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Asymptotic Formulae for Bernstein-Schnabl Operators and Smoothness

FRANCESCO ALTOMARE (*)

dedicated to Professor Paul L. Butzer
on the occasion of his 80th birthday

Abstract. – *Of concern are Bernstein-Schnabl operators associated with a continuous selection of Borel measures on the unit interval. With respect to these sequences of positive linear operators we determine the classes of all continuous functions verifying a pointwise asymptotic formula or a uniform one. Our methods are essentially based on a general characterization of the domains of Feller semigroups in terms of asymptotic formulae and on the determination of both the saturation class of Bernstein-Schnabl operators and the Favard class of the relevant Feller semigroup.*

1. – Introduction.

Consider the sequence $(B_n)_{n\geq 1}$ of Bernstein operators on $C([0,1])$. It is well-known that, if a function $u \in C([0,1])$ is differentiable in a neighborhood of a point $x \in [0,1]$ and if, in addition, it is two time differentiable at x, then

$$\lim_{n\to\infty} n(B_n(u)(x) - u(x)) = \frac{x(1-x)}{2}u''(x);$$

moreover, if $u \in C^{(2)}([0,1])$, then the limit above is uniform with respect to $x \in [0,1]$.

The main aim of this paper is to determine both the class $\mathcal{S}([0,1])$ of those functions $u \in C([0,1])$ for which there exists $v \in C([0,1])$ such that $\lim\limits_{n\to\infty} n(B_n(u) - u) = v$ pointwise on $[0,1]$ and the class $\mathcal{U}([0,1])$ of those functions $u \in C([0,1])$ for which there exists $\lim\limits_{n\to\infty} n(B_n(u) - u)$ uniformly on $[0,1]$.

In fact we deal with the above mentioned problems for the more general sequences of Bernstein-Schnabl operators which are an interesting generalization of Bernstein operators and which furnish, other than new general ap-

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proximation processes for continuous functions, also some useful tools to approximate the solutions of the initial boundary value problems associated with a class of degenerate diffusion equations on $[0,1]$ ([1], [2], [3], [12], [17]).

Our main results (see Section 5) state that $\mathcal{S}([0,1])$ coincides with the linear subspace of all functions $u \in C([0,1]) \cap C^2(]0,1[)$ such that $\lim\limits_{x\to 0^+} a(x)u''(x) = \lim\limits_{x\to 1^-} a(x)u''(x) = 0$, a being a suitable continuous function on $[0,1]$, vanishing at 0 and 1 ($a(x) = x(1-x)/2$ in the case of Bernstein operators) while $\mathcal{U}([0,1])$ coincides with the linear subspace of all functions $u \in C([0,1]) \cap C^2(]0,1[)$ whose second derivative is bounded on $]0,1[$.

The proofs are essentially based on a general characterization of the domain of Feller semigroups in terms of asymptotic formulae and on the determination of both the saturation class of Bernstein-Schnabl operators and the Favard class of the Feller semigroup generated by the differential operator associated with them which is of the form $Au = au''$ coupled with Ventcel's boundary conditions.

As we explained before, our results apply also for Bernstein operators showing some new properties of these operators together with some hold ones.

2. – Asymptotic formulae for Bernstein-Schnabl operators.

Throughout the paper we shall denote by $\mathcal{C}([0,1])$ the Banach lattice of all real valued continuous functions on the interval $[0,1]$ endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering. For every $x \in [0,1]$ we shall denote by ε_x the point-mass measure concentrated at x, i.e.,

$$\varepsilon_x(B) := \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \notin B, \end{cases} \quad \text{for every Borel subset } B \text{ of } [0,1].$$

The symbol $\mathbf{1}$ stands for the constant function 1 and for every $n \geq 1$, $e_n \in \mathcal{C}([0,1])$ denotes the function $e_n(t) := t^n$ $(0 \leq t \leq 1)$.

A continuous selection of probability Borel measures on $[0,1]$ is a family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ such that for every $f \in C([0,1])$ the function

$$x \longmapsto \int_0^1 f d\mu_x \tag{2.1}$$

is continuous on $[0,1]$. Such a function will be denoted by $T(f)$, i.e.,

$$T(f)(x) := \int_0^1 f d\mu_x, \quad (0 \leq x \leq 1). \tag{2.2}$$

The operator $T : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ is positive (hence continuous) and $T(\mathbf{1}) = \mathbf{1}$ (and hence $\|T\| = 1$).

From now on we shall fix a continuous selection $(\mu_x)_{0\le x\le 1}$ of probability Borel measures on $[0,1]$ satisfying the following additional assumption:

$$\int_0^1 e_1 d\mu_x = x, \quad (0 \le x \le 1) \tag{2.3}$$

(i.e., $T(e_1) = e_1$) and

$$\mu_x \neq \varepsilon_x \quad \text{for every } 0<x<1 \tag{2.4}$$

(i.e. $x^2 < T(e_2)(x)$ for every $0<x<1$).

Setting

$$a(x) := \frac{1}{2}\left(T(e_2)(x) - x^2\right) = \frac{1}{2}\left(\int_0^1 e_2 d\mu_x - x^2\right) \quad (0 \le x \le 1), \tag{2.5}$$

we then have that $a \in \mathcal{C}([0,1])$, $a(0) = a(1) = 0$ and

$$0<a(x) \le \frac{x(1-x)}{2} \quad \text{for every } 0<x<1. \tag{2.6}$$

For every $n \ge 1$, consider the positive linear operator $B_n : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ defined for every $f \in \mathcal{C}([0,1])$ and $x \in [0,1]$ by

$$\begin{aligned} B_n(f)(x) :&= \int_{[0,1]^n} f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) d\mu_x^n(x_1, \ldots, x_n) \\ &= \int_0^1 \cdots \int_0^1 f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) d\mu_x(x_1) \cdots d\mu_x(x_n), \end{aligned} \tag{2.7}$$

where μ_x^n denotes the tensor product of μ_x with itself n-times.

B_n is called the n-th Bernstein-Schnabl operator associated with the selection $(\mu_x)_{0\le x\le 1}$.

This sequence of operators was first introduced by Schnabl ([17]) in the context of sets of probability Radon measures on compact Hausdorff spaces and, subsequently, it was investigated by Grossman ([12]), the author ([1]) and many others in the setting of convex compact subsets (see [2, Chapter 6 and the relevant Notes and References]). More recently a rather complete analysis on the operators (2.7) has been carried out in the paper [3] to which we refer in the sequel without no further mention.

If we consider the selection of measures $\mu_x := x\varepsilon_1 + (1-x)\varepsilon_0$ $(0 \le x \le 1)$, then the operators B_n turn into the classical Bernstein operators given by

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \tag{2.8}$$

(see, e.g., [2], [10], [14]) for some relevant results on these important and interesting operators).

Moreover in this case

$$a(x) = \frac{x(1-x)}{2} \qquad (0 \le x \le 1) \tag{2.9}$$

For other examples see [3] and [2].

Given $x \in [0,1]$, consider the auxiliary function

$$\psi_x(t) := t - x \qquad (0 \le t \le 1). \tag{2.10}$$

Then for every $n \ge 1$

$$B_n(\mathbf{1}) = \mathbf{1},\ B_n(e_1) = e_1, \qquad B_n(\psi_x) = \psi_x \tag{2.11}$$

and

$$B_n(\psi_x^2)(x) = \frac{2a(x)}{n}, \qquad B_n(\psi_x^4)(x) \le \frac{C}{n^2}, \tag{2.12}$$

where C is a constant independent of $n \ge 1$.

Moreover, for every $f \in C([0,1])$

$$B_n(f)(0) = f(0) \quad \text{and} \quad B_n(f)(1) = f(1). \tag{2.13}$$

From [2, Theorem 5.1.2 and the subsequent Remark] it also follows that, if $f \in C^1([0,1])$, then for every $x \in [0,1]$

$$|B_n(f)(x) - f(x)| \le 2\sqrt{\frac{2a(x)}{n}}\,\omega\left(f', \sqrt{\frac{2a(x)}{n}}\right), \tag{2.14}$$

where ω denotes the usual modulus of continuity.

In particular, if $f' \in Lip(M,1)$, i.e., f' is Lipschitz continuous and $|f'(x) - f'(y)| \le M\,|x-y|$ for every $x, y \in [0,1]$, then

$$|B_n(f)(x) - f(x)| \le \frac{4M}{n} a(x). \tag{2.15}$$

In [3, Theorem 3.1] it was shown that, if $u \in C^2([0,1])$, then

$$\lim_{n\to\infty} n(B_n(u) - u) = au'' \text{ uniformly on } [0,1]. \tag{2.16}$$

Below we state some further asymptotic formulae. We shall denote by $C_b^2(]0,1[)$ the linear subspace of all continuous functions on $[0,1]$ which possess a bounded continuous second derivative on $]0,1[$.

PROPOSITION 1. – *Let* $u \in C([0,1])$. *Then*

(*i*) *if* u *is differentiable in a neighborhood of a point* x_0 *of*]0,1[*and if, in* *addition, it is two times differentiable at it, then*

$$\lim_{n\to\infty} n(B_n(u)(x_0) - u(x_0)) = a(x_0)u''(x_0). \tag{2.17}$$

(*ii*) *if* $u \in C_b^2(]0,1[)$, *then*

$$\lim_{n\to\infty} n(B_n(u) - u) = au''$$

uniformly on each compact subinterval of]0,1[.

PROOF. – (*i*). By the Peano form of Taylor's formula we may consider a function $\omega : [0,1] \to \mathbb{R}$ such that $\lim\limits_{x\to x_0} \omega(x) = 0$ and

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{u''(x_0)}{2}(x - x_0)^2 + \omega(x)(x - x_0)^2 \tag{1}$$

for every $x \in [0,1$, i.e.,

$$u = u(x_0) + u'(x_0)\psi_{x_0} + \frac{u''(x_0)}{2}\psi_{x_0}^2 + \omega\psi_{x_0}^2.$$

Therefore for any $n \geq 1$, on account of (2.11) and (2.12), it follows that

$$n(B_n(u)(x_0) - u(x_0)) = a(x_0)u''(x_0) + nB_n(\omega\psi_{x_0}^2)(x_0).$$

Thus (*i*) will be proved if we show that $\lim\limits_{n\to\infty} nB_n(\omega\psi_{x_0}^2)(x_0) = 0$. To this end fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|\omega(x)| \leq \varepsilon$ for any $x \in [0,1], |x - x_0| \leq \delta$. Note also that, if $x \in [0,1], |x - x_0| \geq \delta$, then (1) shows that

$$\begin{aligned} |\omega(x)| &\leq \frac{2\|u\|_\infty}{|x - x_0|^2} + \frac{u'(x_0)}{|x - x_0|} + \frac{u''(x_0)}{2} \\ &\leq \frac{2\|u\|_\infty}{\delta^2} + \frac{u'(x_0)}{\delta} + \frac{u''(x_0)}{2} =: M_\delta. \end{aligned}$$

Therefore

$$\omega\psi_{x_0}^2 \leq \varepsilon\psi_{x_0}^2 + \frac{M_\delta}{\delta^4}\psi_{x_0}^4$$

and, for $n \geq 1$, by (2.12)

$$nB_n(\omega\psi_{x_0}^2) \leq 2\varepsilon a(x_0) + \frac{M_\delta C}{\delta^4 n^2}$$

which gives the desired result.

(*ii*) Let [a,b] a compact subinterval of]0,1[. If $x \in [a,b]$ and $y \in [0,1]$, by Taylor's formula there exists ξ in the interval $I(x,y)$ having x and y as end points such that

$$u(y) = u(x) + u'(x)(y - x) + \frac{u''(x)}{2}(y - x)^2 + \frac{u''(\xi) - u''(x)}{2}(y - x)^2. \tag{2}$$

Setting $\omega_u(x,y) := \dfrac{u''(\xi) - u''(x)}{2}$, we then have

$$u(y) = u(x) + u'(x)(y-x) + \frac{u''(x)}{2}(y-x)^2 + \omega_u(x,y)(y-x)^2 \tag{3}$$

and

$$| \omega_u(x,y) | \leq M := \sup_{0<t<1} | u''(t) | \, . \tag{4}$$

Since u'' is uniformly continuous on each compact subinterval of $]0,1[$, we also have that

$$\lim_{y \to x} \omega_u(x,y) = 0 \text{ uniformly with respect to } x \in [a,b]. \tag{5}$$

Consider, indeed, $\delta_1 > 0$ such that $[a-\delta_1, b+\delta_1] \subset]0,1[$. Since u'' is uniformly continuous on $[a-\delta_1, b+\delta_1]$, for $\varepsilon > 0$ there exists $0<\delta<\delta_1$ such that $| u''(s) - u''(t) | \leq 2\varepsilon$ for every $s,t \in [a-\delta_1, b+\delta_1], | s-t | \leq \delta$. Therefore, if $x \in [a,b]$ and $y \in [0,1], | x-y | \leq \delta$, then $I(x,y) \subset [a-\delta_1, b+\delta_1]$ and hence $| \omega_u(x,y) | \leq \varepsilon$.

Given now $x \in [a,b]$, from (3) we infer that

$$u = u(x)\mathbf{1} + u'(x)\psi_x + \frac{u''(x)}{2}\psi_x^2 + \omega_u(x,\cdot)\psi_x^2$$

so that for any $n \geq 1$

$$B_n(u)(x) = u(x) + \frac{a(x)u''(x)}{n} + B_n(\omega_u(x,\cdot)\psi_x^2)(x).$$

Accordingly, to get the result it is sufficient to show that

$$\lim_{n \to \infty} nB_n(\omega_u(x,\cdot)\psi_x^2)(x) = 0$$

uniformly with respect to $x \in [a,b]$.

Consider therefore $\varepsilon > 0$; then by (5) there exists $\delta > 0$ such that $| \omega_u(x,y) | \leq \varepsilon$ for every $x \in [a,b]$ and $y \in [0,1], | x-y | \leq \delta$. By (2.12) we may choose $\nu \geq 1$ such that for $n \geq \nu$ and $x \in [0,1]$

$$nB_n(\psi_x^4)(x) \leq \frac{\varepsilon\delta^2}{2(1+M)},$$

where M is defined by (4). Note also that, if $x \in [a,b]$ and $y \in [0,1]$, then

$$| \omega_u(x,y)\psi_x^2(y) | \leq \begin{cases} 2\varepsilon\psi_x^2(y) & \textit{if} \quad | x-y | \leq \delta, \\ M\psi_x^2(y) \leq \dfrac{M}{\delta^2}\psi_x^4(y) & \textit{if} \quad | x-y | \geq \delta, \end{cases}$$

so that, in any case,

$$| \omega_u(x,\cdot)\psi_x^2 | \leq 2\varepsilon\psi_x^2 + \frac{M}{\delta^2}\psi_x^4$$

and hence for $n \geq \nu$ and $x \in [a, b]$

$$n \mid B_n(\omega_u(x, \cdot)\psi_x^2)(x) \mid \leq nB_n(\mid \omega_u(x, \cdot)\psi_x^2 \mid)(x) \leq 2\varepsilon nB_n(\psi_x^2)(x) + \frac{M}{\delta^2} nB_n(\psi_x^4)(x) \leq \varepsilon.$$

□

The previous result can be further refined.

If $u \in C_b^2(]0,1[)$ we denote by $A(u)$ the function on $[0,1]$ defined by

$$A(u)(x) := \begin{cases} a(x)u''(x) & \text{if } 0<x<1, \\ 0 & \text{if } x = 0, 1. \end{cases}$$

Clearly $A(u) \in C([0,1])$.

THEOREM 2. – *If $u \in C_b^2(]0,1[)$, then*

$$\lim_{n \to \infty} n(B_n(u) - u) = A(u) \quad \textit{uniformly on } [0,1]. \tag{2.18}$$

PROOF. – On account of (2.13) and Proposition 1, clearly $\lim_{n\to\infty} n(B_n(u) - u) = A(u)$ pointwise on [0,1]. Therefore, in order to get the desired result, it is sufficient to show that the sequence $(n(B_n(u) - u))_{n\geq 1}$ is equicontinuous on [0,1]. This is certainly true on $]0,1[$ by virtue of Proposition 1, part (ii). As concerns the endpoints 0 and 1, setting $M := \sup_{0<x<1} \mid u''(x) \mid$, clearly $u' \in Lip(M,1)$. Therefore by (2.15)

$$n|B_n(u)(x) - u(x)| \leq 4Ma(x) \qquad (0 \leq x \leq 1).$$

So, given $\varepsilon > 0$, choose $0<\delta<1$ such that $a(x) \leq \varepsilon/4M$ for every $x \in [0, \delta] \cup [1-\delta, 1]$ and hence

$$n|B_n(u)(x) - u(x)| - n|B_n(u)(0) - u(0)| = n|B_n(u)(x) - u(x)| \leq \varepsilon$$

and

$$n|B_n(u)(x) - u(x)| - n|B_n(u)(1) - u(1)| = n|B_n(u)(x) - u(x)| \leq \varepsilon.$$

□

In the last section of the paper we shall show a converse of Theorem 2.

REMARK 3. – When the operators B_n, $n \geq 1$, are the classical Bernstein operators, then part (i) of Proposition 1 gives the well-known Voronovskaja formula ([19]). However, also in this particular case, part (ii) of Proposition 1 and Theorem 2 seem to be new.

3. – Generators of Feller semigroups and asymptotic formulae.

In this section we shall present a characterization of the domains of Feller semigroups in terms of pointwise asymptotic formulae.

Consider a locally compact Hausdorff space X which also is countably at infinity. As usual we shall denote by $C_0(X)$ the Banach lattice of all real-valued continuous functions on X vanishing at infinity, endowed with the sup-norm and the natural pointwise ordering.

A Feller semigroup on $C_0(X)$ is a strongly continuous semigroup $(T(t))_{t\geq 0}$ of positive linear operators on $C_0(X)$ which also are contractive, i.e., $\| T(t) \| \leq 1$ for every $t \geq 0$.

For more details on Feller semigroups we refer, e.g., to [11], [5], [18].

We finally recall that a linear operator $B : D(B) \to C_0(X)$ defined on a linear subspace $D(B)$ of $C_0(X)$, is said to verify the *positive maximum principle* if $B(u)(x_0) \leq 0$ for every $u \in D(B)$ and $x_0 \in X$ satisfying $\sup\limits_{x\in X} u(x) = u(x_0) > 0$ ([5], [18]).

If this is the case, then necessarily $(B, D(B))$ is dissipative, i.e.,

$$\| \lambda u - Bu \| \geq \lambda \| u \| \qquad \text{for all } u \in D(B) \text{ and } \lambda > 0.$$

The next result is well-known but we present the (short) proof for the reader convenience.

LEMMA 4. – *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach space E. Then $(A, D(A))$ does not admit any (non-trivial) dissipative extension.*

PROOF. – Let $(B, D(B))$ be a dissipative extension of $(A, D(A))$, i.e., $D(A) \subset D(B)$ and $B = A$ on $D(A)$. Given a sufficiently large $\lambda > 0$ and $u \in D(B)$, since $\lambda I - A$ is bijective (here the symbol I stands for the identity operator), there exists $v \in D(A) \subset D(B)$, such that $\lambda v - Av = \lambda u - Bu$.

Since $\lambda I - B$ is injective on $D(B)$, it follows that $u = v \in D(A)$. Therefore $D(B) = D(A)$. □

THEOREM 5. – *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on $C_0(X)$ and consider a net $(L_i)^{\leq}_{i\in I}$ of positive linear contractions on $C_0(X)$ and a net $(\varphi(i))^{\leq}_{i\in I}$ of positive real numbers such that $\lim\limits_{i\in I} \varphi(i) = +\infty$ and $\lim\limits_{i\in I} \varphi(i)(L_i(u) - u) = Au$ pointwise on X for every $u \in D(A)$.*

Then $D(A)$ coincides with the subspace of all functions $u \in C_0(X)$ for which there exists $v \in C_0(X)$ such that $\lim\limits_{i\in I} \varphi(i)(L_i(u) - u) = v$ pointwise on X.

In particular, if $u \in C_0(X)$ and $\lim\limits_{i\in I} \varphi(i)(L_i(u) - u) = 0$ pointwise on X, then $u \in D(A)$ and $Au = 0$.

PROOF. – Denote by $D(B)$ the subspace of all functions $u \in C_0(X)$ for which there exists $v \in C_0(X)$ such that $\lim\limits_{i\in I} \varphi(i)(L_i(u) - u) = v$ pointwise on X and consider the linear operator

$$B(u) := \lim_{i\in I} \varphi(i)(L_i(u) - u) \qquad (u \in D(B)).$$

By assumption $D(A) \subset D(B)$ and $B = A$ on $D(A)$. On account of Lemma 4 to get the result it is sufficient to show that $(B, D(B))$ is dissipative.

In fact we shall show that $(B, D(B))$ verifies the positive maximum principle and, to this end, fix $u \in D(B)$ and $x_0 \in X$ satisfying $\sup_{x\in X} u(x) = u(x_0) > 0$

Since X is countable at infinity, we may choose an increasing sequence $(\varphi_n)_{n\geq 1}$ of positive continuous functions on X having compact support such that $\sup_{n\geq 1} \varphi_n = 1$ on X.

Given $i \in I$ and $x \in X$, an application of the Riesz representation theorem to the positive linear functional $f \longmapsto L_i(f)(x)$ on $C_0(X)$, shows that there exists a bounded Borel measure $\mu_{x,i}$ on X such that

$$L_i(f)(x) = \int_X f d\mu_{x,i} \qquad (f \in C_0(X)).$$

Therefore, by also appealing to the Beppo Levi theorem, we get

$$\begin{aligned} L_i(u)(x) &= \int_X u d\mu_{x,i} \leq u(x_0) \int_X \sup_{n\geq 1} \varphi_n d\mu_{x,i} \\ &= u(x_0) \sup_{n\geq 1} \int_X \varphi_n \, d\mu_{x,i} \leq u(x_0). \end{aligned}$$

In particular, $B(u)(x_0) = \lim_{i\in I} \varphi(i)(L_i(u)(x_0) - u(x_0)) \leq 0$ and this completes the proof. □

COROLLARY 6. – *Let $(A, D(A))$ be the generator of a Feller semigroup $(T(t))_{t\geq 0}$ on $C_0(X)$ and fix $u \in C_0(X)$. Assume that*

(i) there exists a net $(t(i))_{i\in I}^{\leq}$ in $]0, +\infty[$ converging to 0 such that the limit $\lim_{i\in I} \dfrac{T(t(i))u - u}{t(i)} = v \in C_0(X)$ exists pointwise on X.

Then $u \in D(A)$ (and $Au = v$).

In particular, if $\lim_{i\in I} \dfrac{T(t(i))u - u}{t(i)} = 0$ pointwise on X, then $u \in D(A)$ and $Au = 0$, i.e., $T(t)u = u$ for every $t \geq 0$.

PROOF. – It is sufficient to apply Theorem 5 to the nets $L_i := T(t(i))$ and $\varphi(i) := 1/t(i)$ $(i \in I)$. □

We finally point out that the previous assumption (i) is satisfied, for instance, if there exists the limit $\lim_{t\to 0^+} \frac{T(t)u-u}{t} = v \in C_0(X)$ pointwise on X or, under the additional hypothesis that X has a countable base, if the family $\left(\frac{T(t)u-u}{t}\right)_{t\geq 0}$ is equicontinuous and pointwise bounded on X and, provided X is non compact, if $\lim_{x\to\infty} \frac{T(t)u(x)-u(x)}{t} = 0$ uniformly with respect to $t \in]0,1]$ (apply a version of the Ascoli-Arzelà theorem for the space $C_0(X)$).

4. – The saturation class of Bernstein-Schnabl operators and the Favard class of their limit semigroups.

Consider the differential operator $A : D_V(A) \to C([0,1])$ defined by

$$A(u)(x) := \begin{cases} a(x)u''(x) & \text{if } 0<x<1, \\ 0 & \text{if } x = 0, 1. \end{cases} \tag{4.1}$$

on the linear subspace $D_V(A)$ of all functions $u \in C([0,1] \cap C^2(]0,1[)$ such that

$$\lim_{x\to 0^+} a(x)u''(x) = \lim_{x\to 1^-} a(x)u''(x) = 0. \tag{4.2}$$

The boundary conditions (4.2) which we have incorporated in the domain $D_V(A)$ are often called Ventcel's boundary conditions.

The operator $(A, D_V(A))$ is the generator of a Feller semigroup $(T(t))_{t\geq 0}$ on $C([0,1])$ ([8, Theorem 2]). Moreover, for every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$

$$T(t)f = \lim_{n\to\infty} B_n^{k(n)} f \qquad \text{uniformly on } [0,1] \tag{4.3}$$

$(f \in C([0,1]))$([3, Theorem 3.5 and final Note]).

By the next result we determine the saturation class of Bernstein-Schnabl operators and the Favard class of the semigroup $(T(t))_{t\geq 0}$ which are, respectively, the linear subspaces of all functions $f \in C([0,1])$ such that $\sup_{n\geq 1} n \parallel B_n f - f \parallel_\infty < +\infty$, resp. $\sup_{0<t} \frac{\parallel T(t)f - f \parallel_\infty}{t} < +\infty$ (see, e.g., [7, Section 2.1]). Among other things, the characterization of the Favard class reveals some "spatial regularity" properties preserved under the evolution governed by the semigroup $(T(t))_{t\geq 0}$.

However, apart its own interest, the next result will be fruitfully used to obtain a complete characterization of functions verifying asymptotic formulae.

THEOREM 7. – *Given* $f \in C([0,1])$, *the following statements are equivalent:*

(*i*) *There exists* $M_1 \geq 0$ *such that for every* $x \in [0,1]$ *and* $n \geq 1$

$$| B_n(f)(x) - f(x) | \leq \frac{M_1 a(x)}{n}.$$

(*ii*) *There exists* $M_2 \geq 0$ *such that for every* $n \geq 1$

$$\| B_n f - f \|_\infty \leq \frac{M_2}{n}.$$

(*iii*) *There exists* $M_3 \geq 0$ *such that for every* $t \geq 0$

$$\| T(t)f - f \|_\infty \leq M_3 t.$$

(*iv*) $f \in C^1([0,1])$ *and* f' *is Lipschitz continuous with some Lipschitz constant* $M \geq 0$.

If, in addition, a *is concave, then statements* (*i*)-(*iv*) *are also equivalent to*

(*v*) *There exists* $M_4 \geq 0$ *such that for every* $t \geq 0$ *and* $x \in [0,1]$

$$| T(t)(f)(x) - f(x) | \leq M_4 t a(x).$$

Finally, if one of the previous statements holds true, then

$$M_2 = M_3 = M_4 \| a \|_\infty = 4M \| a \|_\infty = M_1 \| a \|_\infty .$$

PROOF. – (*i*) $\Rightarrow$ (*ii*). It is sufficient to set $M_2 := M_1 \| a \|_\infty$.

(*ii*) $\Rightarrow$ (*iii*). For $n \geq 1$ and $p \geq 1$ we get

$$B_n^p f - f = \sum_{k=0}^{p-1} B_n^{k-1}(B_n f - f), \tag{1}$$

so that

$$\| B_n^p f - f \| \leq \sum_{k=0}^{p-1} \| B_n^{k-1} \| \| (B_n f - f) \| \leq \frac{p}{n} M_2.$$

Now, given $t \geq 0$ and considering a sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$, we obtain

$$\| B_n^{k(n)} f - f \| \leq \frac{k(n}{n} M_2.$$

and hence $\| T(t)(f) - f \| \leq M_2 t$ by (4.3).

(*iii*) $\Rightarrow$ (*iv*). By definition, if $u \in D_V(A)$, then $\lim_{t\to 0^+} \frac{T(t)u - u}{t} = Au$ uniformly on $[0,1]$. Therefore, given $x \in [0,1]$, since $\psi_x^2 \in D_V(A)$ and $\psi_x^4 \in D_V(A)$, we get

$$\lim_{t\to 0^+} \frac{T(t)(\psi_x^2)(x)}{t} = \lim_{t\to 0^+} \frac{T(t)(\psi_x^2)(x) - \psi_x^2(x)}{t} = 2a(x)$$

and, similarly,

$$\lim_{t\to 0^+} \frac{T(t)(\psi_x^4)(x)}{t} = 0.$$

Moreover, by (4.2) and (2.11), $T(t)\mathbf{1} = \mathbf{1}$ and $T(t)(\psi_x) = \psi_x$ for every $t \geq 0$.

Therefore, the same proof of part (i) of Proposition 1 adapted to the family $(T(t))_{t\geq 0}$, shows that, if a function $u \in C([0,1])$ is differentiable in a neighborhood of a point x_0 of]0,1[and if, in addition, it is two times differentiable at it, then

$$\lim_{t\to 0^+} \frac{T(t)u(x_0) - u(x_0)}{t} = a(x_0)u''(x_0).$$

We may then apply Theorem 1 of [4] to the sequence of positive linear operators $(T(1/n))_{n\geq 1}$ and hence we obtain that $f \in C^1([0,1])$ and f' is Lipschitz continuous.

$(iv) \Rightarrow (i)$. The result follows directly from (2.15).

Assume now that a is concave. Then $B_n(a) \leq a$ for every $n \geq 1$ ([16]; see also [2, Theorem 6.1.13]).

$(i) \Rightarrow (v)$. By assumption and by formula (1) above we get

$$\mid B_n^p f - f \mid \leq \sum_{k=0}^{p-1} B_n^{k-1}(\mid B_n f - f \mid) \leq \frac{M_1}{n} \sum_{k=0}^{p-1} B_n^{k-1}(a) \leq \frac{pM_1}{n} a$$

for any $n \geq 1$ and $p \geq 1$. So, if $t \geq 0$ and if $k(n)/n \to t$, from the previous estimate and from (4.3) the result follows.

$(v) \Rightarrow (iii)$. It is sufficient to set $M_3 := M_4 \parallel a \parallel_\infty$. □

REMARK 8. – In the case of classical Bernstein operators the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) are due to G.G. Lorentz ([13, Theorem 11, p.102]; see also [9]) and the equivalences $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ are due to C. A. Micchelli ([15, Theorem 3.2]).

As a consequence of the previous result we have an indication of some"spatial regularity" properties which are preserved by the semigroup $(T(t))_{t\geq 0}$, i.e., by the solutions of the corresponding evolution problem (see (5.1)). For other properties preserved by the semigroup see also [3, Corollary 3.6].

COROLLARY 9. – *Let* $f \in C^1([0,1])$ *with* f' *Lipschitz continuous. Then, for every* $s \geq 0$, $T(s)f$ *is continuous differentiable on* [0,1] *and its first derivative is Lipschitz continuous.*

PROOF. – For every $t \geq 0$ we get

$$\parallel T(t)T(s)f - T(s)f \parallel_\infty \leq \parallel T(s) \parallel \parallel T(t)f - f \parallel_\infty \leq \parallel T(t)f - f \parallel_\infty$$

and hence the result follows by applying Theorem 7 both to f and $T(s)f$. □

The characterization of the Favard class as stated in Theorem 7 can be reformulated for a large class of Feller semigroups which implicitly are represented by iterates of Bernstein-Schnabl operators.

Consider $a \in C([0,1])$ such that

$$a(0) = a(1) = 0, \ 0 < a(x) \quad \text{for every } 0 < x < 1 \tag{4.4}$$

and

$$a_0 := \sup_{0<x<1} \frac{a(x)}{x(1-x)} < +\infty, \tag{4.5}$$

and consider the differential operator $(A, D_V(A))$ defined by (4.1) and (4.2). By [8, Theorem 2] $(A, D_V(A))$ generates a Feller semigroup $(T(t))_{t\geq 0}$ on $C([0,1])$.

COROLLARY 10. – *The Favard class of the semigroup $(T(t))_{t\geq 0}$ is the linear subspace of all $f \in C^1([0,1])$ such that f' is Lipschitz continuous.*

PROOF. – The function $\tilde{a} := \dfrac{a}{2a_0} \in C([0,1])$ vanishes at 0 and 1 and satisfies $0 < \tilde{a}(x) \leq \dfrac{x(1-x)}{2}$ for every $0 < x < 1$.

By [3, Theorem 3.10] there exists a continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ satisfying (2.3) and (2.4) such that $\tilde{a}(x) = \dfrac{1}{2}\left(\int_0^1 e_2 d\mu_x - x^2\right)$ $(0 \leq x \leq 1)$. Therefore considering the differential operator

$$\tilde{A}(u)(x) := \begin{cases} \tilde{a}(x)u''(x) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, 1. \end{cases}$$

and the relevant semigroup $(\tilde{T}(t))_{t\geq 0}$, by Theorem 7 its Favard class is the linear subspace of all $f \in C^1([0,1])$ such that f' is Lipschitz continuous. Since $A = 2a_0\tilde{A}$, clearly $T(t) = \tilde{T}(2a_0 t)$ for every $t \geq 0$. Therefore the Favard class of $(T(t))_{t\geq 0}$ is the same of that one of $(\tilde{T}(t))_{t\geq 0}$ and hence the result follows. □

5. – Converse results for asymptotic formulae.

Combining the results of the previous sections we shall characterize those functions $u \in C([0,1])$ for which formulae (2.16) (or (2.18)) and (2.17) may hold.

Consider the linear operator $(A, D_V(A))$ defined by (4.1) and the Feller semigroup $(T(t))_{t\geq 0}$ generated by it. From Proposition 1 it also follows that

$$\lim_{n\to\infty} n(B_n(u) - u) = A(u) \text{ pointwise on } [0,1]$$

for every $u \in D_V(A)$ and, finally, the Bernstein-Schnabl operators are positive and contractive. Therefore Theorem 5 applies.

THEOREM 11. – *Given* $u \in C([0,1])$, *the following statements are equivalent:*

(*i*) *There exists* $v \in C([0,1]$ *such that* $\lim_{n\to\infty} n(B_n(u)-u) = v$ *pointwise on* $[0,1]$.

(*ii*) $u \in D_V(A)$, *i.e.* $u \in C^2(]0,1[)$ *and*

$$\lim_{x\to 0^+} a(x)u''(x) = \lim_{x\to 0^+} a(x)u''(x) = 0.$$

(*iii*) *There exists* $w \in C([0,1]$ *such that* $\lim_{t\to 0^+} \frac{T(t)u-u}{t} = w$ *pointwise on* $[0,1]$.

Moreover, if one of the previous statements holds true, then $v = w = Au$. *In particular, if* $\lim_{n\to\infty} n(B_n(u)-u) = 0$ *pointwise on* $[0,1]$ *or if* $\lim_{t\to 0^+} \frac{T(t)u-u}{t} = 0$ *pointwise on* $[0,1]$, *then* u *is linear.*

REMARK 12. – It seems to be worth stressing that the functions $u \in C([0,1])$ satisfying one of the previous conditions (i), (ii) or (iii) are the only ones for which the following initial boundary value problem

$$(5.1)\qquad \begin{cases} \dfrac{\partial W(x,t)}{\partial t} = a(x)\dfrac{\partial^2 W(x,t)}{\partial x^2} & 0<x<1,\ t\geq 0, \\ \lim\limits_{x\to 0^+} a(x)\dfrac{\partial^2 W(x,t)}{\partial x^2} = \lim\limits_{x\to 1^-} a(x)\dfrac{\partial^2 W(x,t)}{\partial x^2} = 0 & t \geq 0, \\ \lim\limits_{t\to 0^+} W(x,t) = u(x) & 0 \leq x \leq 1, \end{cases}$$

has a unique solution $W : [0,1]\times[0,+\infty[\to \mathbb{R}$ which is given by

$$(5.2)\qquad W(x,t) = T(t)u(x) = \lim_{n\to\infty} B_n^{k(n)}u(x)$$

$(0 \leq x \leq 1,\ t \geq 0)$ where $(k(n))_{n\geq 1}$ is a sequence of positive integers such that $k(n)/n \to t$, and the limit above is uniform with respect to $x \in [0,1]$.

This follows at once from the general theory of strongly continuous semigroups ([11]) and from the representation formula (4.3) of the semigroup $(T(t))_{t\geq 0}$ in terms of iterates of Bernstein-Schnabl operators.

We refer to [3] and to [2, Section 6.3.4] for several applications of representation formula (4.3) and for a discussion of a stochastic model from genetics governed by (5.1).

A "uniform" counterpart of Theorem 10 is stated below. For Bernstein operators this result can be also find in [6, p. 703] where, however, completely different methods are used.

THEOREM 13. – *Given* $u \in C([0,1])$, *the following statements are equivalent:*

(i) *There exists* $\lim_{n\to\infty} n(B_n(u) - u)$ *uniformly on* $[0,1]$.

(ii) $u \in C_b^2(]0,1[)$.

(iii) *There exists* $w \in C([0,1]$ *such that* $\sup_{0<x<1} \frac{w(x)}{a(x)} < +\infty$ *and* $\lim_{t\to 0^+} \frac{T(t)u - u}{t} = w$ *uniformly on* $[0,1]$.

PROOF. – As regards the equivalence (i) $\Leftrightarrow$ (ii), on account of Theorem 2 we have only to prove that (i) implies (ii).

Under assumption (i) we get, in particular, that $\sup_{n\geq 1} n \parallel B_n(u) - u \parallel < +\infty$. By Theorem 7, $u \in C^1([0,1])$ and u' is Lipschitz continuous. On the other hand, by Theorem 10, $u \in C^2(]0,1[)$ and hence u'' is bounded on $]0,1[$.

The equivalence (ii) $\Leftrightarrow$ *(iii)* is obvious. □

We finally point out that some of the results we have established in this paper seem to be extensible to other sequences of positive linear operators acting on continuous function spaces defined on a not necessarily compact interval. We shall develop such an analysis in a forthcoming paper.

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