Silvano Delladio

A Sufficient Condition for the C^2-Rectifiability of the Set of Regular Values (in the Sense of Clarke) of a Lipschitz Map

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.3, p. 695–707.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_3_695_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana,
2008.

Bollettino U. M. I.
(9) **I** (2008), 695-707

A Sufficient Condition for the C^2-Rectifiability of the Set of Regular Values (in the Sense of Clarke) of a Lipschitz Map

Silvano Delladio

Abstract. – *We prove a result about the rectifiability of class C^2 of the set of regular values (in the sense of Clarke) of a Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ (with $n < N$).*

1. – Introduction and statement of main result.

1.1 – C^H-*rectifiable sets*

A Borel subset S of $\mathbb{R}^N$ is said to be a $(\mathcal{H}^n, n)$ rectifiable set of class C^H if there exist countably many n-dimensional submanifolds M_j of $\mathbb{R}^N$ of class C^H such that

$$\mathcal{H}^n\left(S \backslash \bigcup_j M_j\right) = 0.$$

Observe that for $H = 1$ this is equivalent to say that S is a countably n-rectifiable set, e.g. by [15, Lemma 11.1]. Such a notion has been introduced in [3] and provides a natural setting for the description of singularities of convex functions and convex surfaces, [1, 2]. More generally, it can be used to study the singularities of surfaces with generalized curvatures, [2]. Rectifiability of class C^2 is strictly related to the context of Legendrian rectifiable subsets of $\mathbb{R}^N \times \mathbb{S}^{N-1}$, [12, 13, 8, 9]. The level sets of a $W^{k,p}_{\mathrm{loc}}$ mapping between manifolds are rectifiable sets of class C^k, [4].

Some of the papers mentioned above put the problem of finding conditions for the C^2-rectifiability of a set, which could be interesting for applications in a measure-theoretical approach to geometric variational problems. This subject presents many difficulties and no really satisfactory result has been obtained. Even about simple matters, intuition is misleading. As a paradigmatic instance, we recall from [3] the example of the C^1 function

$$f(x) := \int_0^x \operatorname{dist}(t, E)^{1/2} dt, \qquad x \in \mathbb{R}$$

where E is a suitable positive measure Cantor-like subset of $[0,1]$. In [3] the authors prove that the graph of f is not C^2-rectifiable, despite of the fact that $f|(\mathbb{R}\backslash E)$ is of class $C^{1,1}$ and $f'|E = 0$ (in particular the approximate derivative of f' exists almost everywhere). Investigations related to this phenomenon have been developed in the framework of Legendrian rectifiable sets and currents, compare [6, 7, 12, 13]. One of the conclusions that can be drawn is that a Legendrian rectifiable set R is not necessarily mapped to a C^2-rectifiable set, by the orthogonal projection π onto the base space (compare [13, Proposition 4]). However it seems natural to conjecture that $\pi(R)$ is C^2-rectifiable, whenever R carries a Legendrian integral current. In the special one-dimensional case, this fact has been proved in [8]. A result about one-dimensional C^H-rectifiability (reducing to the previous one, for $H = 2$) is provided in [9].

1.2 – *Statement of the main result.*

Let us introduce some notation. First of all, consider a Lipschitz map

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N),$$

a multi-index γ in

$$I(n,N) := \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \,|\, 1 \le \gamma_1 < \ldots < \gamma_n \le N\}$$

and $s \in \mathbb{R}^n$. Then, according to [5, p. 133], let $\partial\varphi^\gamma(s)$ denote the generalized Jacobian (in the sense of Clarke) of the map

$$\varphi^\gamma := (\varphi^{\gamma_1}, \ldots, \varphi^{\gamma_n}) : \mathbb{R}^n \to \mathbb{R}^n$$

at s, that is

$$\partial\varphi^\gamma(s) := \mathrm{co}\left\{ \lim_{i\to\infty} D\varphi^\gamma(s_i) \,\middle|\, D\varphi^\gamma(s_i) \text{ exists}, s_i \to s \right\}$$

where “co” stands for the convex hull. In particular, one has

$$D\varphi^\gamma(s) \in \partial\varphi^\gamma(s)$$

whenever φ^γ is differentiable at s. Recall that $\partial\varphi^\gamma(s)$ is said to be “nonsingular” if every matrix in $\partial\varphi^\gamma(s)$ is of rank n. Finally let

$$\mathcal{R}_\varphi := \{s \in \mathbb{R}^n \,|\, \partial\varphi^\gamma(s) \text{ is nonsingular for some } \gamma \in I(n,N)\}.$$

THEOREM 1.1. – *Consider a Lipschitz map*

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

Then the image $\varphi(\mathcal{R}_\varphi)$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2, provided the following condition is met:

There exist a family of Lipschitz maps $\psi_i : \mathbb{R}^n \to \mathbb{R}^N$ $(i = 1, \dots, n)$ *and a family of bounded functions* $c_i : \mathbb{R}^n \to \mathbb{R}\backslash\{0\}$ $(i = 1, \dots, n)$*, such that*

$$D_i\varphi = c_i\psi_i \qquad (i = 1, \dots, n)$$

almost everywhere in $\mathbb{R}^n$*.*

REMARK 1.1. – The condition above implies the existence of a Lipschitz field of simple n-vectors parallel to the field

$$t \mapsto D_1\varphi(t) \wedge \cdots \wedge D_n\varphi(t)$$

orienting the image of φ. Thus, the condition above yields a Legendrian-type property which just extends to any dimension the one-dimensional assumption of [8] (and of [9]).

2. – Extended technical statement and reduction to graphs.

As a matter of fact we will prove the following result, whose statement looks quite technical but implies immediately Theorem 1.1 (actually it is strictly stronger).

THEOREM 2.1. – *Consider a Lipschitz map*

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

Let be given a family of bounded functions

$$c_i : \mathbb{R}^n \to \mathbb{R}\backslash\{0\} \qquad (i = 1, \dots, n),$$

a family of Lipschitz maps

$$\psi_i : \mathbb{R}^n \to \mathbb{R}^N \qquad (i = 1, \dots, n)$$

and denote by A the set of points $t \in \mathbb{R}^n$ *satisfying the following conditions:*

(i) *The map* φ *and all the maps* ψ_i *are differentiable at* t*;*
(ii) *The equality*

$$D_i\varphi(t) = c_i(t)\psi_i(t) \tag{2.1}$$

holds for all $i = 1, \dots, n$*.*

Also assume that

(iii) *For almost every* $a \in A$ *there exists a non-trivial ball B centered at a and such that*

$$\mathcal{L}^n(B\backslash A) = 0.$$

Then $\varphi(A \cap \mathcal{R}_\varphi)$ *is a* $(\mathcal{H}^n, n)$ *rectifiable set of class* C^2*.*

REMARK 2.1. – Let E be any subset of $\mathcal{R}_\varphi$, $\gamma \in I(n,N)$ and define

$$E^\gamma := \{s \in E \mid \partial\varphi^\gamma(s) \text{ is nonsingular}\}. \tag{2.2}$$

Then one obviously has

$$\bigcup_{\gamma \in I(n,N)} E^\gamma = E.$$

REMARK 2.2. – Let $\gamma \in I(n,N)$ and $\mathcal{R}_\varphi^\gamma$ be defined by (2.2). For $s \in \mathcal{R}_\varphi^\gamma$, the Lipschitz inverse function Theorem (e.g. [5, Theorem 3.12]) implies the existence of a neighborhood U of s and of a neighborhood V of $\varphi^\gamma(s)$ such that

- $V = \varphi^\gamma(U)$ and $\varphi^\gamma|U : U \to V$ is invertible;
- $(\varphi^\gamma|U)^{-1}$ is Lipschitz.

Let $\overline{\gamma}$ denote the multi-index in $I(N-n,N)$ which complements γ in $\{1,2,\ldots,N\}$ in the natural increasing order and set (for $x \in \mathbb{R}^N$)

$$x^\gamma := (x^{\gamma_1},\ldots,x^{\gamma_n}), \qquad x^{\overline{\gamma}} := (x^{\overline{\gamma}_1},\ldots,x^{\overline{\gamma}_{N-n}}).$$

Then the map

$$f := \varphi^{\overline{\gamma}} \circ (\varphi^\gamma|U)^{-1} : V \to \mathbb{R}^{N-n}$$

is Lipschitz and its graph

$$G_f^\gamma := \{x \in \mathbb{R}^N \mid x^\gamma \in V \text{ and } x^{\overline{\gamma}} = f(x^\gamma)\}$$

coincides with $\varphi(U)$.

By virtue of Remark 2.1 (with $E = A \cap \mathcal{R}_\varphi$) and Remark 2.2, and recalling that the graph of a Lipschitz map is a rectifiable set (e.g. [15, Theorem 5.3]), the proof of Theorem 2.1 is reduced to prove the following claim.

THEOREM 2.2. – *Under the assumptions of Theorem* 2.1, *let* $\gamma \in I(n,N)$ *and consider a map*

$$g : \mathbb{R}^n \to \mathbb{R}^{N-n}$$

of class C^1. *Then* $\varphi((A \cap \mathcal{R}_\varphi)^\gamma) \cap G_g^\gamma$ *is a* $(\mathcal{H}^n, n)$ *rectifiable set of class* C^2.

REMARK 2.3. – The remainder of our paper is devoted to proving Theorem 2.2. With no loss of generality, we can restrict our attention to the particular case when $\gamma = \{1,\ldots,n\}$.

3. – Preliminaries (under the assumptions of Theorem 2.2, with $\gamma = \{1, \ldots, n\}$).

3.1 – *Further reduction of the claim.*

From now on, for simplicity, $G_g^{\{1,\ldots,n\}}$, $(A \cap \mathcal{R}_\varphi)^{\{1,\ldots,n\}}$ and $\varphi^{\{1,\ldots,n\}}$ will be donoted by G_g, F and λ, respectively.

Consider the set

$$L := \varphi^{-1}(G_g) \cap F$$

and observe that it is measurable. Without loss of generality, we can assume that $\mathcal{L}^n(L) < \infty$. Then, by a well-known regularity property of $\mathcal{L}^n$, for any given real number $\varepsilon > 0$ there exists a closed subset L_ε of $\mathbb{R}^n$ with

$$L_\varepsilon \subset L, \qquad \mathcal{L}^n(L \backslash L_\varepsilon) \leq \varepsilon, \tag{3.1}$$

compare e.g. [14, Theorem 1.10].

Moreover, since L_ε is closed, one has

$$L_\varepsilon^* \subset L_\varepsilon \tag{3.2}$$

where L_ε^* is the set of density points of L_ε. Recall that

$$\mathcal{L}^n(L_\varepsilon \backslash L_\varepsilon^*) = 0 \tag{3.3}$$

by a well-known result of Lebesgue. In the special case that L has measure zero, we define $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* := \emptyset$.

Observe that

$$G_g \cap \varphi(F) \backslash \varphi(L_\varepsilon^*) \subset \varphi\big(\varphi^{-1}(G_g) \cap F \backslash L_\varepsilon^*\big) = \varphi(L \backslash L_\varepsilon^*)$$

hence

$$\begin{aligned} \mathcal{H}^n\big(G_g \cap \varphi(F) \backslash \varphi(L_\varepsilon^*)\big) &\leq \mathcal{H}^n\big(\varphi\big(L \backslash L_\varepsilon^*\big)\big) \\ &\leq \mathrm{Lip}(\varphi)^n \mathcal{L}(L \backslash L_\varepsilon^*) \\ &\leq \varepsilon \, \mathrm{Lip}(\varphi)^n \end{aligned}$$

by the area formula (compare [11, § 3.2.], [15, § 8]), (3.1), (3.2) and (3.3). It follows that

$$\mathcal{H}^n\left(G_g \cap \varphi(F) \backslash \bigcup_{j=1}^{\infty} \varphi(L_{1/j}^*)\right) = 0.$$

Thus, to prove Theorem 2.2, it suffices to show that

$$\varphi(L_\varepsilon^*) \text{ is a } (\mathcal{H}^n, n) \text{ rectifiable set of class } C^2$$

for all $\varepsilon > 0$.

3.2 – *Further notation.*

Let us consider the projection

$$\Pi : \mathbb{R}^N \to \mathbb{R}^{N-n}, \qquad (x_1, \dots, x_N) \mapsto (x_{n+1}, \dots, x_N).$$

For $i \in \{1, \dots, n\}$ and $s, \sigma \in \mathbb{R}^n$, define

$$\Phi_{i;s}(\sigma) := \Pi\psi_i(\sigma) - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s))\psi_i^j(\sigma), \tag{3.4}$$

$$R_s^{(0)}(\sigma) := g(\lambda(\sigma)) - g(\lambda(s)) - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s))\big[\varphi^j(\sigma) - \varphi^j(s)\big]$$

and

$$R_{i;s}^{(1)}(\sigma) := \frac{\partial g}{\partial x^i}(\lambda(\sigma)) - \frac{\partial g}{\partial x^i}(\lambda(s)).$$

REMARK 3.1. – All the maps $\sigma \mapsto \Phi_{i;s}(\sigma)$ are Lipschitz.

4. – Lemmas (under the assumptions of Theorem 2.2, with $\gamma = \{1, \dots, n\}$).

LEMMA 4.1. – *Consider the square-matrix field*

$$\rho \mapsto M(\rho) := \begin{pmatrix} \psi_1^1(\rho) & \cdots & \psi_1^n(\rho) \\ \vdots & & \vdots \\ \psi_n^1(\rho) & \cdots & \psi_n^n(\rho) \end{pmatrix}, \qquad \rho \in \mathbb{R}^n.$$

and let $t \in F$. Then there exists a nontrivial ball B, centered at t, such that

- *The matrix $M(\rho)$ is invertible for all $\rho \in B$;*
- *The map*

$$\rho \mapsto M(\rho)^{-1}, \qquad \rho \in B$$

  is Lipschitz.

PROOF. – One has

$$M(t) = \left(\prod_{i=1}^{n} c_i(t)\right)^{-1} \begin{pmatrix} D_1\varphi^1(t) & \cdots & D_1\varphi^n(t) \\ \vdots & & \vdots \\ D_n\varphi^1(t) & \cdots & D_n\varphi^n(t) \end{pmatrix}$$

by (2.1). Since $D\lambda(t) \in \partial\lambda(t)$ and $t \in \mathcal{R}_\varphi^{\{1,\dots,n\}}$, one has

$$\det M(t) \neq 0.$$

But the function $\rho \mapsto \det M(\rho)$ is continuous, hence there exists a nontrivial ball B centered at t and such that

$$|\det M(\rho)| \geq \frac{|\det M(t)|}{2}$$

for all $\rho \in B$, hence the two claims easily follow. □

LEMMA 4.2. – *If $s \in L_\varepsilon^*$ then*

(1) *One has*

$$\Phi_{i;s}(s) = 0$$

for all $i \in \{1, \ldots, n\}$;

(2) *Moreover, for $l \in \{1, \ldots, N-n\}$*

$$\frac{\partial g^l}{\partial x^i}(\lambda(s)) = \left[M(s)^{-1}\right]_i \bullet \psi_*^{n+l}(s)$$

where $[\,\cdot\,]_i$ denotes the i^{th} row in the argument matrix and

$$\psi_*^{n+l} := (\psi_1^{n+l}, \ldots, \psi_n^{n+l}).$$

PROOF. – (1) First of all, observe that

$$g(\lambda(t)) = \Pi\varphi(t)$$

for all $t \in \varphi^{-1}(G_g)$. Since $L_\varepsilon^* \subset A$ the two members of this equality are both differentiable at s. Moreover s is a limit point of $L_\varepsilon \subset \varphi^{-1}(G_g)$. It follows that (for $i = 1, \ldots, n$)

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) D_i\varphi^j(s) = \Pi D_i\varphi(s)$$

namely

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) c_i(s)\psi_i^j(s) = c_i(s)\Pi\psi_i(s)$$

by (2.1). Recalling that $c_i(s) \neq 0$, we get

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s))\psi_i^j(s) = \Pi\psi_i(s) \tag{4.1}$$

i.e. $\Phi_{i;s}(s) = 0$.

(2) The system (4.1) is equivalent to

$$M(s)\nabla g^l(\lambda(s)) = \psi_*^{n+l}(s)^T, \qquad l \in \{1, \ldots, N-n\}$$

hence the conclusion follows. □

LEMMA 4.3 (Main lemma). – *Let $s \in L_\varepsilon^*$ and $t \in A$ be such that*

$$\mathcal{H}^1([s;t]\backslash A) = 0 \tag{4.2}$$

where $[s;t]$ denotes the segment joining s and t. Define the map parametrizing $[s;t]$ as

$$\sigma : [0,1] \to \mathbb{R}^n, \qquad \rho \mapsto s + \rho(t-s).$$

If $t \in \varphi^{-1}(G_g)$ then

$$R_s^{(0)}(t) = \sum_{i=1}^{n} (t^i - s^i) \int_0^1 c_i(\sigma(\rho))\, \Phi_{i;s}(\sigma(\rho))\, d\rho;$$

PROOF. – First of all, observe that:

- Since $s, t \in \varphi^{-1}(G_g)$ one has $g(\lambda(s)) = \Pi\varphi(s)$ and $g(\lambda(t)) = \Pi\varphi(t)$;
- The function $\rho \mapsto \varphi(\sigma(\rho))$ is Lipschitz, hence it is differentiable almost everywhere in $[0,1]$. Moreover the assumption (4.2) implies that

$$(\varphi \circ \sigma)'(\rho) = \sum_{i=1}^{n} (t^i - s^i) D_i\varphi(\sigma(\rho))$$

at a.e. $\rho \in [0,1]$.

Recalling also (2.1), we obtain

$$\begin{aligned} R_s^{(0)}(t) &= \Pi\varphi(t) - \Pi\varphi(s) - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s))\left[\varphi^j(t) - \varphi^j(s)\right] \\ &= \sum_{i=1}^{n} (t^i - s^i) \int_0^1 \left[\Pi D_i\varphi(\sigma(\rho)) - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) D_i\varphi^j(\sigma(\rho)) \right] d\rho \\ &= \sum_{i=1}^{n} (t^i - s^i) \int_0^1 c_i(\sigma(\rho)) \left[\Pi\psi_i(\sigma(\rho)) - \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) \psi_i^j(\sigma(\rho)) \right] d\rho. \end{aligned}$$

The conclusion follows at once from (3.4). □

LEMMA 4.4. – *Let Z be a null-measure subset of $\mathbb{R}^n$ and $s \in \mathbb{R}^n$. Then there exists a null-measure subset W of $\mathbb{R}^n$ such that*

$$\mathcal{H}^1(Z \cap [s;t]) = 0 \tag{4.3}$$

for all $t \in \mathbb{R}^n \backslash W$.

PROOF. – Let ψ_Z denote the characteristic function of Z. By a standard application of the coarea formula (e.g. [10, § 3.4.4], [11, § 3.2.13]), we obtain

$$0 = \int_{\mathbb{R}^n} \psi_Z = \int_{\mathbb{S}^{n-1}} \left(\int_0^{+\infty} \psi_Z(s + \rho u)\rho^{n-1} d\rho \right) d\mathcal{H}^{n-1}(u)$$

hence

$$\int_0^{+\infty} \psi_Z(s+\rho u)\rho^{n-1}d\rho = 0 \tag{4.4}$$

for all $u \in \mathbb{S}^{n-1}\backslash Q$, where Q is a measurable subset of $\mathbb{S}^{n-1}$ such that $\mathcal{H}^{n-1}(Q) = 0$. Define

$$W := s + \mathbb{R}^+ Q = \{s + \rho u \mid \rho \in \mathbb{R}^+,\, u \in Q\}.$$

By invoking again the coarea formula, we find (denoting with $B(0, R)$ the ball of radius R centered at the origin)

$$\begin{aligned}\mathcal{L}^n(W \cap B(0,R)) &= \int_{B(0,R)} \psi_W = \int_{\mathbb{S}^{n-1}} \left(\int_0^R \psi_W(s+\rho u)\rho^{n-1}d\rho \right) d\mathcal{H}^{n-1}(u) \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_0^R \psi_Q(u)\rho^{n-1}d\rho \right) d\mathcal{H}^{n-1}(u) = \frac{R^n}{n} \int_{\mathbb{S}^{n-1}} \psi_Q d\mathcal{H}^{n-1} \\ &= 0\end{aligned}$$

for all $R > 0$. It follows that $\mathcal{L}^n(W) = 0$. Finally the formula (4.3) follows at once from (4.4). □

5. – Proof of Theorem 2.2.

As we observed in Remark 2.3 above, we can assume $\gamma = \{1, \ldots, n\}$ and the notation introduced in sections 3, 4. Moreover let A' be the set of $a \in A$ such that there exists a non-trivial ball B centered at a satisfying

$$\mathcal{L}^n(B\backslash A) = 0.$$

One has

$$\mathcal{L}^n(A\backslash A') = 0 \tag{5.1}$$

by assumption (iii) in Theorem 2.1.

For each positive integer j define $\Gamma_{\varepsilon,j}$ as the set of $s \in L^*_\varepsilon \cap A'$ such that

$$\|R_s^{(0)}(t)\| \le j\|\lambda(t) - \lambda(s)\|^2 \tag{5.2}$$

and

$$\|R_{i;s}^{(1)}(t)\| \le j\|\lambda(t) - \lambda(s)\| \qquad (i = 1, \ldots, n) \tag{5.3}$$

for all $t \in L_\varepsilon^*$ satisfying

$$\|t - s\| \le \frac{1}{j}.$$

PROPOSITION 5.1. – *One has*

$$\bigcup_j \Gamma_{\varepsilon,j} = L_\varepsilon^* \cap A'.$$

PROOF. – Since (obviously!)

$$\Gamma_{\varepsilon,j} \subset \Gamma_{\varepsilon,j+1} \subset L_\varepsilon^* \cap A'$$

for all positive integers j, we get at once

$$\bigcup_j \Gamma_{\varepsilon,j} \subset L_\varepsilon^* \cap A'.$$

In order to prove the opposite inclusion, consider $s \in L_\varepsilon^* \cap A'$ and let U and V be as in Remark 2.2. Observe that

$$\|t - s\| = \left\|(\lambda|U)^{-1}(\lambda(t)) - (\lambda|U)^{-1}(\lambda(s))\right\| \le \mathrm{Lip}\,(\lambda|U)^{-1}\,\|\lambda(t) - \lambda(s)\| \tag{5.4}$$

for all $t \in U$. Since $s \in A'$, there exists a non-trivial ball B centered at s such that

$$B \subset U, \qquad \mathcal{L}^n(B \backslash A) = 0.$$

By applying Lemma 4.4 with $Z := B\backslash A$, we find

$$\mathcal{H}^1([s;t]\backslash A) = \mathcal{H}^1(Z \cap [s;t]) = 0$$

for a.e. $t \in B$. Then Lemma 4.3 and Lemma 4.2(1) imply

$$\begin{aligned}
\|R_s^{(0)}(t)\| &\le \sum_{i=1}^n |t^i - s^i| \left\| \int_0^1 c_i(\sigma(\rho))\big[\Phi_{i;s}(\sigma(\rho)) - \Phi_{i;s}(s)\big]\,d\rho \right\| \\
&\le \sum_{i=1}^n \mathrm{Lip}\big(\Phi_{i;s}\big)\,|t^i - s^i|\,\|c_i\|_\infty \int_0^1 \|\sigma(\rho) - s\|\,d\rho \\
&= \frac{\|t - s\|}{2} \sum_{i=1}^n \mathrm{Lip}\big(\Phi_{i;s}\big)\,|t^i - s^i|\,\|c_i\|_\infty \\
&\le C\,\|t - s\|^2
\end{aligned}$$

for a.e. $t \in B \cap \varphi^{-1}(G_g)$, where C is a suitable number which does not depend on t. By continuity we get

$$\|R_s^{(0)}(t)\| \le C\,\|t - s\|^2$$

for all $t \in B \cap \varphi^{-1}(G_g)$. Recalling (5.4) we conclude that

$$\|R_s^{(0)}(t)\| \leq C_0 \, \|\lambda(t) - \lambda(s)\|^2, \qquad C_0 := C \left[\mathrm{Lip}(\lambda|U)^{-1}\right]^2$$

for all $t \in B \cap \varphi^{-1}(G_g)$. By shrinking B (if need be!) we can also deduce the existence of a number C_1 which does not depend on t and is such that

$$\|R_{i;s}^{(1)}(t)\| \leq C_1 \, \|\lambda(t) - \lambda(s)\| \qquad (i = 1, \ldots, n)$$

for all $t \in L_\varepsilon^* \cap B$, by Lemma 4.1, Lemma 4.2(2) and (5.4). Hence

$$s \in \Gamma_{\varepsilon,j}$$

provided j is big enough. □

Since $L_\varepsilon^* \subset A$, from Proposition 5.1 it follows that

$$\varphi(L_\varepsilon^*) = \varphi(L_\varepsilon^* \cap A) = \varphi(L_\varepsilon^* \cap (A \backslash A')) \cup \varphi(L_\varepsilon^* \cap A') = \varphi(L_\varepsilon^* \cap (A \backslash A')) \cup \bigcup_j \varphi(\Gamma_{\varepsilon,j})$$

where $\varphi(L_\varepsilon^* \cap (A \backslash A'))$ has measure zero, by (5.1). Hence it will be enough to prove that (for all ε and j)

(5.5) $\qquad \varphi(\Gamma_{\varepsilon,j})$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2.

To prove this claim, first consider a countable measurable covering $\{Q_l\}_{l=1}^\infty$ of $\Gamma_{\varepsilon,j}$ such that

$$\mathrm{diam}\, Q_l \leq \frac{1}{j}$$

for all l, and define

$$F_l := \overline{\lambda(\Gamma_{\varepsilon,j} \cap Q_l)}.$$

If $\xi, \eta \in F_l$, then there exist two sequences

$$\{s_k\}, \{t_k\} \subset \Gamma_{\varepsilon,j} \cap Q_l$$

such that

$$\lim_k \lambda(s_k) = \xi, \qquad \lim_k \lambda(t_k) = \eta.$$

By (5.2) and (5.3) we get

$$\|R_{s_k}^{(0)}(t_k)\| \leq j \|\lambda(t_k) - \lambda(s_k)\|^2$$

and

$$\|R_{i,s_k}^{(1)}(t_k)\| \leq j \|\lambda(t_k) - \lambda(s_k)\| \qquad (i = 1, \ldots, n)$$

for all k. Letting $k \to \infty$, we conclude that

$$\left\| g(\eta) - g(\xi) - \sum_{h=1}^{n} \frac{\partial g}{\partial x^h}(\xi)(\eta^h - \xi^h) \right\| \le j \|\eta - \xi\|^2$$

and

$$\left\| \frac{\partial g}{\partial x^i}(\eta) - \frac{\partial g}{\partial x^i}(\xi) \right\| \le j \|\eta - \xi\| \qquad (i = 1, \ldots, n)$$

for all $\xi, \eta \in F_l$. By the Whitney extension Theorem [16, Ch. VI, § 2.3] it follows that each $g|F_l$ can be extended to a map in $C^{1,1}(\mathbb{R}^n, \mathbb{R}^{N-n})$. Then the Lusin type result [11, § 3.1.15] implies that $\varphi(\Gamma_{\varepsilon,j} \cap Q_l)$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2. Finally, claim (5.5) follows observing that

$$\varphi(\Gamma_{\varepsilon,j}) = \bigcup_l \varphi(\Gamma_{\varepsilon,j} \cap Q_l).$$

REFERENCES

[1] G. Alberti, *On the structure of singular sets of convex functions.* Calc. Var., **2** (1994), 17-27.

[2] G. Anzellotti - E. Ossanna, *Singular sets of surfaces with generalized curvatures.* Manuscripta Math., **86** (1995), 417-433.

[3] G. Anzellotti - R. Serapioni, *$\mathcal{C}^k$-rectifiable sets.* J. reine angew. Math., **453** (1994), 1-20.

[4] B. Bojarski - P. Hajlasz - P. Strzelecki, *Sard's theorem for mappings in Hölder and Sobolev spaces.* Manuscripta Math., **118**, n. 3 (2005), 383-397.

[5] F.H. Clarke - Yu.S. Ledyaev - R.J. Stern - P.R. Wolenski, *Nonsmooth analysis and control theory.* Graduate Texts in Mathematics, Springer Verlag 1998.

[6] S. Delladio, *Taylor's polynomials and non-homogeneous blow-ups.* Manuscripta Math., **113**, n. 3 (2004), 383-396.

[7] S. Delladio, *Non-homogeneous dilatations of a functions graph and Taylors formula: some results about convergence.* Real Anal. Exchange, **29**, n. 2 (2003/2004), 687-712.

[8] S. Delladio, *A result about C^2-rectifiability of one-dimensional rectifiable sets. Application to a class of one-dimensional integral currents.* Boll. Un. Matem. Italiana, **10-B** (2007), 237-252.

[9] S. Delladio, *A sufficient condition for the C^H-rectifiability of Lipschitz curves.* To appear on J. Geom. Anal. [PDF available at the page http://eprints.biblio.unitn.it/archive/00000934/].

[10] L.C. Evans - R.F. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions.* (Studies in Advanced Math.) CRC Press 1992.

[11] H. Federer, *Geometric Measure Theory.* Springer-Verlag 1969.

[12] J.H.G. Fu, *Some Remarks On Legendrian Rectifiable Currents.* Manuscripta Math., **97**, n. 2 (1998), 175-187.

[13] J.H.G. Fu - Erratum to "*Some Remarks On Legendrian Rectifiable Currents*". Manuscripta Math., **113**, n. 3 (2004), 397-401.

[14] P. MATTILA, *Geometry of sets and measures in Euclidean spaces.* Cambridge University Press, 1995.
[15] L. SIMON, *Lectures on Geometric Measure Theory.* Proceedings of the Centre for Mathematical Analysis, Canberra, Australia, vol. 3, 1984.
[16] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1970.

Department of Mathematics,
University of Trento, Via Sommarive, 14 38050 Povo,Trento, Italy
E-mail: delladio@science.unitn.it

Received December 17, 2007 and in revised form April 30, 2008