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The Martingale Problem in Hilbert Spaces

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Dedicated to the memory of Guido Stampacchia

Abstract. – *We consider an SPDE in a Hilbert space H of the form $dX(t) = (AX(t) + b(X(t)))dt + \sigma(X(t))dW(t)$, $X(0) = x \in H$ and the corresponding transition semigroup $P_t f(x) = \mathbb{E}[f(X(t,x))]$. We define the infinitesimal generator $\bar{L}$ of P_t through the Laplace transform of P_t as in* [1]. *Then we consider the differential operator $L\varphi = \frac{1}{2}\mathrm{Tr}\,[\sigma(x)\sigma^*(x)D^2\varphi] + \langle b(x), D\varphi\rangle$ defined on a suitable set V of regular functions. Our main result is that if V is a core for $\bar{L}$, then there exists a unique solution of the martingale problem defined in terms of L. Application to the Ornstein-Uhlenbeck equation and to some regular perturbation of it are given.*

1. – Introduction.

Let us first recall the classical martingale problem. Consider a diffusion (or Kolmogorov) operator in $\mathbb{R}^d$,

$$L\varphi = \frac{1}{2}\ \mathrm{Tr}\ [\sigma(x)\sigma^*(x)D^2\varphi] + \langle b(x), D\varphi\rangle, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to L(\mathbb{R}^r, \mathbb{R}^d)$ are continuous and bounded. Here d and r are two fixed positive integers.

For any separable Hilbert space H we denote by Ω the space of all continuous functions from $[0, +\infty)$ to H endowed with the distance

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{\|\omega_1 - \omega_2\|_{C([0,k];H)}}{2^k(1 + \|\omega_1 - \omega_2\|_{C([0,k];H)})}\ , \quad \omega_1, \omega_2 \in \Omega.$$

The space (Ω, d) is a complete metric space. We denote by $\mathcal{F}$ the σ-algebra of all Borel subsets of Ω and, for any $t \geq 0$, by η_t the *evaluation function*, that is the random variable in $(\Omega, \mathcal{F})$ defined by

$$\eta_t(\omega) = \omega(t), \quad \omega \in \Omega,\ t \geq 0.$$

Finally, for any $t \geq 0$, $\mathcal{F}_t$ will represent the smallest σ-algebra in $\mathcal{F}$ such that all functions η_s with $s \leq t$ are measurable.

We take now $H = \mathbb{R}^d$ and give the following

DEFINITION 1.1. – *A probability measure μ on $(\Omega, \mathcal{F})$ solves the martingale problem with initial point $x \in \mathbb{R}^d$ if*

(i) $\mu(\Omega^x) = 1$, *where* $\Omega^x = \{\omega \in \Omega : \omega(0) = x\}$.

(ii) *For any $\varphi \in C_0^\infty(\mathbb{R}^d)$ (called the space of test functions) the family of random variables $\{M_t(\varphi)\}_{t\geq 0}$, defined by*

$$M_t(\varphi) = \varphi(\eta_t) - \int_0^t L\varphi(\eta_s)\,ds, \tag{1.2}$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

There is a close relationship between the martingale problem and the stochastic differential equation,

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t), & t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{1.3}$$

where $W(t)$ is an r-dimensional standard Wiener process.

Assume in fact that equation (1.3) has a unique strong solution for all $x \in \mathbb{R}^d$ (this holds in particular when the coefficients b and σ are Lipschitz continuous). Then the martingale problem starting from x has a unique solution given by the law of $X(\cdot, x)$ in Ω, see [16].

When b and σ are merely continuous it is not difficult to show existence of a solution of the martingale problem, whereas the uniqueness of such a solution holds in general only when L is nondegenerate, see [16].

The aim of this paper is to extend the above classical results to a Kolmogorov operator in a separable Hilbert space H of the form,

$$L\varphi = \frac{1}{2}\,\mathrm{Tr}\,[\sigma(x)\sigma^*(x)D^2\varphi] + \langle Ax + b(x), D\varphi\rangle, \quad x \in D(A) \cap D(b). \tag{1.4}$$

Here $A{:}\,D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H, $b{:}\,D(b) \subset H \to H$ and $\sigma : H \to L(H)$ are nonlinear mappings. "Tr" means the trace.

The first problem which arises now is how to choose a "natural" space of test functions. Notice that in order that the operator (1.4) acts in a space of continuous functions, one has to require that

(i) φ is of class C^2,

(ii) The function

$$D(A) \cap D(b) \to \mathbb{R},\ x \to \langle Ax + b(x), D\varphi\rangle,$$

has a unique extension to a continuous function on H,

(iii) $\sigma(x)\sigma^*(x)D^2\varphi(x)$ is of trace class for any $x \in H$ and the function $\mathrm{Tr}\,[\sigma\sigma^* D^2\varphi]$ is continuous.

We also note that a general theory for elliptic operators with infinitely many variables as the operator (1.4) is not available so far; for some results in this direction, see [5].

There are in the literature several generalization of the martingale problem to infinite dimensions, using different settings. We mention only a few: Yor [18], Viot [17], Metivier [13], Mikulevicius-Rozovskii [14], Flandoli [9], Zambotti [20], see also the monograph [4] and references therein.

Our approach will be the following. First, we choose a suitable space of test function V, for instance of cylindrical functions (related to the particular problem under examination) such that the expression of L is meaningful for $\varphi \in V$; then we give the following definition, analogous to Definition 1.1,

DEFINITION 1.2. – *A probability measure μ on $(\Omega, \mathcal{F})$ solves the martingale problem with initial point $x \in H$ with respect to (L, V) if*

(i) $\mu(\Omega^x) = 1$, *where* $\Omega^x = \{\omega \in \Omega : \omega(0) = x\}$.

(ii) *For any $\varphi \in V$ the family of random variables $\{M_t(\varphi)\}_{t \geq 0}$, defined by*

$$M_t(\varphi) = \varphi(\eta_t) - \int_0^t L\varphi(\eta_s)\,ds, \tag{1.5}$$

is a martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

In similar way as in the finite dimensional case there is a natural stochastic differential equation in the Hilbert space H related to the operator (1.4), namely

$$\begin{cases} dX(t) = (AX(t) + b(X(t)))dt + \sigma(X(t))dW(t), \quad t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{1.6}$$

where $W(t)$ is a cylindrical Wiener process in H. We notice that several equations in the applications have the form (1.6).

In this paper we shall assume that problem (1.6) has a unique mild or generalized solution $X(\cdot, x)$ for any $x \in H$, see Definition 2.2 below. In this case we say that problem (1.6) is well posed.

We notice, however, that this solution is *almost never* strong and so, we cannot directly generalize the finite dimensional proof and conclude that the law of $X(\cdot, x)$ is the solution of the martingale problem starting from x.

So, we shall proceed as follows,

(i) We assume that problem (1.6) is well posed.

(ii) We consider the transition semigroup (which we assume to be Feller),

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(H),\ x \in H, \tag{1.7}$$

and its infinitesimal generator $(\bar{L}, D(\bar{L}))$, defined through the Laplace transform of P_t. We notice that P_t is not strongly continuous in general in $C_b(H)$ ([1]) but it enjoys the so-called *pointwise bounded convergence* introduced in [7] and considered by several authors, see Section 2 below.

(iii) We show that $(\bar{L}, D(\bar{L}))$ is an extension of (L, V) and that V is a core for $(\bar{L}, D(\bar{L}))$ with respect to the pointwise bounded convergence.

(iv) We prove existence and uniqueness of the martingale problem.

It would be interesting to consider the case when problem (1.6) is not well posed, for instance when a weak solution exists (in particular a Markov selection). We shall consider this problem later.

In Section 2 we shall study basic properties of the transition semigroup P_t. Section 3 is devoted to prove existence and uniqueness of the martingale problem and Section 4 to some applications. We notice that in the applications the construction of a core V satisfying Hypothesis 2.3 is the difficult part of the job. In this note we limit ourselves to consider the Ornstein-Uhlenbeck equation and some bounded perturbation of this equation.

2. – The transition semigroup P_t.

We start with the definition of π-semigroup following [15]. For this we need the notion of π-convergence see [15] (see also [7], [8] where it is called *bp-convergence*, [10] and [11]).

A sequence $(\varphi_n) \subset C_b(H)$ is said to be *π-convergent* to a function $\varphi \in C_b(H)$ (we shall write $\varphi_n \xrightarrow{\pi} \varphi$) if for any $x \in H$ we have $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ and if $\sup_{n\in\mathbb{N}} \|\varphi_n\|_0 < +\infty$.

A subset Λ of $C_b(H)$ is said to be *π-dense* in H if for any $\varphi \in C_b(H)$ there exists a sequence $(\varphi_n) \subset \Lambda$ such that $\varphi_n \xrightarrow{\pi} \varphi$.

DEFINITION 2.1. – *A semigroup P_t of linear bounded operators on $C_b(H)$ is called a π-semigroup (of contractions) if*

(i) *For any $\varphi \in C_b(H)$ and any $t \geq 0$ we have,*

$$\|P_t\varphi\|_0 \leq \|\varphi\|_0 \,.$$

(ii) *If $\varphi_n \xrightarrow{\pi} \varphi$ then $P_t\varphi_n \xrightarrow{\pi} P_t\varphi, \quad \forall\, t \geq 0.$*

(iii) *For all $\varphi \in C_b(H)$ and for all $x \in H$ the function $[0, +\infty) \to \mathbb{R}$, $t \to P_t\varphi(x)$ is continuous.*

We now define a concept of solution for problem (1.6).

([1]) By $C_b(H)$ we denotes the Banach space of all real uniformly continuous and bounded functions in H endowed with the norm $\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|, \ \varphi \in C_b(H)$.

DEFINITION 2.2. –

(i) *Let* $x \in D(b)$. *A mild solution of problem* (1.6) *is an adapted mean square continuous process* $X(\cdot, x)$ *in* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that for all* $t \geq 0$, $\mathbb{P}$-*a.s. we have* $X(t,x) \in D(b)$ *and*

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s,x))\, ds + \int_0^t e^{(t-s)A} \sigma(X(s,x)) dW(s).$$

(ii) *Let* $x \in H$. *A generalized solution of problem* (1.6) *is an adapted mean square continuous process* $X(t,x)$ *in* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that there exists a sequence* $(x_n) \subset D(b)$ *convergent to* x *and*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E}|X(t,x) - X(t,x_n)|^2 = 0,$$

for all $T > 0$.

In the following we shall assume that,

HYPOTHESIS 2.3. – *For any* $x \in H$ *there exists a unique generalized solution* $X(\cdot, x)$ *of problem* (1.6) *such that*

(i) *For all* $x \in H$, $X(\cdot, x)$ *is a continuous homogeneous Markov process.*

(iii) *There exists* $a \in \mathbb{R}$ *such that*

$$\mathbb{E}(|X(t,x) - X(t,y)|^2) \leq e^{2at}|x-y|^2, \quad \forall x, y \in H,\ t \geq 0. \tag{2.1}$$

Hypothesis 2.3 is fulfilled by several stochastic PDEs. We shall give some application in Section 4.

We can now consider the transition semigroup P_t defined by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H),\ x \in H, \tag{2.2}$$

where $B_b(H)$ is the space of all bounded Borel functions from H into $\mathbb{R}$.

We recall that the semigroup P_t is called *Feller* if $P_t\varphi \in C_b(H)$ for all $\varphi \in C_b(H)$ and all $t \geq 0$.

PROPOSITION 2.4. – P_t *is Feller and it is a* π-*semigroup.*

PROOF. – Since $C_b^1(H)$ is dense in $C_b(H)$ (see e.g. [5]),(2) to prove that P_t is Feller it is enough to show that P_t maps $C_b^1(H)$ into $C_b(H)$. Assume that $\varphi \in C_b^1(H)$.

(2) We denote by $C_b^k(H)$, $k \in \mathbb{N}$, the set of all elements of $C_b(H)$ whose derivatives of order lesser than k are uniformly continuous and bounded.

Then for any $x, y \in H$ we have, taking into account (2.1),

$$\begin{aligned} |P_t\varphi(x) - P_t\varphi(y)| &\leq \|D\varphi\|_0 \mathbb{E}|X(t,x) - X(t,y)| \\ &\leq \|D\varphi\|_0 \mathbb{E}(|X(t,x) - X(t,y)|^2)^{1/2} \\ &\leq e^{at}|D\varphi|_0\|x-y\|. \end{aligned} \tag{2.3}$$

So, $P_t\varphi \in C_b(H)$.

It remains to show that P_t is a π-semigroup. Condition (i) of Definition 2.1 is obviously fulfilled, conditions (ii) follows from the continuity of $X(\cdot, x)$ whereas conditions (iii) follows from the dominated convergence theorem. □

Let us now define the *infinitesimal generator* of P_t following [1]. For this we introduce the family of bounded operators in $C_b(H)$,

$$F_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt, \quad f \in C_b(H),\ x \in H,\ \lambda > 0.$$

PROPOSITION 2.5. – *For any $f \in C_b(H)$ and any $\lambda > 0$ we have $F_\lambda f \in C_b(H)$ and*

$$\|F_\lambda f\|_0 \leq \frac{1}{\lambda}\ \|f\|_0. \tag{2.4}$$

Moreover there exists a unique closed operator $\bar{L}$: $D(\bar{L}) \subset C_b(H) \to C_b(H)$ such that for any $\lambda > 0$ and any $f \in C_b(H)$ we have $F_\lambda f = (\lambda - \bar{L})^{-1} f$. Finally, if $f_n \xrightarrow{\pi} f$ then $F_\lambda f_n \xrightarrow{\pi} F_\lambda f$.

$(\bar{L}, D(\bar{L}))$ is called the infinitesimal generator of P_t.

PROOF. – Let first take f in $C_b^1(H)$. Then for all $x, y \in H$ we have, taking into account (2.3),

$$|F_\lambda f(x) - F_\lambda f(y)| \leq \frac{1}{\lambda - a}\ \|Df\|_0 |x-y|.$$

This implies that $F_\lambda f \in C_b(H)$, for $\lambda > a$.

Since $C_b^1(H)$ is dense in $C_b(H)$, we can conclude by a straightforward argument that $F_\lambda f \in C_b(H)$ for all $f \in C_b(H)$ and all $\lambda > a$. Let us show that the same conclusion holds for all $\lambda > 0$. First notice that, by a direct computation, F_λ fulfills the resolvent identity

$$F_\lambda - F_\mu = (\mu - \lambda) F_\lambda F_\mu, \quad \lambda, \mu > 0.$$

Consequently, we can write

$$F_\lambda = F_\mu [1 - (\mu - \lambda) F_\mu]^{-1},$$

provided λ and μ are sufficiently close. This implies that $F_\lambda f \in C_b(H)$ for all $\lambda > 0$ by an iterative argument. Moreover, (2.4) obviously holds.

Since for every $f \in C_b(H)$

$$\lim_{\lambda\to\infty} \lambda F_\lambda f(x) = \lim_{\lambda\to\infty} \int_0^{+\infty} e^{-\tau} P_{\frac{\tau}{\lambda}} f(x) d\tau = f(x), \quad x \in H,$$

F_λ is one-to-one. So, by a classical result, see e. g. [19], there exists a unique closed operator $\bar{L}: D(\bar{L}) \subset C_b(H) \to C_b(H)$ such that for any $\lambda > 0$ and any $f \in C_b(H)$ we have $F_\lambda f = (\lambda - \bar{L})^{-1} f$. The last statement easily follows from the fact that P_t is a π-semigroup and from the dominated convergence theorem. □

REMARK 2.6. – When P_t is not strongly continuous the domain $D(\bar{L})$ of its infinitesimal generator $\bar{L}$, is not dense in $C_b(H)$. In this case P_t is an Hille–Yosida operator in the sense of [2].

We shall prove now, following [15], a useful characterization of $D(\bar{L})$.

PROPOSITION 2.7. – *Let $\varphi \in D(\bar{L})$. Then we have*

$$\lim_{h\to 0^+} \frac{P_h\varphi(x) - \varphi(x)}{h} = \bar{L}\varphi(x), \quad \textit{for all } x \in H \tag{2.5}$$

and

$$\sup_{h\in(0,1]} \left\| \frac{P_h\varphi - \varphi}{h} \right\|_0 < +\infty \tag{2.6}$$

Conversely, if there exists $\varphi, g \in C_b(H)$ such that

$$\lim_{h\to 0^+} \frac{P_h\varphi(x) - \varphi(x)}{h} = g(x), \quad \textit{for all } x \in H \tag{2.7}$$

and (2.6) *holds, then we have $\varphi \in D(\bar{L})$ and $\bar{L}\varphi = g$.*

PROOF. – Assume that $\varphi \in D(\bar{L})$. Fix $\lambda > 0$ and set $f = \lambda\varphi - \bar{L}\varphi$. Then $F_\lambda f = \varphi$ and for any $h > 0$ and any $x \in H$ we have

$$P_h\varphi(x) = P_h F_\lambda f(x) = e^{\lambda h} \int_h^{+\infty} e^{-\lambda s} P_s f(x) ds.$$

It follows that

$$\begin{aligned} D_h P_h\varphi(x)|_{h=0} &= \lambda \int_0^{+\infty} e^{-\lambda s} P_s f(x) ds - f(x) \\ &= \lambda F_\lambda f(x) - f(x) = \bar{L} F_\lambda f(x) = \bar{L}\varphi(x), \end{aligned}$$

and so (2.5) follows. Moreover, since

$$|P_h\varphi(x)-\varphi(x)| \le (e^{\lambda h}-1)\left|\int_h^{+\infty} e^{-\lambda s}P_sf(x)\,ds\right|$$
$$+\left|\int_0^h e^{-\lambda s}P_sf(x)\,ds\right| \le \|f\|_0\left[\frac{e^{\lambda h}-1}{\lambda}\,e^{-\lambda h}+\frac{1-e^{-\lambda h}}{\lambda}\right]\le ch,$$

where c is a suitable positive constant, we see that (2.6) follows as well.

Assume now that there exists $\varphi, g \in C_b(H)$ such that (2.7) is fulfilled. Since clearly for any $x\in H$,

$$\frac{d}{dt}\,P_t\varphi(x)=\lim_{h\to 0^+}\frac{P_{t+h}\varphi(x)-P_t\varphi(x)}{h}=P_tg(x),$$

we have

$$\begin{aligned} F_\lambda\varphi(x) &= -\frac{1}{\lambda}\int_0^t P_t\varphi(x)de^{-\lambda t}\\ &=\frac{1}{\lambda}\,\varphi(x)+\frac{1}{\lambda}\int_0^t e^{-\lambda t}P_tg(x)dt\\ &=\frac{1}{\lambda}\,\varphi(x)+\frac{1}{\lambda}\,F_\lambda g(x).\end{aligned}$$

Therefore

$$F_\lambda\varphi(x)=\frac{1}{\lambda}\,\varphi(x)+\frac{1}{\lambda}\,F_\lambda g(x),$$

which implies $\varphi\in D(\bar L)$ and $\bar L\varphi = g$. □

PROPOSITION 2.8. – *Let $\varphi\in D(\bar L)$ and $t\ge 0$. Then $P_t\varphi\in D(\bar L)$ and*

$$\bar L P_t\varphi(x)=P_t\bar L\varphi(x)=\frac{d}{dt}P_t\varphi(x). \tag{2.8}$$

PROOF. – Let us consider the following identities

$$\frac{P_{t+h}\varphi(x)-P_t\varphi(x)}{h}=P_t\frac{P_h\varphi(x)-\varphi(x)}{h}=\frac{P_hP_t\varphi(x)-P_t\varphi(x)}{h}.$$

From the second identity, taking in account that P_t is π-semigroup, we get

$$\frac{d}{dt}\,P_t\varphi(x)=P_t\bar L\varphi(x);$$

then, from the third identity, we conclude that $P_t\varphi(x) \in D(\bar{L})$ and that

$$\frac{d}{dt}\,P_t\varphi(x) = P_t\bar{L}\varphi(x) = \bar{L}P_t\varphi(x), \quad x \in H.$$

□

3. – Existence and uniqueness of the martingale problem.

In this section we shall assume, besides Hypothesis 2.3, the following,

HYPOTHESIS 3.1. – *There exists a linear subspace V of $C_b^2(H)$ such that for all $\varphi \in V$ we have,*

(i) *For any $x \in H$ the linear operator $\sigma(x)\sigma(x)^* D^2\varphi(x)$ is of trace class and the mapping $Tr\,[\sigma\sigma^* D^2\varphi]$ belongs to $C_b(H)$.*

(ii) *The mapping*

$$D(A) \cap D(b) \to H,\ x \to \langle Ax + b(x), D\varphi(x)\rangle,$$

has a unique extension to a function of $C_b(H)$ which we still denote by

$$x \to \langle Ax + b(x), D\varphi(x)\rangle.$$

(iii) *$V \subset D(\bar{L})$ and for any $\varphi \in D(L)$ we have $\bar{L}\varphi = L\varphi$.*

(iv) *For any $\varphi \in D(\bar{L})$ there exists a sequence $(\varphi_n) \subset V$ such that*

$$\varphi_n \xrightarrow{\pi} \varphi, \quad L\varphi_n \xrightarrow{\pi} \bar{L}\varphi.$$

We call V a core of $(\bar{L}, D(\bar{L}))$.

THEOREM 3.2. – *Assume that Hypotheses* 2.3 *and* 3.1 *hold. Then for any $x \in H$ the law of the generalized solution $X(\cdot, x)$ of* (1.6) *is the unique solution of the martingale problem with initial point x.*

PROOF. – For any $x \in H$ we denote by $\mathbb{P}_x$ the law of $X(\cdot, x)$ and by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$ so that

$$P_t\varphi(x) = \mathbb{E}_x[\varphi(X(t, x))], \quad \varphi \in B_b(H),\ t \geq 0.$$

Let us prove existence. Let $x \in H$. We check that for any $\varphi \in V$,

$$M_t(\varphi) := \varphi(\eta_t) - \int_0^t L\varphi(\eta_s)ds, \quad t \geq 0, \tag{3.1}$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $t > r$ and write,

$$\begin{aligned}\mathbb{E}_x[M_t(\varphi)|\mathcal{F}_r] &= \mathbb{E}_x[\varphi(\eta_t)|\mathcal{F}_r] - \int_0^t \mathbb{E}_x[L\varphi(\eta_s)|\mathcal{F}_r]ds \\ &= \mathbb{E}_x[\varphi(\eta_t)|\mathcal{F}_r] - \int_0^r \mathbb{E}_x[L\varphi(\eta_s)|\mathcal{F}_r]ds - \int_r^t \mathbb{E}_x[L\varphi(\eta_s)|\mathcal{F}_r]ds.\end{aligned} \tag{3.2}$$

Clearly $\mathbb{E}_x[L\varphi(\eta_s)|\mathcal{F}_r] = L\varphi(\eta_s)$ for $s \leq r$. Moreover, by the Markov property of P_t we have,

$$\mathbb{E}_x[\varphi(\eta_t)|\mathcal{F}_r] = P_{t-r}\varphi(\eta_r), \quad t > r$$

and

$$\mathbb{E}_x[L\varphi(\eta_s)|\mathcal{F}_r] = (P_{s-r}L\varphi)(\eta_r), \quad s > r.$$

Therefore by (3.2) it follows that

$$\mathbb{E}_x[M_t(\varphi)|\mathcal{F}_r] = P_{t-r}\varphi(\eta_r) - \int_0^r L\varphi(\eta_s)ds - \int_r^t (P_{s-r}L\varphi)(\eta_r)ds. \tag{3.3}$$

Now by Hypothesis 3.1-(iii) we have $\varphi \in D(\bar{L})$ and $\bar{L}\varphi = L\varphi$. Consequently by Proposition 2.8 we have

$$P_{s-r}L\varphi = P_{s-r}\bar{L}\varphi(\eta_r) = \frac{d}{ds}\, P_{s-r}\varphi(\eta_r).$$

Substituting in (3.3) yields,

$$\begin{aligned}\mathbb{E}_x[M_t(\varphi)|\mathcal{F}_r] &= P_{t-r}\varphi(\eta_r) - \int_0^r L\varphi(\eta_s)ds - \int_r^t \frac{d}{ds}\, P_{s-r}\varphi(\eta_r)ds \\ &= P_{t-r}\varphi(\eta_r) - \int_0^r L\varphi(\eta_s)ds - P_{t-r}\varphi(\eta_r) + \varphi(\eta_r) \\ &= \varphi(\eta_r) - \int_0^r L\varphi(\eta_s)ds = M_r(\varphi).\end{aligned} \tag{3.4}$$

This prove that $M_t(\varphi)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

We show now the uniqueness. Let $x \in H$ and let μ be a probability measure on $(\Omega, \mathcal{F})$ which solves the martingale problem which initial point x. We have to prove that $\mu = \mathbb{P}^x$. We start by proving that the one-dimensional distributions of μ and $\mathbb{P}^x$ coincide, that is

$$\mathbb{E}_\mu[f(\eta_t)] = \mathbb{E}_x[f(\eta_t)] = P_t f(x), \quad \forall f \in C_b(H) \tag{3.5}$$

To prove (3.5) we fix $f \in C_b(H)$, $\lambda > 0$ and set $\varphi = (\lambda - \bar{L})^{-1}f$. Since V is a core for

$\bar{L}$, there exists a sequence $(\varphi_n) \subset V$ such that

$$\varphi_n \xrightarrow{\pi} \varphi, \quad L\varphi_n \xrightarrow{\pi} \bar{L}\varphi.$$

We have by assumption

$$\mathbb{E}_\mu \left[\varphi_n(\eta_t) - \int_0^t L\varphi_n(\eta_s)\,ds | \mathcal{F}_r \right] = \varphi_n(\eta_r) - \int_0^r L\varphi_n(\eta_s)\,ds,$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we obtain by the dominated convergence theorem,

$$\mathbb{E}_\mu \left[\varphi(\eta_t) - \int_0^t \bar{L}\varphi(\eta_s)\,ds | \mathcal{F}_r \right] = \varphi(\eta_r) - \int_0^r \bar{L}\varphi(\eta_s)\,ds,$$

which is equivalent to

$$\mathbb{E}_\mu \left[\varphi(\eta_t) - \int_r^t \bar{L}\varphi(\eta_s)\,ds | \mathcal{F}_r \right] = \varphi(\eta_r). \tag{3.6}$$

Next we multiply both sides of (3.6) by $\lambda e^{-\lambda t}$ and integrate over $[r, +\infty)$. Since

$$\int_r^\infty \lambda e^{-\lambda t}\,dt \int_r^t \bar{L}\varphi(\eta_s)\,ds = \int_r^\infty \bar{L}\varphi(\eta_s)\,ds \int_s^\infty \lambda e^{-\lambda t} dt = \int_r^\infty e^{-\lambda s}\bar{L}\varphi(\eta_s)\,ds,$$

we obtain that

$$\mathbb{E}_\mu \left[\int_r^\infty \lambda e^{-\lambda t}\varphi(\eta_t)\,dt - \int_r^\infty e^{-\lambda t}\bar{L}\varphi(\eta_t)dt | \mathcal{F}_r \right] = e^{-\lambda r}\varphi(\eta_r),$$

which is equivalent to

$$\mathbb{E}_\mu \left[\int_r^\infty e^{-\lambda t} f(\eta_t)dt | \mathcal{F}_r \right] = e^{-\lambda r}\varphi(\eta_r).$$

Setting $r = 0$ yields

$$\mathbb{E}_\mu \left[\int_0^\infty e^{-\lambda t} f(\eta_t)dt \right] = \varphi(x),$$

which coincides with (3.5).

Iterating this procedure one can show, by a classical argument (see e.g. [12]), that $\mathbb{E}_\mu(\varphi)$ coincides with $\mathbb{P}^x(\varphi)$ for any cylindrical function, so that the conclusion follows. □

4. – Application to the Ornstein-Uhlenbeck equation.

Let us consider the stochastic differential equation,

$$\begin{cases} dX(t) = AX(t)\,dt + BdW(t), \quad t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{4.1}$$

under the following assumption.

HYPOTHESIS 4.1. –

(i) *A is the infinitesimal generator of a C_0-semigroup e^{tA} on H,*
(ii) *$B \in L(H)$,*
(iii) *$Tr\, Q_t < \infty$ where $Q_t = \int_0^t e^{sA} BB^* e^{sA^*}\, ds$,*

where A^* and B^* are the adjoint operators of A and B respectively.

It is well known that problem (4.1) has a unique mild solution $X(t, x)$ (see e.g. [4]) given by

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0, \ x \in H$$

and that Hypothesis (2.3) is fulfilled. Moreover, the law of $X(t, x)$ is given by $N_{e^{tA}x, Q_t}$ (the Gaussian measure in H with mean $e^{tA}x$ and covariance operator Q_t), so that the corresponding transition semigroup P_t is given by

$$P_t\varphi(x) = \int_H \varphi(y) N_{e^{tA}x, Q_t}(dy) = \int_H \varphi(e^{tA}x + y) N_{0, Q_t}(dy). \tag{4.2}$$

In this case, the Kolmogorov operator looks like,

$$L\varphi = \frac{1}{2}\ \mathrm{Tr}\, [BB^* D^2\varphi] + \langle Ax, D\varphi\rangle.$$

Let us consider the following vector space of complex functions in H,

$$Z = \text{linear span} \left\{ \int_0^a e^{i\langle e^{sA}x, h\rangle} ds : \ a > 0, \ h \in D(A^*) \right\}.$$

We shall denote by V the subspace of $C_b(H)$ consisting of all the real and imaginary parts of functions belonging to Z.

PROPOSITION 4.2. – *V fulfills Hypothesis* 3.1. *Moreover, the martingale problem for (L, V) has a unique solution.*

PROOF. – Let us check Hypothesis 3.1(i). Let

$$\varphi(x) = \int_0^a e^{i\langle e^{sA}x,h\rangle} ds, \quad x \in H,$$

where $a > 0$ and $h \in D(A^*)$. Then we have

$$D\varphi(x) = i\int_0^a e^{i\langle e^{sA}x,h\rangle}\ e^{sA^*}h\ ds, \quad x \in H$$

and

$$D^2\varphi(x) = i\int_0^a e^{i\langle e^{sA}x,h\rangle}\ (e^{sA^*}h) \otimes (e^{sA^*}h)\ ds, \quad x \in H.$$

Consequently the operator $D^2\varphi(x)$ is of trace class and (i) is proved. To check Hypothesis 3.1(ii) it is enough to notice that the function $x \to \langle Ax, D\varphi(x)\rangle$ can be written as

$$x \to \langle x, A^*D\varphi(x)\rangle = -i\int_0^a e^{i\langle e^{sA}x,h\rangle}\langle x, A^*e^{sA^*}h\rangle\ ds,$$

which clearly belongs to $C_b(H)$. Let us check Hypothesis 3.1(iii). Let again

$$\varphi(x) = \int_0^a e^{i\langle e^{sA}x,h\rangle} ds, \quad x \in H,$$

for some $a > 0$ and $h \in D(A^*)$, then, by a straightforward computation we see that,

$$P_t\varphi(x) = \int_0^a e^{-\frac{1}{2}\langle Q_t e^{s^*A}h, e^{s^*A}h\rangle} e^{i\langle e^{(t+s)A}x,h\rangle} ds. \tag{4.3}$$

It follows that

$$\lim_{t\to 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t} = L\varphi(x), \qquad \text{for all } x \in H$$

and

$$\sup_{t\in(0,1]} \left\| \frac{P_t\varphi - \varphi}{t} \right\|_0 < +\infty$$

By Proposition 2.7 this implies that $\varphi \in D(\bar{L})$ and $\bar{L}\varphi = L\varphi$.

It remains to prove Hypothesis 3.1(iv). For this we notice that, in view of (4.3) P_t maps V into itself. So, one can repeat the classical proof on [6] and see that V is a core in the sense of Hypothesis 3.1(iv).

Now the last statement follows from Theorem 3.2. □

4.1 – *Perturbation of the Ornstein-Uhlenbeck equation.*

We are here concerned with a perturbation of problem (4.1),

$$\begin{cases} dX(t) = (AX(t) + b(X(t))dt + dW(t), \quad t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{4.4}$$

where b is continuous and we have taken $B = I$ for simplicity. More precisely, we shall assume that,

HYPOTHESIS 4.3. –

(i) *A and $B = I$ fulfill Hypothesis* 4.1,
(ii) $b \in C_b(H; H)$.

It is well known that, under Hypotheses 4.1 and 4.3, problem (4.4) has a unique mild solution for any $x \in H$ and that Hypothesis 2.3 is fulfilled.

Moreover, the transition semigroup P_t is strong Feller and its infinitesimal generator $(\bar{L}, D(\bar{L}))$ enjoys the properties,

$$D(\bar{L}) \subset C_b^1(H) \tag{4.5}$$

and

$$\bar{L}\varphi = \bar{L}_0 + \langle b(x), D\varphi \rangle, \quad \varphi \in D(\bar{L}), \tag{4.6}$$

where $\bar{L}_0$ is the generator of the Ornstein-Uhlenbeck semigroup introduced before, see [5, Corollary 6.4.3].

In this case the Kolmogorov operator reads as follows

$$L\varphi = \frac{1}{2}\,\text{Tr}\,[BB^* D^2\varphi] + \langle Ax + b(x), D\varphi \rangle := L_0\varphi + \langle b(x), D\varphi \rangle,$$

where we have denoted by L_0 the Kolmogorov operator related to the Ornstein-Uhlenbeck equation.

We still consider the space V defined before.

PROPOSITION 4.4. – *V fulfills Hypothesis* 3.1. *Moreover, the martingale problem for* (L, V) *has a unique solution.*

PROOF. – Hypotheses 3.1(i) and 3.1(ii) are obviously fulfilled. Let us check Hypothesis 3.1(iii). Write

$$\begin{aligned} X(t,x) &= e^{tA}x + \int_0^t e^{(t-s)A} b(X(s,x))\,ds + \int_0^t e^{(t-s)A} dW(s) \\ &=: Z(t,x) + \int_0^t e^{(t-s)A} b(X(s,x))\,ds, \end{aligned}$$

where $Z(t,x)$ is the solution of the equation

$$dZ = AZdt + dW(t), \quad Z(0,x) = x.$$

Let

$$\varphi(x) = \int_0^a e^{i\langle e^{sA}x, h\rangle} ds, \quad x \in H,$$

for some $a > 0$ and $h \in D(A^*)$. Then by the Taylor formula we have

$$\begin{aligned} \varphi(X(t,x)) &= \varphi\left(Z(t,x) + \int_0^t e^{(t-s)A} b(X(s,x))\,ds \right) \\ &= \varphi(Z(t,x)) + \left\langle D\varphi(Z(t,x)), \int_0^t e^{(t-s)A} b(X(s,x))\,ds \right\rangle + o(t). \end{aligned}$$

Taking expectation we have

$$P_t\varphi(x) = \mathbb{E}[\varphi(Z(t,x))] + \mathbb{E}\left\langle D\varphi(Z(t,x)), \int_0^t e^{(t-s)A} b(X(s,x))\,ds \right\rangle + \mathbb{E}[o(t)].$$

Moreover, it is easy to see that

$$\sup_{t\in(0,1]} \left\| \frac{P_t\varphi - \varphi}{t} \right\|_0 < +\infty$$

Now, letting $t \to 0$ yields

$$\lim_{t\to 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t} = L_0\varphi(x) + \langle D\varphi(s), b(x)\rangle = L\varphi(x), \qquad \text{for all } x \in H,$$

which implies by Proposition 2.7 that $\varphi \in D(\bar{L})$ and that $\bar{L}\varphi = L\varphi$.

It remains to prove Hypothesis 3.1(iv). Let $\varphi \in D(\bar{L})$. Then by (4.5) and (4.6), $\varphi \in D(\bar{L}_0) \cap C_b^1(H)$ (recall that $\bar{L}_0$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup). Arguing as in [3], we see that for any $\varphi \in D(\bar{L})$ there exists a sequence $(\varphi_n) \subset V$ such that

$$\varphi_n \xrightarrow{\pi} \varphi, \quad D\varphi_n \xrightarrow{\pi} D\varphi, \quad L_0\varphi_n \xrightarrow{\pi} \bar{L}_0\varphi.$$

Consquently, $(\varphi_n) \subset D(\bar{L})$ and $L\varphi_n \xrightarrow{\pi} \bar{L}\varphi$. So, Hypothesis 3.1(iv) is fulfilled. The last statement follows again from Theorem 3.2. □

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