogeneous coordinates on $\mathbb{CP}^3$, this graph is defined explicitly by the mapping

$$\tilde{J} : \mathbb{R}^4 \to \mathbb{CP}^3, \quad (z_1, z_2) \mapsto [1,\ \xi,\ W_1,\ W_2],$$

in which ξ (and so W_1, W_2) are functions of z_1, z_2. From (18), we obtain

PROPOSITION. – *Given an OCS on an open subset Ω of S^4, its image $\tilde{J}(\Omega)$ is a complex surface in $\mathbb{CP}^3$. Conversely, every complex section in $\mathbb{CP}^3$ (lying over an open set $\Omega \subset \mathbb{R}^4$) has the form $\tilde{J}(\Omega)$ where J is an OCS on Ω.*

No graph can be defined over the whole of S^4 because the latter does not admit a global almost-complex structure for topological reasons (compare the analogous situation in Figure 1!). Other than S^2, the only even-dimensional sphere to admit an almost-complex structure is S^6.

EXAMPLES. – We are now in a position to interpret J_0, J_1, J_2 twistorially.

0) Any projective plane $\mathbb{CP}^2$ in $\mathbb{CP}^3$ contains exactly one fibre $\pi^{-1}(p)$, and so determines a unique point $p \in S^4$. (This gives a dual projection $(\mathbb{CP}^3)^* \to S^4$.) For example,

$$\tilde{J}_0(z_1, z_2) = [1, 0, z_1 - \xi\bar{z}_2, z_2 + \xi\bar{z}_1],$$

and the image of $\tilde{J}(\mathbb{R}^4)$ equals $\mathbb{CP}^2 \setminus \mathbb{CP}^1$ where

$$\mathbb{CP}^1 = \{[0, 0, Z_1, Z_2] : Z_i \in \mathbb{C}\} = \pi^{-1}(\infty)$$

is the fibre over the point "at infinity".

1) J_1 corresponds to the section

$$\tilde{J}(z_1, z_2) = [1, W_1, W_1, W_2] = [1, z_1 - \xi\bar{z}_2, z_1, -\xi\bar{z}_2, z_2 + \xi\bar{z}_1]$$

defined over $S^4 \setminus \{(0, -1)\}$.

2) J_2 corresponds to the quadric

$$\{[\xi_0, \xi, W_1, W_2] : \xi W_1 = \xi_0 W_2\}, \tag{27}$$

which contains $\pi^{-1}(p)$ for p in the circle $\overline{L} \subset S^4$. In this case, we remove the circle and choose one branch of the quadric to define J_2.

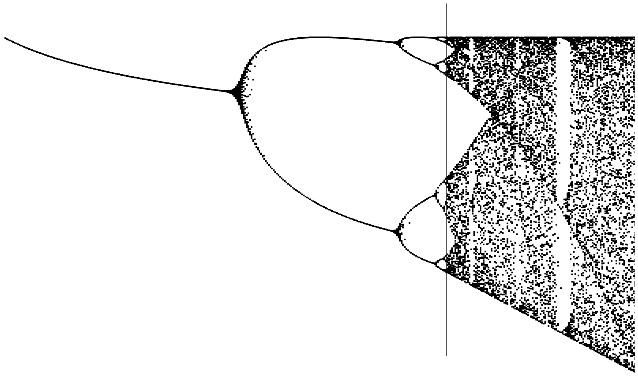

Fig. 3. – The limit of period-doubling.

It is possible to characterize these simple solutions. It was proved in [42] that if J is an OCS defined on all of $\mathbb{R}^4$ then J is necessarily constant (and similar methods were used in [28]). We improve on this result with

THEOREM [36]. – *Let J be an OCS on an open set $S^4 \setminus \Lambda$. If $\mathcal{H}^1(\Lambda) = 0$ then J is conformally constant.*

Here, $\mathcal{H}^d$ denotes d-dimensional Hausdorff measure (see [19] for its definition and properties). Hypothetically, Λ could be a fractal. Although the theorem dispenses us from the study of such sets, we digress briefly to illustrate a non-trivial example.

EXAMPLE. – Figure 3 that depicts iterates of the quadratic map $f(x) = 1 - cx^2$ for $1 < c < 2$. The vertical line represents the exact limit $c = 1.4011\ldots$ of period doubling at the onset of chaos, and contains the so-called *Feigenbaum attractor* A. The latter is a Cantor-type set defined as the closure of the orbit $\{f^{(2^k)}(0) : k \geq 1\}$ and is contained in a union $A_1 \sqcup A_2$ of two disjoint intervals of length ℓ_i, each of which is subdivided ad infinitum in approximately the same way [21, 30]. By a standard argument, the resulting Hausdorff dimension $D = D_0$ is given by the formula $(\ell_1)^D + (\ell_2)^D = 1$, in which ℓ_i is computed using the solution of the universal equation

$$a\, g\!\left(g\!\left(\frac{x}{a}\right)\right) = g(x), \qquad a = -2.5029\ldots$$

to estimate the behaviour of the functions $f^{(2^k)}$, which are not exactly self-similar. The set A is a "multifractal" in the sense that other measures D_q of its dimension do not agree [24].

The result is $D = 0.538\dots$ so, in terms of measure, we can be certain that

$$\mathcal{H}^d(A) = \begin{cases} \infty & \text{if } d < 0.53 \\ 0 & \text{if } d > 0.54. \end{cases}$$

Of course, the theorem tells us that any OCS J defined on the complement in $\mathbb{R}^4$ of a set like A (with Hausdorff dimension less than 1) extends to $\mathbb{R}^4 \setminus \{\mathrm{pt}\}$ or $\mathbb{R}^4$.

SKETCH PROOF OF THE THEOREM. – We shall only outline an argument by means of which potential “singularities” (in the compactification) can be eliminated.

The graph $\tilde{J}(\Omega)$ is a complex analytic set in $\mathbb{CP}^3 \setminus \pi^{-1}(\Lambda)$. A theorem of Shiffman [39] then implies that the closure $\overline{A}$ is analytic in $\mathbb{CP}^3$. Shiffman's theorem is based on Bishop's theorem, itself a generalization of that of Remmert-Stein [9]. Chow's theorem now imply that $\overline{A}$ is algebraic, and it has to be a surface of degree 1. This last step is well explained in the book by Mumford [31]. □

In order to interpret J_2, we first need to discuss the appropriate *real structure* on $\mathbb{CP}^3$. Bearing (17) in mind, multiplication by the unit quaternion j in $\mathbb{H}^2$ induces an antilinear involution $\sigma\colon \mathbb{CP}^3 \to \mathbb{CP}^3$. On each fibre $\mathbb{CP}^1 \cong S^2$, the involution σ acts as the antipodal map sending J (pointwise) to $-J$.

THEOREM [36]. – *Suppose that J is an OCS defined on S^4 minus a round circle S^1 (equivalently, $\mathbb{R}^4$ minus a straight line), and that J does not extend to $S^4 \setminus \{\mathrm{pt}\}$. Then the graph of J in $\mathbb{CP}^3$ lies in a quadric $\mathcal{Q}$ with $\sigma(\mathcal{Q}) = \mathcal{Q}$.*

This is proved directly from Bishop's Theorem [9] by comparing the graph $\tilde{J}$ with a quadric $\mathcal{Q}$ carefully chosen so that it ramifies over the given S^1. The end result is that J is conformally equivalent to $\pm J_2$.

5. – Classification of quadrics.

A complex symmetric 4×4 matrix Q determines a quadric surface

$$\mathcal{Q} = \{[v] \in \mathbb{CP}^3 : v^\top Q\, v = 0\}. \tag{28}$$

Two matrices Q, Q' determine the same set of points $\mathcal{Q}$ if and only if $Q' = cQ$ for some non-zero complex number c. Although this action by $\mathbb{C}^*$ (and, in particular the circle group $U(1)$) seems obvious, it becomes encoded in a non-trivial way in certain canonical forms.

In practice, we may choose an identification

$$\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{29}$$

to reflect a well-known isomorphism of Lie groups

$$SO(4,\mathbb{C}) \cong SL(2,\mathbb{C}) \times_{\mathbb{Z}_2} SL(2,\mathbb{C}). \tag{30}$$

The factors on the right preserve respective skew-symmetric 2-forms ω_1, ω_2 on $\mathbb{C}^2$, and the product $\omega_1 \otimes \omega_2$ is the *symmetric* bilinear form that provides (29) with its complex orthogonal structure. The resulting quadric $\mathcal{Q}_0$ is formed from elements of *rank one* in the tensor product, which explains why it is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Of course, any non-degenerate quadric in $\mathbb{CP}^3$ arises in this way, and is associated (with a suitable choice of basis) to the identity matrix $Q = I$.

The fibration (17) is preserved by the conformal group

$$SO_\circ(5,1) \cong SL(2,\mathbb{H})/\mathbb{Z}_2. \tag{31}$$

This acts on the set of quadratic forms and provides a decomposition

$$S^2(\mathbb{C}^4) \cong \Sigma \oplus i\Sigma, \tag{32}$$

where Σ is a real 10-dimensional vector space consisting of those complex symmetric 4×4 matrices of the form

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}. \tag{33}$$

Such matrices belong to the Lie algebra $\mathfrak{gl}(2,\mathbb{H})$, but being symmetric also satisfy $A^\top = A$ and $\overline{B}^\top = -B$ or $B \in \mathfrak{u}(2)$. If $\mathfrak{m}$ denotes the real 6-dimensional subspace generated by the matrices with $B = 0$, then

$$\Sigma = \mathfrak{u}(2) \oplus \mathfrak{m} \tag{34}$$

can be identified with the Lie algebra $\mathfrak{sp}(2)$, associated to the symmetric space

$$\frac{Sp(2)}{U(2)} \cong \frac{SO(5)}{SO(2) \times SO(3)}, \tag{35}$$

which is both the complex quadric $\mathcal{Q}^3$ and the real Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{R}^5)$.

The matrix (33) is consistent with ordering the inhomogeneous coordinates that we previously adopted for $\mathbb{CP}^3$ as $(1, W_1, \xi, W_2)$, and then identifying this vector with $(1 + j\xi,\ W_1 + jW_2) \in \mathbb{H}^2$. Then (24) boils down to the quaternionic product

$$W_1 + jW_2 = (z_1 + jz_2)(1 + j\xi)$$

that itself determines the fibration (17). With this notation, the mapping discussed at the end of the last section is

$$\sigma : (1, W_1, \xi, W_2) \mapsto (\,-\,\overline{\xi}, -\overline{W}_2, 1, \overline{W}_1).$$

Note that the quadric (27) defining J_2 is indeed σ-invariant.

An alternative way to carry out the identification of $\mathbb{C}^4$ with $\mathbb{H}^2$ is to endow the first factor on the right of (29) with a real structure and the second with a quaternionic structure. Choose an antilinear map $\sigma_1\colon \mathbb{C}^2 \to \mathbb{C}^2$ with $(\sigma_1)^2 = 1$, so that the first $\mathbb{C}^2$ is identified with the complexification of the fixed set $\mathbb{R}^2$ of σ_1. We similarly pick an antilinear map $\sigma_2\colon \mathbb{C}^2 \to \mathbb{C}^2$ with $(\sigma_2)^2 = -1$, so that the second $\mathbb{C}^2$ becomes $\mathbb{H}$. The product $\sigma = \sigma_1 \otimes \sigma_2$ then gives the tensor product (29) a quaternionic structure, and (with some abuse of notation) it becomes $\mathbb{H}^2 = \mathbb{R}^2 \otimes \mathbb{H}$. Moreover, (30) reduces to

$$SO(2,\mathbb{H}) \cong SL(2,\mathbb{R}) \times_{\mathbb{Z}_2} SU(2), \tag{36}$$

and in these terms

$$\{\mathbb{R}\mathbf{p} \otimes \mathbb{H} \,:\, \mathbf{p} \in \mathbb{R}^2,\, \mathbf{p} \neq 0\}/\mathbb{C}^* = \mathbb{RP}^1 \times \mathbb{CP}^1 \tag{37}$$

is the subset of $\mathcal{Q}_0$ consisting of entire fibres of π.

The description (34) can be used to show that any two non-degenerate σ-invariant quadrics (ones arising from matrices in Σ) are equivalent under $GL(2,\mathbb{H})$. The stabilizer at $Q = I$ decomposes Σ into the direct sum of $\mathbb{R}Q$ and the tensor product

$$S^2\mathbb{R}^2 \otimes S^2\mathbb{H} \cong \mathfrak{sl}(2,\mathbb{R}) \otimes \mathfrak{su}(2) \tag{38}$$

of Lie algebras.

PROBLEM. – Describe the orbits of the conformal group on the quadrics (28), or equivalently the orbits of $\mathbb{C}^* \times SL(2,\mathbb{H}) = U(1) \times GL(2,\mathbb{H})$ on the vector space (32).

The generic stabilizer is zero-dimensional, so the orbits will be described by $20 - 15 - 2 = 3$ free parameters. To solve the problem, it is sufficient to study the action of (36) on (38). Passing to a $\mathbb{Z}_2$ quotient, this is equivalent to the action of

$$SO(2,\mathbb{H})/\mathbb{Z}_2 \cong SO_\circ(2,1) \times SO(3) \tag{39}$$

on the set of real 3×3 matrices X. If X has rank 3 then a suitable version of the singular value decomposition (SVD) implies that X can be diagonalized by (39). This is the main step in the proof of the

THEOREM. – *Under the action of the conformal group on $\mathbb{CP}^3$, a generic quadric is equivalent to the one defined by a matrix*

$$Q_{\lambda,\mu,\nu} = \begin{pmatrix} e^{\lambda+i\nu} & 0 & 0 & 0 \\ 0 & e^{\mu-i\nu} & 0 & 0 \\ 0 & 0 & e^{-\lambda+i\nu} & 0 \\ 0 & 0 & 0 & e^{-\mu-i\nu} \end{pmatrix} \tag{40}$$

with $\lambda, \mu, \nu \in \mathbb{R}$.

Note that (40) has unit determinant, so this fixes the matrix in its $\mathbb{C}^*$ orbit up to multiplication by powers of i. In this way, we have "recovered" the $\mathbb{C}^*$ action. Replacing μ by $\mu+\pi$ has the effect of multiplying $Q_{\lambda,\mu,\nu}$ by -1, so it is obvious that we may restrict μ to lie in the interval $[0,\pi)$. But more is true: multiplication by i results in an equivalence $Q_{\lambda,\mu,\nu}\sim Q_{\mu,\lambda,\nu+\frac{\pi}{2}}$. Similar equivalences allow us to choose a unique matrix (40) with $0\le\lambda\le\mu$ and $0\le\nu<\pi/2$, except that $Q_{\lambda,\lambda,\nu}\sim Q_{\lambda,\lambda,\frac{\pi}{2}-\nu}$.

Not all non-degenerate quadrics belong to the 3-parameter family (40). There is a 1-parameter family of non-diagonalizable quadrics that arise from 3×3 matrices X of lower rank.

Qiven a quadric $\mathcal{Q}$ in $\mathbb{CP}^3$, a generic fibre $\pi^{-1}(m)\cong\mathbb{CP}^1$ will intersect $\mathcal{Q}$ in two distinct points. For special points $p\in S^4$, there are however two other possible scenarios: the fibre may lie in $\mathcal{Q}$ or be tangent to it. This leads us to define

$$D_0=\{p\in S^4:\pi^{-1}(p)\subset\mathcal{Q}\},$$
$$D_1=\{p\in S^4:\#(\pi^{-1}(p)\cap\mathcal{Q})=1\},$$

and state the

DEFINITION. – *The discriminant locus of $\mathcal{Q}$ is the disjoint union $D=D_0\sqcup D_1$.*

If both D_0, D_1 are non-empty, one expects D_0 to be the singular locus of D.

Combining the previous theorem with some special cases yields

THEOREM. – *Let $\mathcal{Q}$ be a non-degenerate quadric in $\mathbb{CP}^3$. There are four possibilities under the action of the conformal group on $\mathbb{CP}^3$:*
0) *$D_0=\emptyset$ and $D=D_1$ is a smooth unknotted torus in S^4.*
1) *$D_0=\{p\}$ is one point, and D is a torus in S^4 pinched at p.*
2) *$D_0=\{p,q\}$ consists of two points, and D is a torus in S^4 pinched at p,q.*
∞) *D_0 is a round circle S^1 in S^4 and $D_1=\emptyset$.*

In the last case, the S^1 coincides with the $\mathbb{RP}^1$ in (37), and we are effectively deaing with J_2. On the other hand, Case (1) arises when $\mathcal{Q}$ is not diagonalizable.

In $\mathbb{R}^4$, it makes no sense to talk about the "inside" of a torus. Indeed, in cases (1) and (2) we may view the discriminant locus D as a *spindle torus* and *horn torus* obtained by rotating a circle about an axis that either touches or intersects it. The truncated views shown in Figure 4 were obtained from [22].

In the generic case, the smooth torus D will not disconnect $\mathcal{Q}$ into two leaves, but there is a canonical way of extending D to a solid torus $S^1\times\overline{U}$ (where $\overline{U}$ denotes the closed disc in $\mathbb{R}^2$), and we can then define a single-valued OCS on its complement in S^4. The inverse image $\pi^{-1}\overline{U}$ consists of two disjoint disks glued

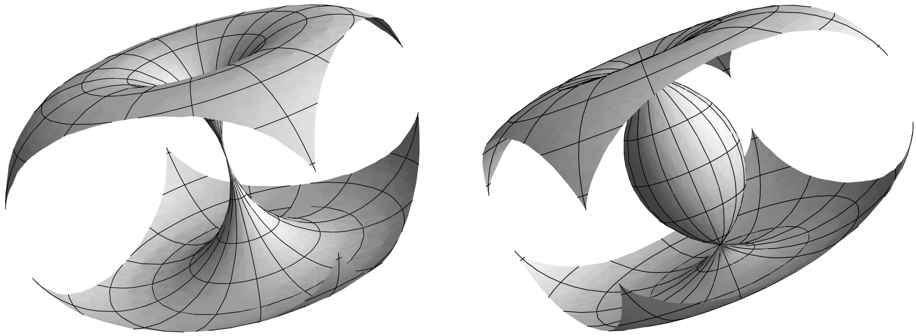

Fig. 4. – Singular tori of revolution.

along their common boundary to form a 2-sphere. Compared to the case of J_2, we have therefore again cut out an $S^1 \times S^2$ from $\mathcal{Q}$, but it is configured differently relative to the Penrose fibration (17).

6. – Higher dimensions.

We include this final section, in part as an introduction to the use of coordinates on twistor space in higher dimensions.

We can easily generalize the approach of Section 4. Consider the almost-complex structure J on $\mathbb{R}^{2n}$ whose space $\Lambda^{1,0}(J)$ is generated by

$$\Xi_i = \xi_0 dz_i + \sum_j \xi_{ij} d\overline{z}_j, \qquad i = 1, \ldots, n. \tag{41}$$

Then J is orthogonal if and only if

$$\xi_{ij} = -\xi_{ji}, \qquad \forall i,j, \tag{42}$$

so (ξ_{ij}) is a skew-symmetric matrix, and $\omega = \xi_{ij} e^i \wedge e^j$ a complex 2-form. The annihilator of (41) is the space of $(0,1)$ vectors

$$\overline{D}_i = \xi_0 \frac{\partial}{\partial \overline{z}_i} + \sum_j \xi_{ij} \frac{\partial}{\partial z_j}, \qquad i = 1, \ldots, n, \tag{43}$$

best thought of as differential operators.

If we set $\xi_0 = 1$, then the ξ_{ij} determine local coordinates on the subset of Z_n consisting of structures that satisfy (23). These correspond precisely to those pure spinors that can be written in the form (15) with $\omega = (\xi_{ij})$. However, to cover Z_n with charts more effectively, one needs to introduce quantities $\xi_{ijkl}, \xi_{ijklmn}, \cdots$ to parametrize the susbpaces λ^{2k} of (14) with $k > 1$ (this is explained in [10]).

The integrability condition (11) is

$$\overline{D}_i\xi_{jk} - \overline{D}_j\xi_{ik} = 0, \qquad \forall i,j,k. \tag{44}$$

Given the skew-symmetry (42), this equation asserts that $\overline{D}_i\xi_{jk}$ belongs to the kernel of the linear mapping

$$\Lambda^2 \otimes \Lambda^1 \subset \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^2,$$

well known to be an isomorphism. Hence, (44) is equivalent to

$$\overline{D}_i\xi_{jk} = 0, \qquad \forall i,j,k. \tag{45}$$

The operators (43) can be regarded as covariant derivatives, and the same argument underlies the uniqueness of the Levi Civita connection, and the equivalence of (19) and (20).

In view of (45), it makes sense to define

$$W_i = \xi_0 z_i - \sum_j \xi_{ij}\,\overline{z}_j, \qquad i = 1,\dots,n. \tag{46}$$

If $\xi_0 = 1$, the functions W_i and ξ_{ij} determine local holomorphic coordinates on a dense open set of the twistor space $\mathscr{Z} = Z_{n+1}$ of S^{2n}. Note however that the fibre $\pi^{-1}(\infty)$ (for which the z_i are not finite) is completely excluded.

Let us illustrate the set-up in six dimensions, so that $\mathscr{Z}$ is a subset of

$$\mathbb{P}(\Delta_+) = \mathbb{P}(\lambda^0 \oplus \lambda^2 \oplus \lambda^4).$$

Pretend that the symbols $\xi_1, \xi_2, \xi_3, \xi_4$ (even though we have not defined them) form a basis of $\lambda^1 = \mathbb{C}^4$. Then we identify ξ_{i4} with W_i and decree

$$W_1, \quad W_2, \quad W_3, \quad \xi_{23}, \quad \xi_{31}, \quad \xi_{12}$$

to be a basis of λ^2. Relative to the Klein quadric in $\mathbb{P}(\lambda^2)$ generated by simple 2-forms, the first three elements span an "α plane", and the last three a "β plane". Moreover, we take ξ_0 to span λ^0, and

$$W_0 = -(z_1\xi_{23} + z_2\xi_{31} + z_3\xi_{12}) \tag{47}$$

to span λ^4. It follows that

$$W_0\xi_0 + W_1\xi_{23} + W_2\xi_{31} + W_3\xi_{12} = 0, \tag{48}$$

and we take this to be the defining relation for the quadric $\mathscr{Z}$ in $\mathbb{P}(\Delta_+)$. This is consistent with the exponential description (15) in which we see one quadratic relation between the components.

We have seen that an orthogonal complex structure J on an open set of $\mathbb{R}^6$ is defined, via (41), by functions ξ_{ij} satisying (42). Referring back to (46) and (47), we can now identity the graph of J:

LEMMA. – $\tilde{J}(z_1, z_2, z_3) = [\xi_0, \xi_{23}, \xi_{31}, \xi_{12}, W_0, W_1, W_2, W_3]$.

Standard projective geometry tells us that $\mathscr{Z}$ incorporates two families of linear subspaces $\mathbb{CP}^3$ also parametrized by the quadric Z_4 (this is a special feature of the triality phenomenon associated to $\mathfrak{so}(8)$). In our twistorial context, these were first described by Slupinski [40]. They are:

a) the subspaces $\mathbb{CP}^3$ consisting of fibres of π, as well as those that are twistor spaces of S^4's conformally embedded in S^6 (we call these *reduced twistor spaces*).

β) subspaces $\mathbb{CP}^3$ that surject to S^6 and are ramified only over one point $p \in S^6$, which is "blown up" into a $\mathbb{CP}^2 \subset \pi^{-1}(p)$.

Two spaces in the same family are either disjoint (such as distinct fibres) or intersect in a $\mathbb{CP}^1$ (such as fibre and reduced twistor space). Spaces in different families intersect in a single point or a $\mathbb{CP}^2$ (necessarily the latter if one contestant is a reduced twistor space).

Given an OCS J on $\mathbb{R}^6$, let $\Gamma = \tilde{J}(\mathbb{R}^6)$ denote (the image of) its graph in $\mathscr{Z} = Z_4$. We consider the case in which the closure $\overline{\Gamma}$ is an analytic subset of $\mathbb{CP}^7$, and so algebraic [31]. By Bishop's Theorem [9], this will be true if the L^6 norm of ∇J is finite, provided the norm and covariant derivative are taken relative to S^6. With this assumption, we may define the *degree* $d(J)$ by taking the cup product between the homology classes defined by $\overline{\Gamma}$ and any $\mathbb{CP}^3$ of the (β) family. The first step in classifying such an OCS J is to show that Γ has the same behaviour as such a "horizontal" $\mathbb{CP}^3$.

In joint work with L. Borisov and J. Viaclovsky, it is shown that if $\overline{\Gamma}$ is analytic then $\overline{\Gamma} \cap \pi^{-1}(\infty)$ is a projective plane. It does not however follow that $\overline{\Gamma}$ is itself a projective subspace $\mathbb{CP}^3$ or (equivalently) that J is conformally constant. It turns out that this is only true if $d(J) = 1$. Orthogonal complex structures on $\mathbb{R}^6$ for which $\overline{\Gamma}$ is analytic are classified in [10], and we refer the reader to this paper (completed after the author's lectures). If $d(J) > 1$ then $\overline{\Gamma} \cap \pi^{-1}(\infty)$ necessarily contains singular points, and all the examples arise in some sense from the following construction.

EXAMPLE. – Set $\xi_{13} = \xi_{23} = 0$ and (for convenience) $\xi_0 = 1$. Suppose further that $\xi_{12} = \xi$ is a polynomial in z_3. Then

$$W_0 = -\xi z_3, \quad W_1 = z_1 - \xi\overline{z}_2, \quad W_2 = z_2 + \xi\overline{z}_1, \quad W_3 = z_3,$$

the condition (44) is satisfied, and we have

$$\tilde{J}(z_1, z_2, z_3) = [1,\ 0,\ 0,\ \xi,\ -\xi z_3,\ z_1 - \xi\overline{z}_2,\ z_2 + \xi\overline{z}_1,\ z_3]. \tag{49}$$

Hence Γ belongs to a complex 4-dimensional quadric $\mathcal{Q}^4$, and

$$\overline{\Gamma} \cap \pi^{-1}(\infty) = \{[0,0,0,0,W_0,W_1,W_2,0] : W_i \in \mathbb{C}\} \cong \mathbb{CP}^2,$$

as predicted. Observe also that $J = J(z_3) \oplus J_0$, where $J(z_3)$ is a constant OCS on $\mathbb{R}^4 = \{z_3 = 0\}$ and J_0 is the fixed OCS on $\mathbb{R}^2$ given by z_3. An example of this type was discovered by Wood in the context of harmonic morphisms [42, 6].

To conclude, we return to a construction in Section 1, by reducing $SO(7)$ to the exceptional Lie group G_2. This enables us to identify $\mathbb{R}^7$ with the space Im $\mathbb{O}$ of imaginary octonians and adopt the cross product defined by Cayley multiplication. The latter induces, exactly as in (3), an almost-complex structure J on $S^6 \subset \mathbb{R}^7$ and a corresponding section $\tilde{J}$ of the twistor space $\mathscr{Z} = Z_4$. There is a complementary subbundle $\tilde{\mathscr{Z}}$ of $\mathscr{Z}$ whose fibre $\mathbb{CP}^2$ consists of almost-complex structures K for which $J + K$ has rank 2. Although J is not integrable (the resulting structure is *strictly nearly-Kähler*), $\tilde{\mathscr{Z}}$ *is* a complex submanifold of $\mathscr{Z}$, a fact implicit in [33, 12]. Indeed, $\tilde{\mathscr{Z}}$ can be identified with the complex 5-dimensional quadric $\mathcal{Q}^5$.

The example (49) involved the hypersurface of a 4-quadric. An even more obvious choice is a complex 3-quadric $\mathcal{Q}^3$, which can be identified with the real Grassmannian (35) relative to some fixed splitting $\mathbb{R}^7 = \mathbb{R}^2 \oplus \mathbb{R}^5$. Retaining the G_2 structure, we map a 2-subspace $\langle u, v\rangle$ (with u, v an orthonormal pair in $\mathbb{R}^5$) to $u \times v \in S^6$. It is easy to see that this restricts to a diffeomorphism

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^5) \setminus \mathbb{G}\mathrm{r}_2(\mathbb{R}^4) \longrightarrow S^6 \setminus S^2.$$

The resulting OCS J_3 has discriminant locus S^2 (covered by $\mathbb{G}\mathrm{r}_2(\mathbb{R}^4) \cong S^2 \times S^2$ in $\mathcal{Q}^3$), and is an analogue of the OCS J_2 characterized at the end of Section 4. Whilst $\mathbb{CP}^3$ is in some non-standard sense the blow-up of S^6 at a point (relative to a constant OCS on $\mathbb{R}^6$ undefined at that point), so $\mathcal{Q}^3$ is the blow-up of S^6 along a $\mathbb{CP}^1$ (relative to J_3).

REFERENCES

[1] V. Apostolov - P. Gauduchon - G. Grantcharov, *Bi-Hermitian structures on complex surfaces*, Proc. London Math. Soc. (3), **79**, 2 (1999), 414-428.

[2] V. Apostolov - G. Grantcharov - S. Ivanov, *Orthogonal complex structures on certain Riemannian 6-manifolds*, Differ. Geom. Appl., **11** (1999), 279-296.

[3] V. Apostolov - M. Gualtieri, *Generalized Kähler manifolds, commuting complex structures, and split tangent bundles*, Comm. Math. Phys., **271** (2007), 561-575.

[4] M. F. Atiyah - N. J. Hitchin - I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, **362**, 1711 (1978), 425-461.

[5] M. F. Atiyah - R. S. Ward, *Instantons and algebraic geometry*, Comm. Math. Phys., **55** (1977), 117-124.

[6] P. Baird - J. C. Wood, *Harmonic Morphisms between Riemannian Manifolds*, volume 29 of London Mathematical Society Monographs, Oxford University Press (Oxford, 2003).

[7] L. Bérard Bergery - T. Ochiai, *On some generalizations of the construction of twistor spaces*. In Global Riemannian geometry (Durham, 1983) (Ellis Horwood, Chichester, 1984), 52-59.

[8] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.

[9] E. BISHOP, *Conditions for the analyticity of certain sets*, Michigan Math. J., **11** (1964), 289-304.
[10] L. BORISOV - S. SALAMON - J. VIACLOVSKY, *Orthogonal complex structures in Euclidean spaces*, In preparation.
[11] R. L. BRYANT, *Submanifolds and special structures on the octonians*, J. Differ. Geom., **17** (1982), 185-232.
[12] R. L. BRYANT, *Lie groups and twistor spaces*, Duke Math. J., **52** (1985), 223-261.
[13] F. E. BURSTALL - J. H. RAWNSLEY, *Twistor Theory for Riemannian Symmetric Spaces*, Lecture Notes Math. 1424 (Springer-Verlag Berlin, 1990).
[14] E. CALABI, *Construction and properties of some 6-dimensional almost complex manifolds*, Trans. Amer. Math. Soc., **87** (1958), 407-438.
[15] É. CARTAN, *The Theory of Spinors*, Dover Publications Inc. (New York, 1981). With a foreword by Raymond Streater, A reprint of the 1966 English translation, Dover Books on Advanced Mathematics.
[16] S. S. CHERN, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc., **6** (1955), 771-782.
[17] S. K. DONALDSON - J. FINE, *Toric anti-self-dual 4-manifolds via complex geometry*, Math. Ann., **336** (2006), 281-309.
[18] J. EELLS - S. SALAMON, *Twistorial constructions of harmonic maps of surfaces into four-manifolds*, Ann. Sc. Norm. Sup. Pisa, **12** (1985), 589-640.
[19] L. C. EVANS - R. F. GARIEPY, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1992).
[20] A. FUJIKI - M. PONTECORVO, *On Hermitian geometry of complex surfaces*. In Complex, contact and symmetric manifolds, Progr. Math. 234 (Birkhäuser, Boston, 2005), 153-163.
[21] P. GRASSBERGER, *On the Hausdorff Dimension of Fractal Attractors*, J. Stat. Phys, **26** (1981), 173-179.
[22] A. GRAY - E. ABBENA - S. SALAMON, *Modern Differential Geometry of Curves and Surfaces, with Mathematica*, CRC Press, Taylor and Francis (2006).
[23] M. GUALTIERI, Generalized complex geometry. arXiv:math/0703298.
[24] R. HILBORN, *Chaos and Nonlinear Dynamics. An Introduction for Scientists and Engineers*, Oxford University Press (New York, 1994).
[25] N. HITCHIN, *Bihermitian metrics on Del Pezzo surfaces*, J. Symplectic Geom., **5**, 1 (2007), 1-8.
[26] D. JOYCE, *The hypercomplex quotient and the quaternionic quotient*, Math. Ann., **290** (1991), 323-340.
[27] P. KOBAK, *Explicit doubly-Hermitian metrics*, Differential Geom. Appl., **10** (1999), 179-185.
[28] C. LEBRUN - Y. S. POON, *Self-dual manifolds with symmetry*. In Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. 54, Amer. Math. Soc. (Providence, RI, 1993), 365-377.
[29] C. R. LEBRUN, *Explicit self-dual metrics on* $\mathbb{CP}^2\#\cdots\#\mathbb{CP}^2$, J. Differ. Geom., **34** (1991), 223-253.
[30] J. MILNOR, *On the concept of attractor*, Commun. Math. Phys., **99** (1985), 177-195.
[31] D. MUMFORD, *Algebraic Geometry. I*, Classics in Mathematics (Springer-Verlag, Berlin, 1995). Complex projective varieties.
[32] A. NEWLANDER - L. NIRENBERG, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (2) (1957), 391-404.
[33] N. R. O'BRIAN - J. R. RAWNSLEY, *Twistor spaces*, Ann. Global Anal. Geom., **3** (1985), 29-58.

[34] M. PONTECORVO, *Uniformization of conformally flat Hermitian surfaces*, Differential Geom. Appl., **2**, 3 (1992), 295-305.
[35] Y. S. POON, *Compact self-dual manifolds with positive scalar curvature*, J. Differ. Geom., **24** (1986), 97-132.
[36] S. SALAMON - J. VIACLOVSKY, *Orthogonal complex structures on domains of* $\mathbb{R}^4$, to appear in Math. Ann.
[37] S. M. SALAMON, *Orthogonal complex structures*. In Differential geometry and applications (Brno, 1995) (Masaryk Univ., Brno, 1996), 103-117.
[38] S. M. SALAMON, *Hermitian geometry*. In Invitations to Geometry and Topology, Oxf. Grad. Texts Math. 7, (Oxford University Press, 2002), 233-291.
[39] B. SHIFFMAN, *On the removal of singularities of analytic sets*, Michigan Math. J., **15** (1968), 111-120.
[40] M. J. SLUPINSKI, *The twistor space of the conformal six sphere and vector bundles on quadrics*, J. Geom. Phys., **19** (1996), 246-266.
[41] F. TRICERRI - L. VANHECKE, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc., **267** (1981), 365-397.
[42] J. C. WOOD, *Harmonic morphisms and Hermitian structures on Einstein 4-manifolds*, Internat. J. Math., **3**, 3 (1992), 415-439.

Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italia
e-mail: simon.salamon@polito.it

Received May 9, 2008 and in revised form July 1, 2008