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Solitary Waves and Electromagnetic Fields (*)

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Abstract. – *Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet; by soliton, we mean a solitary wave which exhibits some form of stability. In this respect solitary waves and solitons have a particle-like behavior and they occur in many questions of mathematical physics, such as superconductivity, phase transition, classical and quantum field theory, non linear optics,* (*see e.g.* [37], [50], [56]).
We are not interested in the study of a particular model. Here we shall be concerned with the existence of solitary waves for a class of variational field equations which exhibit suitable symmetry properties, namely equations which are invariant for the Poincarè group and the gauge group. In particular we shall describe two results obtained jointly with V. Benci in [17], [18]. *These results state the existence of three dimensional vortices for Abelian gauge theories describing the interaction of electrically charged solitary waves with the electromagnetic field.*

1. – The nonlinear wave equation.

We start from a basic variational equation which is invariant with respect to the Poincarè group and the gauge group. Let us consider the nonlinear wave equation for a complex valued field ψ defined on the spacetime $\mathbb{R}^4$

$$\square\psi + W'(\psi) = 0 \tag{1}$$

where

$$\square\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

and $W : \mathbb{C} \to \mathbb{R}$ satisfies

$$W(\psi) = F(|\psi|) \tag{2}$$

for some smooth function $F : \mathbb{R}_+ \to \mathbb{R}$. Hence

$$W'(\psi) = F'(|\psi|)\frac{\psi}{|\psi|}.$$

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Hereafter $x = (x_1, x_2, x_3)$ and t will denote the space and time variables.

The field $\psi : \mathbb{R}^4 \to \mathbb{C}$ will be called "matter field".

Equation (1) is variational, namely its solutions are the stationary points of the action functional

$$\int \mathcal{L}_0 \, dx \, dt$$

where $\mathcal{L}_0$ denotes the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\left(\left| \frac{\partial \psi}{\partial t} \right|^2 - |\nabla \psi|^2 \right) - W(\psi). \tag{3}$$

If $W'(\psi)$ is linear, $W'(\psi) = m_0^2 \psi$, $m_0 \neq 0$, equation (1) reduces to the Klein-Gordon equation. Among the solutions of the Klein-Gordon equation there are those which start as localized *wave packets*, but disperse in space as time goes on (see e.g. [74]). On the contrary, if W' is nonlinear, the wave packets could not disperse, actually equation (1) could possess solitary waves solutions which behave as relativistic particles (see the review articles [12], [13], [4], [15]).

1.1 – *Symmetry and conservation laws.*

The Lagrangian $\mathcal{L}_0$ defined by (3) is invariant with respect to the action of the Poincarè group.

We recall that the Poincaré group $\mathfrak{P}$ is the transformation group in $\mathbb{R}^4$ which preserves the Minkowski quadratic form

$$|z|_M^2 = -t^2 + \sum_{i=1}^{3} x_i^2, \;\; z = (t, \; x_1, \; x_2, \; x_3)$$

The Poincaré group is a ten parameter Lie group generated by the following transformations:

- **Time translations:**

  $$t' = t + \bar{t}, \;\; \bar{t} \in \mathbb{R}, \, x' = x$$

  They form a one parameter group. This invariance guarantees that time is homogeneous, namely that the laws of physics are independent of time: if an experiment is performed earlier or later, it gives the same results.

- **Space translations:**

  $$x_i' = x_i + \bar{x}_i, \;\; \bar{x}_i \in \mathbb{R} \;\, i = 1, 2, 3, \;\; t' = t$$

They form a three parameter group. This invariance guarantees that space is homogeneous, namely that the laws of physics are independent of space: if an experiment is performed here or there, it gives the same results.

- **Space rotations:**

$$x' = gx,\ g \in O(3),\ \ t' = t$$

They form a three parameter group. This invariance guarantees that space is isotropic, namely that the laws of physics are independent of orientation.

- **Lorentz transformations:**

$$\begin{cases} x_1' = \gamma_1(x_1 - v_1 t) \\ x_2' = x_2 \\ x_3' = x_3 \\ t' = \gamma_1(t - v_1 x_1) \end{cases};\quad \begin{cases} x_1' = x_1 \\ x_2' = \gamma_2(x_2 - v_2 t) \\ x_3' = x_3 \\ t' = \gamma_2(t - v_2 x_2) \end{cases};\quad \begin{cases} x_1' = x_1 \\ x_2' = x_2 \\ x_3' = \gamma_3(x_3 - v_3 t) \\ t' = \gamma_3(t - v_3 x_3) \end{cases}.$$

where

$$\gamma_i = \frac{1}{\sqrt{1 - v_i^2}},\ \ i = 1, 2, 3 \tag{4}$$

with $v_i^2 < 1,\ i = 1, 2, 3.$

They form a three parameter group. This invariance guarantees the principle of relativity, which states that an experiment performed in an inertial frame gives the same results than an experiment performed on another frame moving with constant velocity v_i.

The Lorentz invariance implies the remarkable facts of the Special Theory of Relativity, such as the celebrated formula $E = mc^2$. For a more detailed discussion on this point see e.g. [15].

The Poincarè group $\mathfrak{P}$ acts on a field ψ by the following representation:

$$\left(T_g \psi\right)(t, x_1, x_2, x_3) = \psi\left(t', x_1', x_2', x_3'\right),\ \ \left(t', x_1', x_2', x_3'\right) = g(t, x_1, x_2, x_3),\ g \in \mathfrak{P}.$$

The classical Noether's theorem [52] (see also e.g.[40]) states a deep connection between symmetry and conservation laws. Namely, for a variational system described by a Lagrangian $\mathcal{L}$, it states that any invariance of $\mathcal{L}$ for a continuous one-parameter group implies the existence of an integral of motion for the Euler -Lagrange equations, namely the existence of a quantity which is preserved with time by the solutions. Thus the invariance of $\mathcal{L}_0$ with respect to the Poincaré group implies that (1) has ten integrals of motion.

- **Energy**. The energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian. For a field ψ described by a Lagrangian $\mathcal{L}$, it has the following form (see e.g.[40])

$$\mathcal{E} = \int_{\mathbb{R}^3} \rho_{\mathcal{E}} dx, \ \rho_{\mathcal{E}} = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi}{\partial t} \right)} \cdot \frac{\partial \psi}{\partial t} - \mathcal{L} \tag{5}$$

In particular, if $\mathcal{L} = \mathcal{L}_0$ defined in (3), we get

$$\mathcal{E} = \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(\left| \frac{\partial \psi}{\partial t} \right|^2 + |\nabla \psi|^2 \right) + W(\psi) \right] dx. \tag{6}$$

- **Momentum.** The momentum, by definition, is the quantity which is preserved by the space invariance of the Lagrangian; the invariance for translations along x_i $(i = 1, 2, 3)$ gives rise to the following integrals of motion (see e.g. [40])

$$P_i = \int_{\mathbb{R}^3} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi}{\partial t} \right)} \cdot \frac{\partial \psi}{\partial x_i} dx. \tag{7}$$

In particular, if we take $\mathcal{L} = \mathcal{L}_0$, we get

$$P_i = \operatorname{Re} \int_{\mathbb{R}^3} \frac{\partial \psi}{\partial t} \overline{\frac{\partial \psi}{\partial x_i}} dx, \ i = 1, 2, 3. \tag{8}$$

Then, when $\mathcal{L} = \mathcal{L}_0$, the momentum $\boldsymbol{P} = (P_1, P_2, P_3)$ can be written

$$\boldsymbol{P} = \operatorname{Re} \int_{\mathbb{R}^3} \frac{\partial \psi}{\partial t} \overline{\nabla \psi} \, dx. \tag{9}$$

If we write ψ in polar form

$$\psi(t, x) = u(t, x) e^{iS(t,x)}, u \geq 0, \ S \in \mathbb{R}, \tag{10}$$

we have the following expression for $\boldsymbol{P}$

$$\boldsymbol{P} = \int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial t} \nabla u + \frac{\partial S}{\partial t} \nabla S \, u^2 \right) dx. \tag{11}$$

- **Angular momentum.** The angular momentum, by definition, is the quantity which is preserved by the invariance of the Lagrangian $\mathcal{L}$ under space rotations with respect to the origin (see e.g. [40]).
  When we take $\mathcal{L} = \mathcal{L}_0$, the angular momentum is given by

$$\boldsymbol{M} = \operatorname{Re} \int_{\mathbb{R}^3} \boldsymbol{x} \times \nabla \psi \overline{\frac{\partial \psi}{\partial t}} \, dx \tag{12}$$

where $\boldsymbol{x}$ is the vector whose components are x_1, x_2, x_3 and $\times$ denotes the exterior product.

Using (10) we get the following expression for the angular momentum (12):

$$\boldsymbol{M} = \int_{\mathbb{R}^3} \boldsymbol{x} \times \left(\frac{\partial u}{\partial t} \nabla u + \frac{\partial S}{\partial t} \nabla S \, u^2 \right) dx. \tag{13}$$

- **Ergocenter velocity.** The integrals of motion related to the Lorentz invariance of the Lagrangian $\mathcal{L}$ are the following ones (see e.g.[40])

$$K_i = \int_{\mathbb{R}^3} x_i \rho_{\mathcal{E}} dx - tP_i, \ \ i = 1, 2, 3 \tag{14}$$

where $\rho_{\mathcal{E}}$ and P_i are defined in (5) and (7).

Let us interprete these integrals of motion in a more meaningful way. Define the ergocenter $\boldsymbol{Q}$ as follows

$$\boldsymbol{Q} := \frac{\int_{\mathbb{R}^3} \boldsymbol{x} \rho_{\mathcal{E}} dx}{\int_{\mathbb{R}^3} \rho_{\mathcal{E}} dx} = \frac{\int_{\mathbb{R}^3} \boldsymbol{x} \rho_{\mathcal{E}} dx}{\mathcal{E}}. \tag{15}$$

Taking the derivative with respect to t in (14), when $\boldsymbol{P} = (P_1,\ P_2,\ P_3)$ and $\boldsymbol{K} = (K_1,\ K_2, K_3)$ are constants, we get

$$\boldsymbol{P} = \frac{d}{dt} \left(\int_{\mathbb{R}^3} \boldsymbol{x} \rho_{\mathcal{E}} dx \right). \tag{16}$$

When the energy $\mathcal{E}$ is constant, we get from (15) and (16)

$$\dot{\boldsymbol{Q}} = \frac{\boldsymbol{P}}{\mathcal{E}} \tag{17}$$

Then, if $\mathcal{L}$ is Lorentz invariant and $\boldsymbol{P}$, $\mathcal{E}$ are constants of motion, the three components of the ergocenter velocity $\dot{\boldsymbol{Q}}$ are integrals of motion.

Let the Lagrangian $\mathcal{L}$ be invariant under the $U(1)$ action

$$\psi(t, x) \longrightarrow e^{ia} \psi(t, x), \quad a \in \mathbb{R}, \tag{18}$$

where a is independent of the spacetime location (t, x). Observe that, by (2), the Lagrangian $\mathcal{L}_0$ in (3) is invariant for (18). This symmetry (global gauge symmetry), by Noether's theorem, gives rise to another conservation law, namely the conservation of the charge.

Charge. The charge of a complex valued field ψ described by the Lagrangian

$\mathcal{L}$ has the following expression (see e.g. [13])

$$\sigma = \mathrm{Im} \int_{\mathbb{R}^3} \frac{\partial \mathcal{L}}{\partial (\frac{\partial \psi}{\partial t})} \overline{\psi}\, dx.$$

In particular, if $\mathcal{L} = \mathcal{L}_0$, we have

$$\sigma = \mathrm{Im} \int_{\mathbb{R}^3} \frac{\partial \psi}{\partial t} \overline{\psi}\, dx. \tag{19}$$

Using (10), the charge (19) takes the following expression:

$$\sigma = \int_{\mathbb{R}^3} \frac{\partial S}{\partial t}\, u^2 dx. \tag{20}$$

Along the solutions of (1) the following continuity equation holds

$$\frac{\partial \rho_\psi}{\partial t} + \nabla \cdot \boldsymbol{j}_\psi = 0. \tag{21}$$

where

$$\rho_\psi = u^2 \frac{\partial S}{\partial t} \text{ and } \boldsymbol{j}_\psi = -u^2 \nabla S \tag{22}$$

are the charge and current densities of the matter field ψ.

If we interpret ψ as a quantum field describing a *swarm* of particles, the charge σ represents the total number of particles counted algebrically (namely an antiparticle counts for -1). Following [7], σ will be called *hylenic charge*. So, when each particle in the *swarm* carries an electric charge q, we have

$$Q = q\sigma \tag{23}$$

where Q denotes the total electric charge.

1.2 – *Standing waves.*

The easiest way to produce solitary waves of (1) consists in finding static solutions of (1), i.e. finite energy solutions $\psi = \psi(x_1, x_2, x_3)$ independent on time. Clearly ψ solves the elliptic equation

$$-\triangle \psi + W'(\psi) = 0 \tag{24}$$

Then, making a Lorentz boost with velocity $\boldsymbol{v} = (v, 0, 0)$ ($|v| < 1$) along the x_1 axis, we get

$$\psi_v(t, x_1, x_2, x_3) = \psi\left(\frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3\right); \tag{25}$$

Since (1) is invariant under Lorentz transformations, $\psi_v(t,x)$ is also solution of equation (1). This solution represents a bump which travels in the x_1-direction with speed v.

In [55], [67] it has been proved that (24) has finite energy nontrivial solutions provided that W has the following form:

$$W(s) = \frac{1}{2}m_0^2 s^2 - \frac{1}{p}s^p,\ s \geq 0, m_0^2 > 0,\ 2 < p < 6, \tag{26}$$

However it would be interesting to assume

$$W \geq 0; \tag{27}$$

in fact, if (27) holds, then the energy (6) is positive. The positivity of the energy, not only is an important request for the physical models related to the equation, but it provides good *a priori* estimates for the solutions of the relative Cauchy problem. These estimates allow to prove existence and well-posedness results under very general assumptions on W.

However Derrick [38] has proved that the request (27) implies that the only finite energy solution of (24) is the trivial one. His proof is based on the following equality which, in a different form, was also found by Pohozaev [55] (see also [13]). The Derrick-Pohozaev identity states that any finite energy solution ψ of (24) satisfies the equality

$$\frac{1}{6}\int_{\mathbb{R}^3} |\nabla\psi|^2 dx + \int_{\mathbb{R}^3} W(\psi)dx = 0. \tag{28}$$

Clearly (28) and (27) imply that $\psi \equiv 0$.

However, we can try to prove the existence of solitary waves of (1), with $W \geq 0$, exploiting the possible existence of *standing waves*, since this fact is not prevented by (28). In fact a standing wave is a finite energy solution of (1) having the following form

$$\psi_0(t,x) = u(x)e^{-i\omega_0 t},\ u\ \geq 0, \omega_0 \in \mathbb{R}. \tag{29}$$

Substituting (29) in (1), we are reduced to solve the following elliptic equation

$$-\Delta u + W'(u) = \omega_0^2 u,\ u \in H^1(\mathbb{R}^3) \tag{30}$$

Now, as for the static solutions, we can obtain a travelling solitary wave $\psi_v(t,x)$, just making a Lorentz boost with velocity $\boldsymbol{v} = (v,0,0)$ ($|v| < 1$) on the standing wave $\psi_0(t,x) = u(x)e^{-i\omega_0 t}$.

Thus, if $u(x) = u(x_1,x_2,x_3)$ is any solution of (30) and $\gamma = \left(1 - v^2\right)^{-\frac{1}{2}}$, then

$$\psi_v(t,x_1,x_2,x_3) = u(\gamma(x_1 - vt), x_2, x_3)e^{i(kx_1 - \omega t)} \tag{31}$$

is a solution of (1) provided that

$$\omega = \gamma\omega_0 \text{ and } k = \gamma\omega_0 v. \tag{32}$$

Notice that (25) is a particular case of (31) when $\omega_0 = 0$.

The following result holds:

THEOREM 1. – *Assume that W is a C^2 function s.t.*

i) $W \geq 0$,
ii) $W(0) = W'(0) = 0$; $W''(0) = m_0^2$, $m_0 > 0$,
iii) *there exists* $s_0 \in \mathbb{R}_+$ *such that* $W(s_0) < \frac{1}{2}m_0^2 s_0^2$.

Then (1) *has a nontrivial standing wave* $\psi_0(t,x) = u(x)e^{-i\omega_0 t}$ *for* $|\omega_0| \in (\Omega_0, m_0)$ *where*

$$\Omega_0 = \inf\left\{\Omega > 0 : \exists s \in \mathbb{R}^+ s.t.\ W(s) - \frac{1}{2}\Omega^2 s^2 < 0\right\}.$$

Notice that, by iii), $\Omega_0 < m_0$; then the interval (Ω_0, m_0) is not empty.

The existence of standing waves when $W \geq 0$ has been established by Rosen in [57] under more restrictive assumptions than i), ii), iii). Starting from this pionering paper, many physicists have studied nonlinear wave equations or gauge theories with W satisfying i), ii), iii) (see for example [27], [43], [46], and [70] with the reference therein contained). Theorem 1 has been proved in [27].

Coleman [27] calls *Q-balls* those radially symmetric solitary waves which solve (1) with W satisfying i), ii), iii) and this is the name usually used in the physics literature. The study of Q-balls was mainly motivated by the dark matter problem in cosmology [39], [46], [48].

We point out that Theorem 1 can be derived by a well known general existence result due to Berestycki and P. L. Lions [20]. In fact, by the previous discussion, it is sufficient to solve (30). Now equation (30) can be written as follows:

$$-\Delta u + G'(u) = 0,\ u \in H^1(\mathbb{R}^3) \tag{33}$$

where

$$G(s) = W(s) - \frac{1}{2}\omega_0^2 s^2$$

By [20], the existence of nontrivial solutions of (33) is guaranteed by the following assumptions on G:

$$G(0) = G'(0) = 0 \text{ and } G''(0) > 0 \tag{34}$$

$$\limsup_{s\to+\infty} \frac{G'(s)}{s^5} \geq 0 \tag{35}$$

$$\exists s_0 \in \mathbb{R}^+ :\ G(s_0) < 0. \tag{36}$$

It can be checked that, for $|\omega_0| \in (\Omega_0, m_0)$, the above assumptions are consequences of the assumptions i), ii) and iii) on W.

The stability of the standing waves of (1) has been studied in [64], [42], [65], [68], [6]. In particular in [6] it has been proved that the assumptions i), ii), iii) of Theorem1 guarantee the existence of an orbitally stable solitary wave (*hylomorphic soliton*) for (1).

A numerical approach to the construction of hylomorphic solitons and a detailed discussion on some of their mathematical features are contained in [7].

2. – Vortices in Abelian Gauge theories.

Now we want to analyse the interaction of an electrically charged matter field ψ with the electromagnetic field. Coupling equation (1) with the Maxwell's equations gives rise to an Abelian gauge theory which permits to model various physical phenomena.

2.1 – *Basic facts on Maxwell's equations.*

First we recall some elementary facts. Maxwell's equations for an electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ can be written as follows in the usual three vector notations

$$\nabla \cdot \boldsymbol{E} = \rho \text{ Gauss's law} \tag{37}$$

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j} \text{ Ampère's law} \tag{38}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0 \text{ Faraday's law} \tag{39}$$

$$\nabla \cdot \boldsymbol{H} = 0 \text{ absence of magnetic poles.} \tag{40}$$

Here $\nabla\cdot$ and $\nabla\times$ denote respectively the divergence and the curl operators.

We look for solutions of the type

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}, \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi, \tag{41}$$

where the maps

$$\boldsymbol{A} : \mathbb{R}^4 \to \mathbb{R}^3, \ \phi : \mathbb{R}^4 \to \mathbb{R}$$

are called gauge potentials ($\boldsymbol{A}$ is the vector potential, ϕ is the scalar potential).

Substituting (41) in Maxwell's equations we see that (39) and (40) are satisfied; moreover (37) and (38) become

$$\nabla \cdot \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \right) = \rho \tag{42}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) + \frac{\partial}{\partial t} \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right) = \boldsymbol{j} \ . \tag{43}$$

In the electrostatic case (i.e. when $\boldsymbol{j} = 0$, $\boldsymbol{A} = 0$ and $\frac{\partial \phi}{\partial t} = 0$) the system (42), (43) reduces to the Poisson equation

$$-\triangle \phi = \rho.$$

In the magnetostatic case (i.e. when $\rho = 0$, $\phi = 0$ and $\frac{\partial \boldsymbol{A}}{\partial t} = 0$) the system (42), (43) reduces to the equation

$$\nabla \times (\nabla \times \boldsymbol{A}) = \boldsymbol{j}.$$

Observe that the role of $-\triangle$ in the electrostatic case is taken in the magnetostatic case by the operator $\nabla \times \nabla \times$.

When $\boldsymbol{j} = 0$ and $\rho = 0$ Maxwell's equations (42), (43) are the Euler-Lagrange equations of the functional

$$S = \int \mathcal{L}_{\max} \, dx dt,$$

where

$$\mathcal{L}_{\max} = \frac{1}{2} \left(\left| \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right|^2 - |\nabla \times \boldsymbol{A}|^2 \right). \tag{44}$$

Observe that $\mathcal{L}_{\max}$ satisfies the following *gauge invariance* property:

for all smooth $\chi = \chi(t, x)$ we have $\mathcal{L}_{\max}(\phi, \boldsymbol{A}) = \mathcal{L}_{\max}(\phi - \frac{\partial \chi}{\partial t}, \boldsymbol{A} + \nabla \chi)$.

2.2 – *The interaction with the electromagnetic field.*

A gauge theory provides an elegant way to describe an electrically charged matter field ψ interacting with an electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$. Let $\boldsymbol{A} = (A_1, A_2, A_3)$, ϕ be gauge potentials of $\boldsymbol{E}$, $\boldsymbol{H}$.

Let

$$\mathcal{L}_0 = \frac{1}{2} \left(\left| \frac{\partial \psi}{\partial t} \right|^2 - |\nabla \psi|^2 \right) - W(\psi)$$

be the Lagrangian of the field ψ and assume that W satisfies (2). Then the

Lagrangian describing the influence of the electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ on ψ is

$$\mathcal{L}_1 = \frac{1}{2}\left|D_t\psi\right|^2 - \frac{1}{2}\left|\boldsymbol{D}_x\psi\right|^2 - W(\psi),$$

where D_t, $\boldsymbol{D}_x$ denote the so called Weyl covariant derivatives, namely

$$D_t = \frac{\partial}{\partial t} + iq\phi, \quad \boldsymbol{D}_x\psi = (D_1\psi, D_2\psi, D_3\psi) \tag{45}$$

where i is the imaginary unit and

$$D_j = \frac{\partial}{\partial x_j} - iqA_j, \ j = 1,2,3.$$

The constant q denotes the elementary charge and it represents the size of the interaction. If Q and σ denote respectively the electric charge and the hylomorphic charge of ψ, we have $Q = q\sigma$ (see (23)).

On the other hand ψ is not only influenced by $\boldsymbol{E}, \boldsymbol{H}$, but it also is a source for the electromagnetic field. If we want to take into account this action, also the gauge potentials $\boldsymbol{A}, \phi$ have to be considered unknowns of the problem. So the total system consisting of the charged field ψ and the electromagnetic field is described by the total Lagrangian

$$\mathcal{L}_{tot} = \mathcal{L}_1 + \mathcal{L}_{max}$$

where $\mathcal{L}_{max}$ is defined in (44).

Then total action $\mathcal{S}$ is

$$\mathcal{S}(\psi, \phi, \boldsymbol{A}) = \int \mathcal{L}_{tot}\ dxdt. \tag{46}$$

Making the variation of $\mathcal{S}$ with respect to ψ, ϕ and $\boldsymbol{A}$ we get the following system of equations

$$D_t^2\psi - \boldsymbol{D}_x^2\psi + W'(\psi) = 0 \tag{47}$$

$$\nabla \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right) = q\left(\operatorname{Im}\frac{1}{\psi}\frac{\partial\psi}{\partial t} + q\phi\right)|\psi|^2 \tag{48}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right) = q\left(\operatorname{Im}\frac{\nabla\psi}{\psi} - q\boldsymbol{A}\right)|\psi|^2 \tag{49}$$

When $q = 0$, (47) reduces to the nonlinear wave equation (1); moreover (48) and (49) become the Maxwell's equations (42) and (43) with $\rho = 0, \boldsymbol{j} = 0$. The Cauchy problem for the system of equations (47), (48), (49) has been studied in [45], where the existence of a global solution has been proved when $W = 0$.

It will be useful to write ψ in polar form

$$\psi(t,x) = u(t,x)\, e^{iS(t,x)}, \quad u \geq 0, \quad S \in \mathbb{R}/2\pi\mathbb{Z}.$$

Then

$$\begin{aligned} \mathcal{L}_{tot} &= \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 - |\nabla u|^2\right) - W(u) \\ &+ \frac{1}{2}\left(\left(\frac{\partial S}{\partial t} + q\phi\right)^2 - |\nabla S - q\boldsymbol{A}|^2\right) u^2 \\ &+ \frac{1}{2}\left(\left|\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right|^2 - |\nabla \times \boldsymbol{A}|^2\right) \end{aligned} \tag{50}$$

and (46) takes the following form

$$\begin{aligned} \mathcal{S}(u,S,\phi,\boldsymbol{A}) &= \int \left(\frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 - |\nabla u|^2\right) - W(u)\right) dxdt \\ &+ \frac{1}{2}\int \left(\left(\frac{\partial S}{\partial t} + q\phi\right)^2 - |\nabla S - q\boldsymbol{A}|^2\right) u^2 dxdt \\ &+ \frac{1}{2}\int \left(\left|\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right|^2 - |\nabla \times \boldsymbol{A}|^2\right) dxdt. \end{aligned}$$

We point out that the Lagrangian $\mathcal{L}_{tot}$ defined in (50) is invariant under the *gauge* transformations T_χ defined by

$$T_\chi \psi = \psi e^{iq\chi}; \tag{51}$$

$$T_\chi \phi = \phi - \frac{\partial \chi}{\partial t}; \tag{52}$$

$$T_\chi \boldsymbol{A} = \boldsymbol{A} + \nabla\chi; \tag{53}$$

where $\chi = \chi(t,x)$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^4$. We recall that the Lagrangian $\mathcal{L}_0$, representing the matter field ψ in absence of interaction, is invariant for (51) (see (18)) only if χ is constant, i.e. when the gauge transformation is identically performed at every point in the spacetime (global gauge symmetry). Hence we conclude that, incorporating the Maxwell field, it is possible to gain a much more strict symmetry (local gauge symmetry), in which the phase change could depend on the point $(t,x) \in \mathbb{R}^4$.

The equations (47), (48), (49) take the form:

$$\Box u + W'(u) + \left(|\nabla S - q\boldsymbol{A}|^2 - \left(\frac{\partial S}{\partial t} + q\phi \right)^2 \right) u = 0 \tag{54}$$

$$\frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t} + q\phi \right) u^2 \right) - \nabla \cdot \left[(\nabla S - q\boldsymbol{A}) u^2 \right] = 0 \tag{55}$$

$$\nabla \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi \right) = q\left(\frac{\partial S}{\partial t} + q\phi \right) u^2 \tag{56}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi \right) = q(\nabla S - q\boldsymbol{A}) u^2 \ . \tag{57}$$

It turns out that (56) and (57) are the Gauss's and Ampère's laws (see (42), (43)) with respect to charge and current densities $\rho, \boldsymbol{j}$ defined by

$$\rho = -\left(\frac{\partial S}{\partial t} + q\phi \right) q u^2 \tag{58}$$

$$\boldsymbol{j} = (\nabla S - q\boldsymbol{A}) q u^2. \tag{59}$$

Finally equation (55) is the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \boldsymbol{j} = 0. \tag{60}$$

where the charge and the current densities $\rho, \boldsymbol{j}$ are defined by (58), (59).

Notice that (55) can be easily deduced from (56) and (57). So we are reduced to study the system (54), (56), (57).

2.3 – *Stationary solutions and vortices.*

Now let us consider the problem of the existence of solitary waves for the Abelian gauge theory described by the system (54), (56), (57). The Lagrangian $\mathcal{L}_{tot}$ is invariant for the following representation of the Lorentz group:

$$\psi_{\boldsymbol{v}}(t, x) = \psi(t', x')$$

$$\phi_{\boldsymbol{v}}(t, x) = \gamma[\phi(t', x') + \boldsymbol{v} \cdot \boldsymbol{A}(t', x')]$$

$$\boldsymbol{A}_{\boldsymbol{v}}(t, x) = \gamma[\boldsymbol{A}(t', x') + \phi(t', x')\boldsymbol{v}]$$

where t', x' are the Lorentz transformed of t, x and $\gamma, \boldsymbol{v}$ are as in subsection 1.2.

Thus, similarly to the case of equation (1), in order to produce solitary waves, it is sufficient to find stationary solutions of (47), (48), (49) and to make a Lorentz boost. By definition, a stationary solution of the system of equations (47), (48), (49), is a solution $(\psi, \phi, \boldsymbol{A})$ such that

$$\psi(t,x) = u(x)e^{i(S_0(x)-\omega t)}, u \geq 0,\ \omega \in \mathbb{R}, S_0 \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = 0,\ \frac{\partial \phi}{\partial t} = 0. \tag{61}$$

Substituting $S = S_0(x) - \omega t$ and taking into account (61), we get from (54), (56), (57) that u, S_0, ω, ϕ, $\boldsymbol{A}$ solve the following system of equations:

$$-\Delta u + \left(|\nabla S_0 - q\boldsymbol{A}|^2 - (q\phi - \omega)^2\right) u + W'(u) = 0 \tag{62}$$

$$-\Delta \phi = q(\omega - q\phi)u^2 \tag{63}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = q(\nabla S_0 - q\boldsymbol{A})u^2\ . \tag{64}$$

It can be shown (see Theorem 1 in [9]) that the energy of a stationary solution $\left(u(x)e^{i(S_0(x)-\omega t)}, \phi, \boldsymbol{A}\right)$ is

$$\mathcal{E} = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 + W(u) + \frac{\rho^2 + \boldsymbol{j}^2}{2q^2u^2} + \frac{|\nabla \phi|^2 + |\nabla \times \boldsymbol{A}|^2}{2}\right) dx \tag{65}$$

where ρ and $\boldsymbol{j}$ are defined in (58) and (59).

Clearly when $u = 0$, the only finite energy gauge potentials, which solve (62), (63), (64), are $\boldsymbol{A} = 0$, $\phi = 0$.

It is possible to have three types of stationary, non trivial solutions of (62), (63), (64):

- electrostatic solutions: $\boldsymbol{A} = 0$, $\phi \neq 0$;
- magnetostatic solutions: $\boldsymbol{A} \neq 0$, $\phi = 0$;
- electromagnetostatic solutions: $\boldsymbol{A} \neq 0$, $\phi \neq 0$.

Under suitable assumptions, all these types of solutions exist. The existence of electrostatic solutions has been largely studied when W satisfies (26) (see [10], [11], [22], [28], [29], [35], [69], [36]). Recently some existence results for the electrostatic solutions have been proved also when $W \geq 0$ (see [16], [49]).

Observe that the angular momentum $\boldsymbol{M}(\psi)$ (see (13)) of the matter field

$$\psi = u(x)e^{i(S_0(x)-\omega t)}$$

is

$$\boldsymbol{M}(\psi) = -\omega \int_{\mathbb{R}^3} \boldsymbol{x} \times \left(u^2 \nabla S_0\right) dx. \tag{66}$$

If $\boldsymbol{A}=0$ (electrostatic case) and $q\neq 0$, by (64) we get

$$u^2\nabla S_0=0.$$

So necessarily $\boldsymbol{M}(\psi)=0$.

Now we give the following definition

DEFINITION 2. – *A finite energy, stationary solution* $(u(x)e^{i(S_0(x)-\omega t)},\phi,\boldsymbol{A})$ *of* (47), (48), (49) *is called vortex, if the matter field* $\psi=u(x)e^{i(S_0(x)-\omega t)}$ *has non trivial angular momentum* $\boldsymbol{M}(\psi)$.

Then, in order to find vortex solutions, we need to look for solutions with $\boldsymbol{A}\neq 0$.

Observe that we have to solve the system of the three equations (62), (63), (64) with respect the unknowns $u,S_0,\omega,\phi,\boldsymbol{A}$. We shall consider the frequency ω as a parameter and make an ansatz for the phase S_0. Then we shall solve (62), (63), (64) with respect $u,\phi,\boldsymbol{A}$.

We set

$$\Sigma=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1=x_2=0\right\}$$

and define the map

$$\theta:\mathbb{R}^3\backslash\Sigma\to\frac{\mathbb{R}}{2\pi\mathbb{Z}}$$

$$\theta(x)=\operatorname{Im}\log\left(x_1+ix_2\right).$$

Observe that $\theta\in C^\infty\left(\mathbb{R}^3\backslash\Sigma,\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)$ and $\nabla\theta\in C^\infty\left(\mathbb{R}^3\backslash\Sigma,\mathbb{R}^3\right)$

$$\nabla\theta(x)=\left(\frac{x_2}{r^2},\ \frac{-x_1}{r^2},\ 0\right),\ r^2=x_1^2+x_2^2$$

We set

$$S_0(x)=l\theta(x),\ l\in\mathbb{Z}. \tag{67}$$

Using this ansatz, the equations (62), (63), (64) become

$$-\Delta u+\left[|l\nabla\theta-q\boldsymbol{A}|^2-(q\phi-\omega)^2\right]u+W'(u)=0 \tag{68}$$

$$-\Delta\phi=q(\omega-q\phi)u^2 \tag{69}$$

$$\nabla\times(\nabla\times\boldsymbol{A})=q(l\nabla\theta-q\boldsymbol{A})u^2. \tag{70}$$

So, if $(u,\phi,\boldsymbol{A})$ solves the system of equations (68), (69), (70), then

$$(\psi,\phi,\boldsymbol{A})\text{ with }\psi=u(x)e^{i(l\theta(x)-\omega t)} \tag{71}$$

is a stationary solution of (47), (48), (49).

Under some mild growth assumptions on W, it can be shown that, if

$$\int\left((1+\frac{1}{r^2})u^2+|\nabla u|^2+|\nabla\phi|^2+|\nabla \boldsymbol{A}|^2\right)dx<\infty, \tag{72}$$

then the energy $\mathcal{E}$ (see (65)) of $\left(u(x)e^{i(l\theta(x)-\omega t)},\phi,\boldsymbol{A}\right)$ is finite.

By (66) the angular momentum of the matter field ψ is

$$\boldsymbol{M}(\psi)=-\omega l\int_{\mathbb{R}^3}\boldsymbol{x}\times\left(u^2\nabla\theta\right)dx.$$

The third component $M_3(\psi)$ of $\boldsymbol{M}(\psi)$ is

$$M_3(\psi)=-\omega l\int_{\mathbb{R}^3}u^2dx. \tag{73}$$

So, if $l,\omega\neq 0$, the nontrivial solutions $u,\phi,\boldsymbol{A}$ of (68), (69), (70) satisfying (72), give rise to vortices.

2.4 – *3D vortices: results and remarks.*

First we state an existence result for vortices in three space dimensions when W satisfies (26):

THEOREM 3. – *Assume that*

$$W(s)=\frac{1}{2}m_0^2s^2-\frac{1}{p}s^p,\ s\geq 0,\ m_0>0,\ 2<p<6 \tag{74}$$

and set

$$\omega_p=m_0\sqrt{\min\left(1,\frac{p-2}{2}\right)}.$$

Then, for any $\omega\in\left(-\omega_p,\omega_p\right)$ *and any* $l\in\mathbb{Z}$, *the system of equations* (47), (48), (49) *admit a finite energy stationary solution* $(u(x)e^{i(l\theta(x)-\omega t)},\phi,\boldsymbol{A})$ *with* $u\neq 0$. *Moreover*

a) If $\omega\neq 0$ *then* $\boldsymbol{E}=-\nabla\phi\neq 0$ *.*

b) If $l\neq 0$ *then* $\boldsymbol{H}=\nabla\times\boldsymbol{A}\neq 0$

c) If $\omega,l\neq 0$ *then* $(u(x)\,e^{i(l\theta(x)-\omega t)},\phi,\boldsymbol{A})$ *is a vortex.*

The proof of this result is contained in [17] (see also [14]).

The existence of magnetostatic vortices in two space dimensions and with W as in (74) has been recently proved in [8].

We recall that the existence of vortices of the type $u(x)\,e^{i(l\theta(x)-\omega t)}$ for the nonlinear Schrödinger equation, with nonlinearities like (74), has been established in [19].

REMARK 4. – Since $u, \phi, \boldsymbol{A}$ solve (68), (69), (70), assertions a), b) in Theorem 3 follow immediately from (69), (70). Assertion c) follows from (73).

REMARK 5. – By the presence of the term $\nabla\theta$ the equations (68), (69), (70) are not invariant under the $O(3)$ group action. The solutions u, ϕ, $\boldsymbol{A}$ we find have only an S^1 (cylindrical) symmetry, namely

$$u = u(r, x_3),\ \ \phi = \phi(r, x_3),\ \ r = \sqrt{x_1^2 + x_2^2}.$$

and $\boldsymbol{A}$ can be written as follows

$$\boldsymbol{A} = b(r, x_3)\nabla\theta = b(r, x_3)\Big(\frac{x_2}{r^2},\ \frac{-x_1}{r^2},\ 0\Big)$$

where $b = b(r, x_3)$ is a real function.

REMARK 6. – The presence of the magnetic potential $\boldsymbol{A} = b(r, x_3)\nabla\theta$ gives rise to a three dimensional vortex. Since the energy is finite the bundle of the vortex lines forms a magnetic tube which is similar to the magnetic field originated by a finite length solenoid.

Theorem 3 is not suitable for physical models since W is not positive and the conservation of the energy does not give the suitable estimates which could guarantee global existence for the initial value problems (see discussion in section 1). Now we examine the existence of vortices for (47), (48), (49) when $W \geq 0$.

Equations (47), (48), (49) have been largely studied when $W \geq 0$ is double well shaped, i.e.

$$W(s) = \big(1 - s^2\big)^2. \tag{75}$$

We point out that in two space dimensions, when W is double well shaped and $\omega = \phi = 0$ (magnetostatic case), the equations (62), (63), (64) reduce to the Ginzburg-Landau equations. For these equations the existence of magnetostatic, planar vortices $(\psi, \boldsymbol{A})$ with

$$\psi = u(r)e^{il\theta(x)}, \boldsymbol{A} = b(r)\nabla\theta(x),\ \ r = \sqrt{x_1^2 + x_2^2} \tag{76}$$

has been proved by Abrikosov [1] in a celebrated paper, where superconductors of the second type have been analyzed. After Nielsen and Olesen [51] have also shown the existence of two dimensional vortices (76) in the context of string and elementary particle theories. However these results cannot be extended to the three dimensional case. In fact it can be proved (see [18]) that, if $W \geq 0$, $W(0) > 0$ and $W(s) = 0$ for some $s > 0$ (as for double well shaped W), then the equations (62), (63), (64) in three space dimensions do not possess vortex type solutions.

Now we shall consider the case of a C^2 function $W \geq 0$ with $W(0) = 0$. More precisely we assume that W satisfies the assumptions of Theorem 1, namely:

i) $W \geq 0$

ii) $W(0) = W'(0) = 0$, $W''(0) = m_0^2 > 0$,

iii) there exists $s_0 \in \mathbb{R}_+$ such that $W(s_0) < \frac{1}{2} m_0^2 s_0^2$.

Observe that by iii) we can write

$$W(s) = \frac{m_0^2}{2} s^2 + N(s)$$

with $N(0) = N'(0) = N''(0) = 0$ and

$$N(s_0) < 0 \text{ for some } s_0 > 0. \tag{77}$$

The following existence result for three dimensional vortices holds [18] :

THEOREM 7. – *Let W satisfy i),ii),iii). Then for any $\ell \in \mathbb{Z}, \ell \neq 0$ and any sufficiently small q there exists $\omega \neq 0$ such that the equations* (47), (48), (49) *posses a finite energy stationary solution which is a vortex of type* $(u(x)e^{i(l\theta(x) - \omega t)}, \phi, \boldsymbol{A})$.

REMARK 8. – When there is no coupling with the electromagnetic field, i.e. when $q = 0$, equation (47) reduces to the nonlinear wave equation (1) with W satisfying i), ii), iii). In this case the existence of three dimensional vortices of the type $u(r, x_3)e^{i(l\theta(x) - \omega t)}$ has been stated in [5] (see also [71], [72] for related results). The existence of planar vortices for (1) has been proved in [43] for a particular class of nonlinearities satisfying assumptions i), ii), iii).

REMARK 9. – The term $N = N(s)$ is negative in some point (see (77)), then, roughly speaking, it produces an "attractive force" which allows the existence of solitary waves. The smallness assumption on q plays a fundamental role in the proof of the theorem 7. This fact can be interpreted as follows: if q is too large, the repulsive electric force becomes too strong with respect to the attractive force represented by N and it is reasonable to conjecture that solitary waves cannot exist.

The proofs of Theorem 3 and Theorem 7 are contained in [17] and [18] respectively. Here we confine ourselves to briefly discuss some features of the system of equations (68), (69), (70).

In order to find solutions $u, \phi, \boldsymbol{A}$ of (68), (69), (70) we look for critical points of the functional

$$J(u,\phi,\boldsymbol{A}) = \frac{1}{2}\int\left(|\nabla u|^2 - |\nabla\phi|^2 + |\nabla\times\boldsymbol{A}|^2\right)dx$$

$$+\frac{1}{2}\int\left[|l\nabla\theta - q\boldsymbol{A}|^2 - (q\phi-\omega)^2\right]u^2dx + \int W(u)dx$$

on the space H defined by

$$H = \left\{(u,\phi,\boldsymbol{A}) : \int\left((1+\frac{1}{r^2})u^2 + |\nabla u|^2 + |\nabla\phi|^2 + |\nabla\boldsymbol{A}|^2\right)dx < \infty\right\}. \tag{78}$$

The study of the functional J exhibits the following main difficulties:

- Due to the presence of the electric field, namely of the term $-\int|\nabla\phi|^2dx$, the functional J is strongly indefinite. This means that the Morse index of any critical point of J is infinite and, as a consequence, the critical points are topologically *invisible:* roughly speaking, crossing a critical value does not give any change in the topological properties of the sublevels .
- J contains the term $\int|\nabla\times\boldsymbol{A}|^2dx$ which is not a Sobolev norm and it does not yield a control on $\int|\nabla\boldsymbol{A}|^2dx$.
- There is a lack of compactness due to the invariance of the functional J under the representation T_L on H of the translations along the x_3-axis: namely for any $U = (u,\phi,\boldsymbol{A}) \in H$ and $L \in \mathbb{R}$ we have

  $$J(T_LU) = J(U)$$

  where

  $$(T_LU)(x_1,x_2,x_3) = U(x_1,x_2,x_3+L). \tag{79}$$

  We point out that compactness cannot be recovered, as usual, by working in the framework of radially symmetric functions; in fact, due to the presence of $\nabla\theta$, J is not invariant under the standard representation of the group $O(3)$ on H (see Remark 5).
- Due to presence of the weight function $\frac{1}{r^2}$, we have

  $$C_0^\infty\left(\mathbb{R}^3\backslash\Sigma,\mathbb{R}^5\right) \subset H \text{ and } C_0^\infty\left(\mathbb{R}^3,\mathbb{R}^5\right) \not\subset H.$$

  Then the critical points of J solve (47), (48), (49), in the sense of distributions, in $\mathbb{R}^3\backslash\Sigma$. However it can be shown that the singular set Σ is, in a suitable sense, "removable" and that the critical points of J are also solutions in $\mathbb{R}^3$.

REMARK 10. – By using the Weyl covariant derivatives (45) it is also possible to represent the interactions of electromagnetic fields with electrically charged matter fields ψ described by other equations, as the nonlinear Schrödinger equation. Also in these cases an Abelian gauge theory can be constructed [9]. The research of electrostatic solutions for this Abelian gauge theory leads to study the so called Schrödinger-Maxwell (or Schrödinger-Poisson) systems. In the last ten years much attention has been devoted to such systems in presence of changing sign nonlinearities W satisfying (74) (see [2], [3], [9], [21], [23], [25], [24], [26], [34], [41], [44], [47], [60], [61], [62], [63], [28], [30], [31], [32], [33], [66], [53], [54]). However very little is known for Schrödinger-Maxwell systems when $W \geq 0$.

REMARK 11. – We have considered matter fields ψ taking values in $\mathbb{C}$ and an Abelian gauge theory, related to $U(1)$, has been constructed. Let $N > 1$ and assume that ψ takes values in $\mathbb{C}^N$, which is the representation space of the non-Abelian Lie group $U(N)$. In this case a non-Abelian gauge theory can be constructed (see e.g.[73]) and it would be interesting to properly extend the above existence results to non-Abelian gauge theories.

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