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Thiemo Kessel, Tristan Rivière

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Singular Bundles with Bounded L^2-Curvatures

THIEMO KESSEL - TRISTAN RIVIÈRE

Dedicated to the memory of Guido Stampacchia

Abstract. – *We consider calculus of variations of the Yang-Mills functional in dimensions larger than the critical dimension 4. We explain how this naturally leads to a class of – a priori not well-defined – singular bundles including possibly "almost everywhere singular bundles". In order to overcome this difficulty, we suggest a suitable new framework, namely the notion of* singular bundles with bounded L^2-curvatures.

1. – Introduction.

1.1 – *Yang-Mills functional and the Uhlenbeck Coulomb gauge extraction result in dimensions* $n \leq 4$.

Let $\pi : P \longrightarrow M$ be a principal G-bundle over a compact n-dimensional Riemannian manifold M. The structure group G of P is assumed to be a compact Lie group with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{D}(P)$ the space of connections on P. For any connection D let $F_D \in \Omega^2(M, \mathrm{ad}(P))$ denote the curvature of D. The Yang-Mills functional is then defined as

$$YM(D) = \int_M |F_D|^2 \, dvol_g, \tag{1.1}$$

where $dvol_g$ denotes the volume form on M induced by the metric g. The norm of F_D is induced by the Killing form on $\mathfrak{g}$ and the Riemannian metric on M.

In order to proceed to calculus of variations (such as finding critical points, minimizers and saddle points) of the Yang-Mills functional, a first approach consists in enlarging the class of smooth connections to the class of Sobolev $W^{1,2}$-connections. The space of these Sobolev $W^{1,2}$-connections is defined to be – modulo the addition of an arbitrary smooth reference connection D_0 – the space $W^{1,2}\big(\Gamma(T^*M \otimes \mathrm{ad}(P))\big)$. The latter consists of $W^{1,2}$-sections of the bundle $T^*M \otimes \mathrm{ad}(P)$ which are in the closure of smooth sections for the $W^{1,2}$-norm (see for instance [Uh1], [FrU] for details). For smooth local trivializations $P = \bigcup_{i \in I} \pi^{-1}(U_i)$, where the U_i form a covering of M by open sets over which the bundle

P is trivial and transition functions of P by $g_{ij} \in C^\infty(U_i \cap U_j, G)$, the previously defined $W^{1,2}$-connections are families of $W^{1,2}$-1-forms A_i on U_i taking values into the Lie algebra $\mathfrak{g}$. Moreover, they are related by the formulae

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \qquad \text{in} \quad W^{1,2}(\wedge^1 (U_i \cap U_j) \otimes \mathfrak{g}).$$

This framework of **Sobolev connections** and **Sobolev gauge transformations** over **smooth bundles** was successful for pursuing calculus of variation questions for the Yang-Mills functional up to the critical dimension $n = 4$. Note that the dimension 4 is critical in the sense that in this dimension the functional is conformally invariant which is related to the fact that the Yang-Mills equation (1.5) is critical precisely in dimension 4. The success of this framework is due to the fundamental *Coulomb gauge extraction* theorem by K. Uhlenbeck.

THEOREM 1.1 [Uhl]. – *Let $n \le 4$, then there exists $\varepsilon(n) > 0$ such that the following holds: Let A be a $W^{1,2}$-1-form over the unit ball B^n taking values into the Lie algebra $\mathfrak{g}$, i.e. $A \in W^{1,2}(\wedge^1 (B^n) \otimes \mathfrak{g})$. Assuming*

$$YM(A) = \int_{B^n} |dA + A \wedge A|^2 \, dx^n < \varepsilon(n), \tag{1.2}$$

there exists a $W^{2,2}$-map g from B^n into G, i.e. a gauge transformation, such that the 1-form A_{coul} given by $A_{coul} := g^{-1} A g + g^{-1} dg$ satisfies the $W^{1,2}$-norm control

$$\|A_{coul}\|^2_{W^{1,2}(B^n)} \le C(n) \int_{B^n} |dA + A \wedge A|^2 \, dx^n \,, \tag{1.3}$$

where $C(n)$ is independent of A, and the Coulomb gauge condition

$$d^* A_{coul} = 0 \quad in \quad B^n. \tag{1.4}$$

□

This result is the main tool in doing calculus of variations with the Yang-Mills functional in dimensions less or equal to 4. For instance, it gives the required coercivity of the Yang-Mills functional in order to get the existence of minimizers under the small energy assumption (1.2) for some boundary data. Indeed, under this assumption, the Coulomb gauge representatives of a minimizing sequence of connections satisfy the $W^{1,2}$-norm control (1.3). One can then extract converging subsequences in $W^{1,2}$ and – using the lower semi continuity of the L^2-norm – one gets the existence of a minimizer. The global existence of minimizers for a given bundle P over a 4-dimensional Riemannian manifold M requires a covering procedure by balls where the condition (1.2) is satisfied (see [Se]). However, this could sometimes fail due to the possibility of pointwise concentration of the Yang-Mills energy which is the famous *concentration-compactness* phenomenon first discovered in [SaU] for the harmonic map setting and extended to Yang-Mills in [Uh1], [Uh2], [Uh3], [DK] (see also [FrU]).

A further important contribution of Theorem 1.1 to the calculus of variations of the Yang-Mills functional in dimensions less or equal to 4 is the following fact: The existence of the Coulomb gauge A_{coul} is fundamental for looking at critical points of *YM* since it "gives" it's elliptic nature to the Yang-Mills equation. More precisely, the intrinsic Yang-Mills equation suffers from a too large symmetry group whereas – once the Coulomb gauge is fixed – it reads

$$\Delta A_{coul} = d^*[A_{coul}, A_{coul}] - *\left[A_{coul}, *\left(dA_{coul} + \frac{1}{2}[A_{coul}, A_{coul}]\right)\right], \tag{1.5}$$

which clearly is a non-linear elliptic equation critical in dimension 4 for the $W^{1,2}$-norm of A.

In recent years, there has been an increasing interest for pursuing calculus of variations of the Yang-Mills functional in dimensions larger than the critical dimension 4. Geometric motivations for looking at the analysis of Yang-Mills fields and, more generally, at the analysis of gauge theories in higher dimensions can be found for instance in [DT] and [Ti]. However, Theorem 1.1 does not hold for the Yang-Mills energy in dimensions larger than 4, it is easy to construct counter examples to it. There have been several attempts for finding higher dimensional versions of the Uhlenbeck result (for instance in [MR], [TT]). These attempts could only be successful through requiring the curvature to be small in an ad-hoc Morrey space. Although such assumptions "naturally" extend hypothesis (1.2) from a "functional analysis" point of view, they are far too strong for looking at critical points of the Yang-Mills Lagrangian with bounded energy in its full generality.

Another difficulty for doing calculus of variations of the Yang-Mills functional in dimensions larger than 4 comes from the fact that **there is a need of enlarging the class of connections and bundles beyond the $W^{1,2}$-connections on smooth bundles**. In order to motivate this, we shall make a digression to the framework of harmonic maps.

1.2 – *The topological singular set of $W^{1,2}$-Sobolev maps into S^2.*

Since the middle of the 20th century, for doing calculus of variations of the Dirichlet energy

$$E(u) = \int_{B^n} |\nabla u|^2 \, dx^n$$

for maps u from the unit ball $B^n \subset \mathbb{R}^n$ into the unit 2-sphere S^2, it has become natural to extend the class of smooth maps to the class of $W^{1,2}$-Sobolev maps defined by

$$W^{1,2}(B^n, S^2) := \Big\{u \in W^{1,2}(B^n, \mathbb{R}^3) \ \text{ s.t. } \ |u|(x) = 1 \ \text{ for a.e. } x \in B^n\Big\}.$$

This class of maps is suitable due to the coercivity and the lower semicontinuity of E on $W^{1,2}(B^n, S^2)$. Critical points of E are called harmonic maps and satisfy the equation

$$\Delta u + u|\nabla u|^2 = 0 \qquad \text{in} \quad \mathcal{D}'(B^n). \tag{1.6}$$

The dimension 2 for E corresponds to the dimension 4 for YM. It is indeed this critical dimension for which E is conformally invariant and for which the corresponding Euler-Lagrange equation (1.6) is critical for the $W^{1,2}$-norm (in the non-linear elliptic PDE terminology). On the other hand, the dimension 2 is also a critical dimension for $W^{1,2}(B^n, S^2)$ due to the following result:

THEOREM 1.2 [Wh], [ScU2]. – *Smooth maps are dense in $W^{1,2}(B^n, S^2)$ if and only if $n \leq 2$.* □

The map $v(x) = x/|x|$ is an example of a map which cannot be approximated strongly by smooth maps in $W^{1,2}(B^3, S^2)$. It has a singularity at the origin of "topological nature". More precisely, the restriction of v to any 2-sphere containing the origin is a map between 2-spheres which is not homotopic to a constant and has topological degree equal to $+1$. Using the integral representation of the degree this reads

$$\int_{\partial B} v^*\omega = +1\,, \tag{1.7}$$

where B is any ball containing the origin 0 and ω is an arbitrary two-form on S^2 whose integral is equal to one. The last equation can also be written in the form

$$d(v^*\omega) = \delta_0 \quad \text{in} \quad \mathcal{D}'(B^3)\,. \tag{1.8}$$

The realization of non-zero homotopic maps on a "full measure" of 2-spheres in B^3 is in fact the obstruction for a map in $W^{1,2}(B^3, S^2)$ to be strongly approximable by smooth maps in the $W^{1,2}$-norm. Precisely, the following theorem holds:

THEOREM 1.3 [Be2], [BCDH]. – *A map u in $W^{1,2}(B^3, S^2)$ is in the closure of $C^\infty(B^3, S^2)$ for the strong $W^{1,2}$-topology if and only if*

$$d(u^*\omega) = 0 \qquad \text{in} \quad \mathcal{D}'(B^3)\,, \tag{1.9}$$

where ω is an arbitrary two-form on S^2 satisfying $\int_{S^2} \omega \neq 0$. □

In the attempt to approximate an arbitrary map in $W^{1,2}(B^3, S^2)$ by maps being "as regular as possible" F. Bethuel introduced the following space:

$$\mathcal{R}^{\infty}_{1,2}(B^3, S^2) = \left\{ \begin{array}{l} u \in W^{1,2}(B^3, S^2) \text{ s.t. } \exists\ a_1, \ldots, a_N \in B^3 \text{ with} \\ u \in C^{\infty}(B^3 \setminus \{a_1, \ldots, a_N\}, S^2) \text{ and } deg(u, a_i) = \pm 1 \end{array} \right\},$$

where $deg(u, a_i)$ is the topological degree of small spheres in B^3 bounding balls containing the point a_i and no other a_j. In other words $\mathcal{R}^{\infty}_{1,2}(B^3, S^2)$ is the subspace of maps in $W^{1,2}(B^3, S^2)$ smooth away from finitely many singular points around which the map has topological degree ± 1. The next result motivates the definition of this space.

THEOREM 1.4 [Be1]. – *The space $\mathcal{R}^{\infty}_{1,2}(B^3, S^2)$ is dense in $W^{1,2}(B^3, S^2)$ for the strong $W^{1,2}$-topology.* □

Now we consider the regularity theory of the critical points to the Dirichlet energy E in the Sobolev space $W^{1,2}(B^n, S^2)$. In dimension 2, for which E is conformally invariant and for which the harmonic map equation (1.6) is critical, it was proved by F. Hélein that $W^{1,2}$-solutions of (1.6) are smooth and even analytic (see [He]). In contrast to this, it was proved by the second author in [Ri1] that – in dimension 3 – there exist solutions to (1.6) which are everywhere discontinuous. The counter examples constructed in this work used the possibility for solutions of (1.6) to realize non-trivial homotopy groups in a dense class of 2-spheres in B^3. Precisely, the support of $d(u^*\omega)$ is the whole ball B^3.

When restricting ourselves to solutions of (1.6) which are minimizing E for some given smooth boundary data ϕ, the singular set of u is made of isolated points with degree ± 1. This is the content of the next theorem.

THEOREM 1.5 [ScU]. – *Let ϕ be in $C^{\infty}(\partial B^3, S^2)$. The minimizers of E among the maps in $W^{1,2}(B^3, S^2)$ equal to ϕ on ∂B^3 are in $\mathcal{R}^{\infty}_{1,2}(B^3, S^2)$.* □

The result is optimal in the following sense: For any $N \in \mathbb{N}$ there exists boundary data ϕ_N of degree zero and a minimizer of E among the maps in $W^{1,2}(B^3, S^2)$ equal to ϕ_N on ∂B^3 which has at least N distinct singular points of degree ± 1 (see [HL]).

1.3 – *Beyond $W^{1,2}$-connections on smooth bundles.*

It is now time to make the link between our digression on S^2-valued maps and connections. For smooth connections this link is given by a theorem of M. S. Narasimhan and S. Ramanan [NR1] and [NR2]. This theorem states that given a connection on a principal $U(k)$-bundle P over a manifold M there exists a smooth map u from M into the complex Grassmannian manifold $G(m,k)$ (for some m depending of k and the dimension of M) such that the given connection is the pull-back under u of the universal canonical connection of the Stiefel bundle $V(m,k)$. (To the knowledge of the authors, no weak version of the Narasimhan-Ramanan theorem is known in the framework of Sobolev connections on smooth bundles which is certainly an interesting open problem). Motivated by the situation in the Narasimhan-Ramanan theorem, we consider first the case $k=1$ of Abelian principal $U(1)$-bundles over B^3. In this setting, the corresponding complex Grassmannian manifold becomes $\mathbb{C}P^1$ and the Stiefel bundle is given by the so-called tautological Hopf fibration $S^3 \to \mathbb{C}P^1$ with universal canonical curvature form given by the volume form ω_{S^2} of S^2. Without appealing to Narasimhan-Ramanan result itself but getting inspired by it, we can follow our intuition guided by the examples given by the pull-back curvatures of the form $F := 2\,u^*\omega_{S^2}$ where u is a map from B^3 into S^2. Note that if u is smooth F is an exact form on B^3.

In trying to minimize

$$YM_1(F) = \int_{B^3} |F|\, dx^3$$

among smooth curvatures $F = dA$ of the Abelian trivial bundle over B^3 for a fixed boundary condition $\imath^*_{\partial B^3} F = \xi$ on ∂B^3 ($\imath_{\partial B^3}$ denotes the inclusion map of ∂B^3 into $\mathbb{R}^3$) one encounters the following difficulties:

i) It is not clear whether this infimum is achieved by a smooth curvature.

ii) For some boundary data ξ there is an energy gap between this infimum and the infimum of YM_1 among curvatures of **singular** bundles over B^3.

The first remark i) is clear from a PDE point of view. Minima of YM_1 satisfy the equation $d^*(dA/|dA|) = 0$ possibly coupled with the Coulomb gauge condition $d^*A = 0$. It is well known that solutions to such PDE can have singularities.

The second remark ii) is motivated by our previous digression on maps. It could be energetically more favorable to include point singularities in the bundle. In order to see this, we consider the pull-back of $2\omega_{S^2}$ by a map u in $\mathcal{R}^\infty_{1,2}(B^3,S^2)$ with singular set $\{a_1,\ldots,a_N\}$. Then $2u^*\omega_{S^2}$ is a smooth curvature of the smooth $U(1)$-bundle P_u over $B^3 \setminus \{a_1,\ldots,a_N\}$ with first Chern class given by the degree of the map u on small spheres surrounding the a_i

$$\left\{\begin{array}{l} c_1(P_u) = \left[\dfrac{1}{4\pi} u^*\omega_{S^2}\right] \quad \text{in} \quad H^2(B^3 \setminus \{a_1,\ldots,a_N\}, \mathbb{R}) \\[2mm] \text{and} \quad d\left(\dfrac{1}{4\pi} u^*\omega_{S^2}\right) = \displaystyle\sum_{i=1}^N deg(u,a_i)\ \delta_{a_i} \qquad \text{in } \mathcal{D}'(B^3)\,. \end{array}\right. \tag{1.10}$$

Now take ξ_n to be a sequence of smooth 2-forms converging in measure to

$(\delta_N - \delta_S)\,\omega_{S^2}$ where δ_N (resp. δ_S) denotes the Dirac masses at the north (resp. south) pole of $\partial B^3 = S^2$. It is not difficult to see that

$$\lim_{n\to+\infty} \inf \left\{ \begin{array}{c} \displaystyle\int_{B^3} |F|\, dx^3 \ ;\ F \in C^\infty(\wedge^2 (\overline{B^3})) \\ dF = 0 \text{ in } \mathcal{D}'(B^3) \text{ and } \imath^*_{\partial B^3} F = \xi_n \end{array} \right\} = |N - S|\,. \tag{1.11}$$

However, by allowing the bundle to have point singularities around which some Chern class is realized, without too much effort one can prove that

$$\lim_{n\to+\infty} \inf \left\{ \begin{array}{c} \displaystyle\int_{B^3} |F|\, dx^3 \ ;\ \exists\, a_1, \cdots, a_N \in B^3\ ,\ d_1, \ldots, d_N \in \{-1, +1\} \\ \text{with } F \in C^\infty(\wedge^2 (\overline{B^3} \setminus \{a_1, \ldots, a_N\}))\,, \\ \displaystyle * \, dF = 4\pi \sum_{i=1}^{N} d_i\ \delta_{a_i} \text{ in } \mathcal{D}'(B^3) \text{ and } \imath^*_{\partial B^3} F = \xi_n \end{array} \right\} = 0\,. \tag{1.12}$$

Comparing (1.11) with (1.12) we obtain the desired energy gap and conclude remark ii).

The corresponding problem for the Yang-Mills 2-energy (1.1) and a non-Abelian structure group G is even more severe. It arises first in dimension $n = 5$. To simplify the presentation we take the simplest non-Abelian setting with $G = SU(2)$. In trying to minimize

$$YM_2(F) = \int_{B^5} |F|^2\, dx^5$$

among smooth curvatures $F = dA + A \wedge A$ of the trivial $SU(2)$-bundle over B^5 – i.e. A is a 1-form with values in $su(2)$ – for a fixed boundary condition $\imath^*_{\partial B^5} F = \Xi$ on ∂B^5 ($\imath_{\partial B^5}$ denotes the inclusion map of ∂B^5 into $\mathbb{R}^5$) one again encounters the following two difficulties:

i) It is not clear whether this infimum is achieved by a smooth curvature.

ii) For some boundary data Ξ there is an energy gap between this infimum and the infimum of YM_2 among curvatures of **singular** bundles over B^5.

The first remark i) is far more problematic in this non-Abelian setting than in the previously described Abelian case. Indeed, in the Abelian situation the Lagrangians YM_p are coercive for all $p \geq 1$. This is because of an "Abelian" Coulomb gauge extraction which is a simple linear problem that can be solved for any amount of YM_p-energy and in arbitrary dimension. However, in the present non-Abelian case – as we already saw in Section 1.1 – due to the fact that even for small YM_2-energy Uhlenbeck's Coulomb gauge extraction result fails in dimensions larger than 4, coercivity is missing. Therefore we cannot immediately conclude the existence of a minimizer – regular or not – of YM_2.

The second remark ii) can be illustrated by an example similar to the one we saw in the Abelian case above. Taking a sequence of smooth boundary data $\Xi_n \in C^\infty(\wedge^2(\partial B^5)\otimes su(2))$ converging in Radon measure on ∂B^5 and in $C^k_{loc}(\partial B^5\setminus\{N,S\})$ to

$$\begin{aligned}\Xi_\infty := 2[&(dx_1\, dx_2 + dx_3\, dx_4)\sigma_1 + (dx_1\, dx_3 - dx_2\, dx_4)\sigma_2\\ &+ (dx_1\, dx_4 + dx_2\, dx_3)\sigma_3]\,,\end{aligned}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices forming an orthonormal basis of $su(2)$ and N and S respectively denote the north and the south pole of the 4-sphere ∂B^5. Restricting to the smooth trivial bundle over B^5 and $W^{1,2}$-connections, it is then not difficult to see that

$$\lim_{n\to+\infty}\inf\left\{\begin{array}{c}\displaystyle\int_{B^5}|dA+A\wedge A|^2\,dx^5\,;\\ A\in W^{1,2}(\wedge^1(\overline{B^5})\otimes su(2))\,,\\ i^*_{\partial B^5}(dA+A\wedge A)=\Xi_n\end{array}\right\} = |N-S|\,. \tag{1.13}$$

On the other hand by allowing the bundle to be singular one can save energy in such a way that

$$\lim_{n\to+\infty}\inf\left\{\begin{array}{c}\displaystyle\int_{B^5}|F_D|^2\,dx^5\,;\exists\, a_1,\ldots,a_N\in B^5,\ d_1,\ldots,d_N\in\{-1,+1\}\\ \text{and } D \text{ is a } C^\infty\text{-connection}\\ \text{on the } SU(2)\text{-bundle over } \overline{B^5}\setminus\{a_1,\ldots,a_N\} \text{ given by}\\ \displaystyle *\,d(tr(F_D\wedge F_D)) = 8\pi^2\sum_{i=1}^N d_i\ \delta_{a_i}\quad \text{in } \mathcal{D}'(B^5)\end{array}\right\} = 0\,. \tag{1.14}$$

At this stage it is important to observe that a connection D on a smooth bundle over $B^5\setminus\{0\}$ with some non-zero second Chern class around the origin does not admit a gauge A in $W^{1,2}(\wedge^1 B^5, su(2))$. Indeed, if such a gauge would exist, one would have

$$*d[tr(F_A\wedge F_A)] = 0\,,$$

where $F_A = dA + A\wedge A$, contradicting the assumption on the topology of the $SU(2)$-bundle. This last fact is a consequence of a density result of smooth connections over S^4 for which we refer to Proposition 3.1.

Conclusion to the Introduction: The calculus of variations of the Dirichlet energy naturally leads to the class of Sobolev maps $W^{1,2}(B^n, S^2)$. In dimensions larger than the critical dimension 2, maps in this class can have “topological-type” singularities even when considering the minimizers to E which are expected to have the highest regularity. The parallel between smooth maps and smooth connections on smooth bundles given by the Narasimhan-Ramanan result – which possibly also works in a non-smooth framework – suggests to extend the class of smooth (or even Sobolev) connections on smooth bundles to an enlarged class of connections on singular bundles. Up to now we introduced singular bundles in critical+1 dimensions – $2+1$ for $U(1)$-bundles and YM_1 and $4+1$ for $SU(2)$-bundles and YM_2 – which are smooth apart from finitely many points in the base. When doing calculus of variations of YM_1 or YM_2, the position and the number of these points have to be arbitrary. Therefore the question of describing the "boundary" of this space of singular bundles, meaning that the number of singular points tends to infinity, arises naturally. Furthermore, one needs to formulate the corresponding Yang-Mills YM_1 and YM_2 variational problems for **singular connections** on **singular bundles** and the Euler-Lagrange equations corresponding to their critical points. (1)

2. – Singular Abelian bundles with bounded L^1-curvatures.

2.1 – *L^1-curvatures of singular $U(1)$-bundles and an approximation result.*

In this section we introduce a "closure" for the L^1-norm of the space of smooth curvatures of $U(1)$-bundles over $B^3 \setminus \{\text{isolated points}\}$.

DEFINITION 2.1 [L^1-curvatures of singular $U(1)$-bundles]. – *An L^1-curvature of a singular $U(1)$-bundle over B^3 is a measurable real-valued 2-form F satisfying*

i)

$$\int_{B^3} |F|\, dx^3 < +\infty\,. \tag{2.1}$$

ii) *For all $x \in B^3$ and for almost every $0 < r < dist(x, \partial B^3)$ we have*

$$\frac{1}{4\pi}\int_{\partial B_r(x)} \imath^*_{\partial B_r(x)} F \in \mathbb{Z}\,, \tag{2.2}$$

where $\imath_{\partial B_r(x)}$ is the inclusion map of the boundary of $B_r(x)$ into B^3. □

(1) An interesting parallel between smooth harmonic maps and smooth Yang-Mills fields, also inspired by the Narasimhan-Ramanan result, can be found in [DV].

Observe that a real-valued 2-form F in L^1 satisfying

$$\int_{\partial B_r(x)} \imath^*_{\partial B_r(x)} F = 0$$

for every x in B^3 and almost every $0 < r < dist(x, \partial B^3)$ is exact, since there exists an L^1-1-form A such that $F = dA$ in distributional sense. This then implies that F can be interpreted as an L^1-curvature of a **smooth** trivial $U(1)$-bundle over B^3. Also note that L^1-curvatures of smooth $U(1)$-bundles over $B^3 \setminus I$, where I is a discrete subset of B^3, are examples of *L^1-curvatures of singular $U(1)$-bundles over B^3*.

In analogy with the situation of $\mathcal{R}^\infty_{1,2}(B^3, S^2)$ in $W^{1,2}(B^3, S^2)$, one can prove the following density result (see [KR]) which also motivates Definition 2.1.

THEOREM 2.1. – *Let F be an L^1-curvature of a singular $U(1)$-bundle over B^3. Then there exists a sequence of finite families of points, $I^k = \{a^k_1, \ldots, a^k_{N^k}\}$, a sequence of finite families of ± 1, $\mathcal{D}^k = \{d^k_1, \ldots, d^k_{N^k}\}$ and a sequence of smooth curvatures F^k of the smooth $U(1)$-bundles over $B^3 \setminus I^k$ given by*

$$* dF^k = 4\pi \sum_{i=1}^{N^k} d^k_i \; \delta_{a^k_i} \qquad \text{in } \mathcal{D}'(B^3)\,, \tag{2.3}$$

such that

$$F^k \longrightarrow F \qquad \text{strongly in } L^1(B^3)\,. \tag{2.4}$$

□

Sketch of the proof of Theorem 2.1. Its structure is modelled after the proofs of approximation results for maps between manifolds similar to Theorem 1.4 given by F. Bethuel in [Be1]. Precisely, we proceed as follows:

i) *Choice of an ε–ball covering :* We choose a covering of B^3 by N_0 regular famillies of disjoint balls of radius ε, where N_0 is an universal number. This covering is chosen by the mean of the mean value and Fubini theorems in such a way that the L^1-difference between F and the two form $\overline{F}$ on B^3, which on each ε-ball is equal to the average of F, tends to zero on the boundary of the corresponding ε-balls as ε goes to zero.

ii) *Good and bad balls:* For each of the N_0 famillies the good ε-balls are the ones for which the L^1-norm of the restriction of F to the boundary is below a certain universal quantity. In particular, this quantity is small enough to ensure the triviality of the bundle over the boundary of the good cubes for which the restriction of F to this boundary is a weak curvature (due to condition (2.2)). The bad balls are the remaining balls and for energy reasons the total volume of their union tends to zero as ε tends to zero.

iii) *Smoothing on the boundary:* On the two-dimensional submanifold given by the union of the boundaries of the ε-balls the L^1-norm of F is critical and we can hence approximate F by smooth curvatures $\tilde{F}$ applying a density result corresponding to Theorem 1.2 for maps.

iv) *Gauge fixing:* On the boundary of each good ε-ball of the first family we consider the "linear Coulomb gauge" for the approximation $\tilde{F}$ of $\iota^*_{\partial ball}F$ given by the previous step, i.e. $\tilde{F} = dA_{coul}$ and $d^*A_{coul} = 0$ on $\partial cube$. At this stage it is important to observe that in the present Abelian situation, changing the gauge does not change the 2-form defining the curvature. This will no longer be the case in the next section and hence will be the main source of new difficulties in the approximation procedure of non-Abelian-type singular curvatures.

v) *Extensions:* On the good ball in the first family we replace F by the exterior derivative of the harmonic extension of the linear Coulomb gauge of the approximation $\tilde{F}$. On the bad balls we take a radial extension of $\tilde{F}$ which gives rise to a topological point singularity in the bundle.

vi) *Iteration* we repeat the procedure for the further families one after the other.

vii) *Smoothing:* We have obtained in this way a family of curvatures of smooth bundles over B^3 minus finitely many points (the centers of the bad balls). We take locally, away from this finite familly of centers of bad balls, a gauge for these curvatures that we smooth by taking convolutions with a smooth approximation of the Dirac mass.

viii) *Passing to the limit as ε tends to zero:* The last step is the checking-test step where we collect the estimates in the previous steps and prove that the constructed sequence strongly converges to F in L^1 as ε tends to zero.

2.2 – *Connecting the topological singularities of L^1-curvatures of singular $U(1)$-bundles.*

The purpose of this subsection is to give a better description of the numbers of topological singularities that a *singular $U(1)$-bundle with bounded L^1-curvature* could have. To that aim we should again mimic the situation for maps in $W^{1,2}(B^3, S^2)$.

First recall that a finite mass integer rectifiable 1-dimensional current $\mathbb{L}$ in $\mathbb{R}^n$ is a linear form on smooth compactly supported 1-forms of $\mathbb{R}^n$ satisfying the following two conditions:

i)

$$\forall \psi \in \Omega^1_0(\mathbb{R}^n) \quad \langle \mathbb{L}, \psi \rangle = \sum_{k=1}^{+\infty} \int_{\Gamma_k} \theta \, \psi \, ,$$

where Γ_k are disjoint measurable subsets of oriented C^1-curves in $\mathbb{R}^n$ with respect to the 1-dimensional Hausdorff measure $\mathcal{H}^1$ and θ is a measurable map on the union of the Γ_k taking values into $\mathbb{Z}$.

ii)

$$M(\mathbb{L}) = \sum_{k=1}^{+\infty} \int_{\Gamma_k} |\theta| \, d\mathcal{H}^1 < +\infty \,,$$

where $M(\mathbb{L})$ is called the *mass of* $\mathbb{L}$.

For any 1-dimensional current $\mathbb{T}$ in $\mathbb{R}^n$, the boundary of $\mathbb{T}$ is the distribution in $\mathcal{D}'(\mathbb{R}^n)$ defined by

$$\forall \varphi \in C_0^\infty(\mathbb{R}^n) \quad \langle \partial \mathbb{T}, \varphi \rangle := \langle \mathbb{T}, d\varphi \rangle \,.$$

Moreover, we shall use the following notation. For any 2-form F in $L^1(\wedge^2(B^3))$ we denote by $[F]$ the 1-dimensional current given by

$$\forall \psi \in \Omega_0^1(\mathbb{R}^n) \quad \langle [F], \psi \rangle := \int_{\mathbb{R}^n} F \wedge \psi \,.$$

In trying to control the number of possible topological singularities for maps in $W^{1,2}(B^3, S^2)$, M. Giaquinta, G. Modica and G. Souček obtained the following result:

THEOREM 2.2 [GMS]. – *Let u be a map in $W^{1,2}(B^3, S^2)$ and let ω be a 2-form on S^2 satisfying $\int_{S^2} \omega = 1$. Then there exists a finite mass integer rectifiable current $\mathbb{L}$ in B^3 such that*

$$\partial [u^* \omega] = \partial \mathbb{L} \quad \text{in } \mathcal{D}'(B^3) \,. \tag{2.5}$$

□

The minimal mass $L(u)$ among all rectifiable currents $\mathbb{L}$ satisfying (2.5) was first introduced under the name of *minimal connections* for maps in $\mathcal{R}_{1,2}^\infty$ in [BCL] and, for arbitrary u in $W^{1,2}(B^3, S^2)$, is given by the following formula:

$$L(u) = \sup \left\{ \int_{B^3} d\xi \wedge u^* \omega - \int_{\partial B^3} \xi \, u^* \omega \quad ; \quad \xi \in Lip(\overline{B^3}) \;,\; \|d\xi\|_\infty \leq 1 \right\} .$$

This quantity is the "energy defect" for strongly approximating u in $W^{1,2}$ by smooth maps in the sense described below. For any smooth map ϕ from ∂B^3 into S^2 with degree 0, we denote by $C^1_\phi(B^3, S^2)$ (resp. $W^{1,2}_\phi(B^3, S^2)$) the maps in $C^1(B^3, S^2)$ (resp. in $W^{1,2}(B^3, S^2)$) equal to ϕ on the boundary. Then it is shown in [BBC] that

$$\inf_{u\in C^1_\phi(B^3,S^2)} \int_{B^3} |\nabla u|^2\, dx^3 = \inf_{u\in W^{1,2}_\phi(B^3,S^2)} \int_{B^3} |\nabla u|^2\, dx^3 + 2L(u)\,. \tag{2.6}$$

Going back to the framework of singular $U(1)$-bundles with bounded L^1-curvature, we have the subsequent result (see the proof in [KR]):

THEOREM 2.3. – *Let F be an L^1-curvature of a singular $U(1)$-bundle over B^3. Then there exists a finite mass integer rectifiable current $\mathbb{L}$ in B^3 such that*

$$\frac{1}{4\pi}\,\partial\,[F] = \partial\,\mathbb{L} \quad \text{in } \mathcal{D}'(B^3)\,. \tag{2.7}$$

□

Let Φ be a smooth 2-form on S^2 such that $\int_{S^2} \Phi \in \mathbb{Z}$. Denote by $\mathcal{F}^\infty(B^3)$ the space of smooth L^1-bounded curvatures on smooth bundles over $B^3 \setminus I$, where I is a discrete subset of B^3, and by $\mathcal{F}^\infty_\Phi(B^3)$ the subspace of 2-forms F in $\mathcal{F}^\infty(B^3)$ whose restriction to ∂B^3 equals Φ. Finally, denote by $\overline{\mathcal{F}^\infty_\Phi}(B^3)$ the closure of $\mathcal{F}^\infty_\Phi(B^3)$ for the L^1-norm. Because of Theorem 2.1 this closure coincides with the space of L^1-curvatures of singular $U(1)$-bundles on any open subset $U \subset\subset B^3$.

Similarly to the case of maps, for an L^1-curvature F in $\overline{\mathcal{F}^\infty_\Phi}$, the minimal mass among the 1-dimensional integer rectifiable currents satisfying (2.7) is given by

$$L(F) = \frac{1}{4\pi}\sup\left\{ \int_{B^3} d\xi \wedge F - \int_{\partial B^3} \xi\ \Phi \quad ; \quad \xi \in Lip(\overline{B^3})\ ,\ \|d\xi\|_\infty \le 1 \right\}. \tag{2.8}$$

Moreover, assuming $\int_{S^2} \Phi = 0$ and denoting by $C^\infty_{d,\Phi}(\wedge^2 B^3)$ the space of smooth closed 2-forms in $\overline{B^3}$ whose restriction to ∂B^3 equals to Φ, we have the next result proved in [KR]:

THEOREM 2.4. – *In the above setting we have*

$$\inf_{F\in C^\infty_{d,\Phi}(\wedge^2 B^3)} \int_{B^3} |F|\, dx^3 = \inf_{F\in\overline{\mathcal{F}^\infty_\Phi}} \int_{B^3} |F|\, dx^3 + 4\pi L(F)\,. \tag{2.9}$$

□

3. – Singular $SU(2)$-bundles with bounded L^2-curvatures.

3.1 – *Definition and approximation problems.*

In trying to extend the previous section to the situation of non-Abelian $SU(2)$-bundles over B^5 one again meets similar difficulties. In the Abelian case of singular $U(1)$-bundles over B^3 there was no global representation of a connection by a 1-form A such that $dA = F$. Likewise, in the non-Abelian case, the presence of topological-type singularities in the base B^5 for the $SU(2)$-bundles, i.e. $d(tr(F_D \wedge F_D)) \neq 0$, prevents the existence of a global representative $A \in \Omega^1(B^5, su(2))$ for a connection D such that $F_D = dA + A \wedge A$. However, the main difficulty in the non-Abelian case comes from the fact that the adjoint action of the Lie group $SU(2)$ on the 2-forms representing the curvature, given by

$$ad_g(F_D) = g^{-1} F_D\, g\,,$$

is non-trivial.

DEFINITION 3.2 [L^2-curvatures of singular $SU(2)$-bundles]. – *A representative of an L^2-curvature of a singular $SU(2)$-bundle over B^5 is a measurable* 2-*form F with values in $su(2)$ satisfying*

i)

$$\int_{B^5} |F|^2\, dx^5 < +\infty\,. \tag{3.1}$$

ii) *For all $x \in B^5$ and for almost every $0 < r < dist(x, \partial B^5)$ the restriction of F to $\partial B_r(x) \simeq S^4$ coincides – modulo the adjoint action of measurable maps into $SU(2)$ – with the curvature of a $W^{1,2}$-connection on a smooth $SU(2)$-bundle over $\partial B_r(x)$.*

An L^2-curvature of a singular $SU(2)$-bundle over B^5 is an equivalence class $[F]$ in the space of 2-forms $F \in L^2(\wedge^2 (B^5) \otimes su(2))$ satisfying i) and ii) for the equivalence relation given by the adjoint action of measurable maps g in $L^\infty(B^5, SU(2))$. □

The second condition ii) can also be stated as follows: For all $x \in B^5$ and for almost every $0 < r < dist(x, \partial B^5)$ there exists a smooth bundle over $\partial B_r(x)$ and a $W^{1,2}$-connection D on this bundle such that for any smooth local trivialisation of the bundle over some contractible open set $\mathcal{U}$ of $\partial B_r(x)$ there exists a measurable map g from $\mathcal{U}$ into $SU(2)$ such that

$$g^{-1} F\, g = dA + A \wedge A \quad \text{ in } \mathcal{U}\,,$$

where A is the 1-form in $W^{1,2}(\wedge^1 (\mathcal{U}) \otimes su(2))$ representing D in this trivialization.

Observe that our assumption ii) corresponds to condition (2.2) in the definition of L^1-curvatures of $U(1)$-bundles, since it implies that for all $x \in B^5$ and for almost every $0 < r < dist(x, \partial B^5)$ we have

$$\frac{1}{8\pi^2} \int_{\partial B_r(x)} \iota^*_{\partial B_r(x)} tr(F \wedge F) \;\in \mathbb{Z}\,. \tag{3.2}$$

Because of the non-trivial adjoint action of the gauge group on forms representing singular L^2-curvatures, we have to adjust the topology on the space of these curvatures. One possibility would be to consider the topology induced by the metric for *L^2-curvatures of singular $SU(2)$-bundles* defined by

$$d([F_1],[F_2]) := \inf_{g\in L^\infty(B^5,SU(2))}\left[\int_{B^5} |F_1 - g^{-1}F_2\,g|^2\,dx^5\right]^{\frac{1}{2}}. \tag{3.3}$$

An alternative is to construct a distance function based on intrinsic quantities. At a point x the norm of the curvature $|F|^2(x) \in \mathbb{R}^+$ and the *Chern form* $tr(F \wedge F)(x) \in \wedge^4\mathbb{R}^5$ are the most commonly used gauge invariant quantities. However, these two objects do not uniquely characterize $F(x)$ – modulo the adjoint action of $SU(2)$. A more complete gauge invariant object is given by

$$tr(F\otimes F)(x) \in \wedge^2(\mathbb{R}^5)\otimes\wedge^2(\mathbb{R}^5)\,. \tag{3.4}$$

This tensor product does characterize $F(x)$ – modulo the adjoint action of $SU(2)$ – in a unique way. Moreover, it encodes the full information about the curvature at x which is a consequence of the next elementary lemma (see [KR]).

LEMMA 3.1. – *Let F and G be two elements of $L^2(\wedge^2(S^4)\otimes su(2))$ and assume that*

$$\textit{for a.e. } x\in S^4 \qquad tr(F\otimes F)(x) = tr(G\otimes G)(x)\,. \tag{3.5}$$

Then there exists $g \in L^\infty(S^4, SU(2))$ such that

$$\textit{for a.e. } x\in S^4 \qquad g^{-1}(x)\,F(x)\,g(x) = G(x)\,. \tag{3.6}$$

□

From this result we obtain that an L^2-curvature of a weak $SU(2)$-bundle is uniquely determined by the tensor field $tr(F\otimes F)$. Then, instead of the metric topology given by (3.3), we could also consider the metric defined by

$$\delta([F_1],[F_2]) := \int_{B^5} |tr(F_1\otimes F_1) - tr(F_2\otimes F_2)|\,dx^5\,. \tag{3.7}$$

One can show that the two metrics d and δ generate equivalent topologies (see [KR]), yet δ is more explicit and thus more convenient to handle.

Because of the above considerations, an element in $L^1(\wedge^2(\mathbb{R}^5)\otimes\wedge^2(\mathbb{R}^5))$ for a given 2-form F in $L^2(\wedge^2(\mathbb{R}^5)\otimes su(2))$ satisfying condition ii) of Definition 3.2 can also be called an *L^2-curvature of a singular $SU(2)$-bundle over B^5*.

The following question is still open and represents the approximation property in Theorem 2.1 for the non-Abelian case.

Open problem 1. *Let $tr(F \otimes F) \in L^1(\wedge^2(B^5) \otimes \wedge^2(B^5))$ be an L^2-curvature of a singular $SU(2)$-bundle over B^5. Does there exist a sequence of finite families of points, $I^k = \{a_1^k, \ldots, a_{N^k}^k\}$, a sequence of finite families of ± 1, $\mathcal{D}^k = \{d_1^k, \ldots, d_{N^k}^k\}$, and a sequence of* <u>*smooth*</u> *connections D^k over the* <u>*smooth*</u> *$SU(2)$-bundles over $B^5 \setminus I^k$ given by*

$$* \, d[tr(F_{D^k} \wedge F_{D^k})] = 8\pi^2 \sum_{i=1}^{N^k} d_i^k \; \delta_{a_i^k} \qquad in \ \mathcal{D}'(B^5)\,, \tag{3.8}$$

such that

$$tr(F_{D^k} \otimes F_{D^k}) \longrightarrow tr(F \otimes F) \quad \text{strongly in } L^1(B^5)? \tag{3.9}$$

A proof of this open problem should follow steps i) to vi) of our proof of Theorem 2.1. For instance, step iii) is a consequence of a proposition proved in [KR].

PROPOSITION 3.1. – *Let P be a principal $SU(2)$-bundle over a compact 4-dimensional Riemannian manifold M. Let D be a $W^{1,2}$-connection on P. Then there exists a sequence of smooth connections D^k on P such that*

$$D^k \longrightarrow D \quad \text{strongly in } W^{1,2}\big(\Gamma(T^*M \otimes \mathrm{ad}(P))\big)\,. \tag{3.10}$$

□

Let ε_0 be a positive constant smaller than $\varepsilon(4)$ in Theorem 1.1 so that for any connection satisfying the small Yang-Mills energy condition $YM(F_D) < \varepsilon_0$, there exists a unique Coulomb gauge with estimate (1.3) - see a proof of this fact in [KR]. The next question is strongly related to the complete solution of Open problem 1 and it is related to the fact that on the space of curvature of $W^{1,2}$ connections over $S^4 \times SU(2)$ satisfying the small energy assumption $YM(F_D) < \varepsilon_0$ **the topology given by the $W^{1,2}-$distance between the Coulomb gauges is not equivalent to the topology generated by** δ. Understanding the difference between these two topologies is, in itself, an interesting analysis problem that should have interesting consequences. Precisely we raise the following question:

Open problem 2. *Identify those $W^{1,2}$-1-forms A on S^4 with values in $su(2)$ which on the one hand satisfy the small energy condition*

$$\int_{S^4} |F_A|^2 \leq \varepsilon_0\,, \tag{3.11}$$

where $F_A = dA + A \wedge A$, and on the other hand have the property that for any sequence A^k in $W^{1,2}(\wedge^1(S^4) \otimes su(2))$ the convergence

$$tr[F_{A^k} \otimes F_{A^k}] \longrightarrow tr[F_A \otimes F_A] \quad in \ L^1(S^4)\,, \tag{3.12}$$

also implies

$$A^k_{coul} \longrightarrow A_{coul} \qquad in\ W^{1,2}\,, \tag{3.13}$$

where A^k_{coul} and A_{coul} are the Coulomb gauges of the connections given by A^k and A respectively. Does there exist a dense family of such A for the $W^{1,2}$-norm in the space of 1*-forms merely satisfying the small energy condition* (3.11)? □

Next denote by $\mathcal{F}^\infty_{SU(2)}(B^5)$ the space of curvatures of smooth connections with finite Yang-Mills energy on smooth bundles over $B^5 \setminus I$, where I is a discrete subset of B^5. Let η be a smooth 1-form on S^4 with values in $su(2)$. Denote by $\mathcal{F}^\infty_{SU(2),\eta}(B^5)$ the space of curvatures in $\mathcal{F}^\infty_{SU(2)}(B^5)$ whose restriction to ∂B^5 is gauge equivalent to $d\eta + \eta \wedge \eta$. Finally, denote by $\overline{\mathcal{F}}^\infty_{SU(2),\eta}(B^5)$ the closure of $\mathcal{F}^\infty_{SU(2),\eta}(B^5)$ for the topology induced by the metric d or equivalently by δ. It is not difficult to show that an element F in $\overline{\mathcal{F}}^\infty_{SU(2),\eta}(B^5)$ is an *L^2-bounded curvature of a singular $SU(2)$-bundle*. In this setting, it is natural to study the following question:

Open problem 3. *Is the infimum given by*

$$\inf_{F\in\overline{\mathcal{F}}^\infty_{SU(2),\eta}(B^5)} \int_{B^5} |F|^2\, dx^5 \tag{3.14}$$

attained? If so, does the singular set of any minimum consist of isolated points, i.e. are the minima in $\mathcal{F}^\infty_{SU(2),\eta}(B^5)$?

3.2 – *The topological singular set of singular $SU(2)$-bundles with bounded L^2-curvatures.*

The topological singular set of a singular $SU(2)$-bundle over B^5 with a bounded L^2-curvature F is the distribution given by

$$* \, d(tr(F \wedge F)) \in \mathcal{D}'(B^5)\,. \tag{3.15}$$

Though the strong approximation property for L^2-bounded curvatures of singular bundles is still an open problem, we can prove the approximability of the topological singular set of L^2-bounded curvatures by the topological singular set of smooth $SU(2)$-bundles over $B^5 \setminus \{\text{singular points}\}$. Precisely, the following result is proved in [KR].

THEOREM 3.1. – *Let $tr(F \otimes F) \in L^1(\wedge^2 (B^5) \otimes \wedge^2(B^5))$ be an L^2-curvature of a singular $SU(2)$-bundle over B^5. Then there exists a sequence of finite families of points, $I^k = \{a^k_1, \dots, a^k_{N^k}\}$, a sequence of finite families of ± 1, $\mathcal{D}^k = \{d^k_1, \dots, d^k_{N^k}\}$,* *and a sequence of smooth connections D^k over the smooth $SU(2)$-bundles over $B^5 \setminus I^k$ given by*

$$* \, d[tr(F_{D^k} \wedge F_{D^k})] = 8\pi^2 \sum_{i=1}^{N^k} d_i^k \; \delta_{a_i^k} \qquad in \ \mathcal{D}'(B^5)\,, \tag{3.16}$$

such that

$$\limsup_{k \to +\infty} \int_{B^5} |tr(F_{D^k} \otimes F_{D^k})| \, dx^5 \leq \int_{B^5} |tr(F \otimes F)| \, dx^5 \tag{3.17}$$

and

$$tr(F_{D^k} \wedge F_{D^k}) \longrightarrow tr(F \wedge F) \quad \text{in } \mathcal{D}'(\wedge^1 (B^5))\,. \tag{3.18}$$

□

This approximation result allows us to describe the topological singular set of singular $SU(2)$-bundles corresponding to Theorem 2.3 for the Abelian ones.

THEOREM 3.2. – *Let F be an L^2-curvature of a singular $SU(2)$-bundle over B^5. Then there exists a finite mass integer rectifiable current $\mathbb{L}$ in B^5 such that*

$$\frac{1}{8\pi^2} \, \partial \, [tr(F \wedge F)] = \partial \, \mathbb{L} \quad \text{ in } \mathcal{D}'(B^5)\,. \tag{3.19}$$

□

Similarly to the Abelian case and the L^1-energy, for a given L^2-bounded curvature F in $\overline{\mathcal{F}}^{\infty}_{SU(2),\eta}(B^5)$, the minimal mass among all 1-dimensional integer rectifiable currents $\mathbb{L}$ satisfying (3.19) is given by

$$L(F) = \frac{1}{8\pi^2} \sup \left\{ \begin{array}{c} \displaystyle\int_{B^5} d\xi \wedge tr(F \wedge F) - \int_{\partial B^5} \xi \; tr(F_\eta \wedge F_\eta) \\ \text{s.t.} \quad \xi \in Lip(\overline{B^3}) \; , \; \|d\xi\|_\infty \leq 1 \end{array} \right\} . \tag{3.20}$$

Denote by $W^{1,2}_\eta(\wedge^1 (B^5) \otimes su(2))$ the space of $W^{1,2}$-1-forms in $\overline{B^5}$ with values in $su(2)$ whose restriction to ∂B^5 is equal to the boundary data η.

THEOREM 3.3. – *In the above setting we have*

$$\inf_{A \in W^{1,2}_\eta(\wedge^1 B^5 \otimes su(2))} \int_{B^5} |dA + A \wedge A|^2 \, dx^5 = \inf_{F \in \overline{\mathcal{F}}^{\infty}_{SU(2),\eta}} \int_{B^5} |F|^2 \, dx^5 + 8\pi^2 \, L(F)\,. \tag{3.21}$$

□

Note that in this result the condition $A \in W^{1,2}_\eta(\wedge^1 (B^5) \otimes su(2))$ can of course be replaced by $A \in C^\infty_\eta(\wedge^1 (B^5) \otimes su(2))$.

The relaxed energy defined by

$$Z(F) = \int_{B^5} |F|^2 \, dx^5 + 8\pi^2 \, L(F)\,, \tag{3.22}$$

was already considered by T. Isobe in [Is1] and [Is2] for connections on smooth $SU(2)$-bundles over $B^5 \setminus I$, where I is a discrete subset of B^5.

Open problem 4. *Is the infimum given by*

$$\inf_{F \in \overline{\mathcal{F}}^{\infty}_{SU(2),\eta}} \int_{B^5} |F|^2 \, dx^5 + 8\pi^2 \, L(F) \tag{3.23}$$

attained?

More generally, one can ask about the existence of minima in $\overline{\mathcal{F}}^{\infty}_{SU(2),\eta}$ of functionals of the form

$$Z_G(F) = \int_{B^5} |F|^2 \, dx^5 + 8\pi^2 \, L(F,G)\,, \tag{3.24}$$

where G is a fixed arbitrary element in $\overline{\mathcal{F}}^{\infty}_{SU(2),\eta}$ and where we use the notation

$$L(F,G) = \frac{1}{8\pi^2} \sup \left\{ \begin{array}{c} \displaystyle\int_{B^5} d\xi \wedge tr(F \wedge F) - \int_{B^5} d\xi \wedge tr(G \wedge G) \\ \text{s.t.} \quad \xi \in Lip(\overline{B^3})\ ,\ \|d\xi\|_\infty \leq 1 \end{array} \right\} . \tag{3.25}$$

A positive answer to Open problem 4 and the generalization following it, would open the door to the possibility of constructing **everywhere discontinuous Yang-Mills fields** on B^5 with a dense topological singular set in B^5, i.e. $\text{supp}(d(tr(F \wedge F))) = \overline{B^5}$, as it was done by the second author in [Ri1] for harmonic maps.

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Thiemo Kessel: Departement Mathematik
HG G 28.2, Rämistrasse, 101-8092 Zürich, Switzerland
e-mail: thiemo.kessel@math.ethz.ch

Tristan Riviere: Department of Mathematics
ETH Zentrum, CH-8093 Zürich, Switzerland
e-mail: riviere@math.ethz.ch

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