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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.3, p. 663–681.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_3_663_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2008.

Bollettino U. M. I.
(9) **I** (2008), 663-681

Systems of Bellman Equations to Stochastic Differential Games with Discount Control

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Dedicated to the memory of Guido Stampacchia

Abstract. – *Abstract: We consider two dimensional diagonal elliptic systems $\Delta u + au = H(x, u, \nabla u)$ which arise from stochastic differential games with discount control. The Hamiltonians H have quadratic growth in ∇u and a special structure which has not yet been covered by regularity theory. Without smallness condition on H, the existence of a regular solution is established.*

1. – Introduction.

In this paper we consider diagonal systems of elliptic partial differential equations

$$L_0 u_i + a_i u_i = H_i(x, u, \nabla u)\,, \qquad i = 1, \ldots, N\,, \quad \text{in } \Omega \tag{1}$$

in u, specially in two space dimensions. Here L_0 is a scalar uniformly elliptic operator

$$L_0 v = -\sum_{i,k=1}^{n} D_i\big(a_{ik}(x) D_k v\big)$$

with Lipschitz or measurable coefficients a_{ik}, in particular

$$L_0 = -\Delta$$

and $H = (H_1, \ldots, H_N)$ is a Caratheodory function

$$H : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}^N\,.$$

The a_i are real numbers > 0. Throughout the paper Ω is a bounded domain of $\mathbb{R}^n$.

The function H has quadratic growth in ∇u, i.e.

$$\big|H(x, u, \nabla u)\big| \le K|\nabla u|^2 + K_0\,. \tag{2}$$

We assume Dirichlet, Neumann, mixed boundary or periodic boundary conditions. Systems of type (1) with the quadratic growth (2) occur in many situations,

e. g. in differential geometry. cf. the survey (1) [Hil82] or [Jos02]; another application are Bellman equations to stochastic differential games with quadratic cost functionals (cf. § 2). The function H is determined by a certain formalism explained in the next chapter. One of the benefits of this application lies in the fact, that regularity theory, i.e. C^{α}-regularity of weak $L^{\infty} \cap H^1$-solution may fail for Hamiltonians with quadratic growth in ∇u, c.f. [Fre73]. Stochastic differential games give surprisingly many examples of Hamiltoneans where C^{α}-regularity takes place and new structure condition for H giving C^{α}-regularity are discovered via examples of control function of the players, c. f [BF84], [BF95], [BF02].

In this paper we study Hamiltonians arising from the situation that the players can influence the discount factor in their cost functionals (see § 2). This leads to Hamiltonians

$$H_i(x, u, \nabla u) = H_{0i}(x, u, \nabla u_i) - \nabla u_i \cdot L(x, u, \nabla u) - u_i F_i(x, u, \nabla u) + f_i(x)\,, \tag{3}$$

where $H_{0i}(x, u, \nabla u_i)$ and F_i has *quadratic* growth in ∇u and $L(x, u, \nabla u)$ linear growth in ∇u, $F_0 \geq 0$.

From the theory of diagonal elliptic systems it is known that there are regular solutions if either $H_{0i} = 0$ and $F_i = F_0, i = 1, \ldots, N$ (Wiegner's Theorem [Wie81]) or $F_i = 0$ (the authors' Theorem [BF84], [BF02]. For treating discount control, one needs a C^{α}-theorem for systems with Hamiltonians satisfying the structure condition (3) which is not yet available. At least, in two dimensions, we succeed to treat the case (3) and obtain existence and regularity for the system (1). This is the purpose of our paper. A part of our considerations work in n-dimension.

The paper was written while the second author was guest of the Nečas center of Prague. It has been supported by the Hausdorff Institute of Mathematics in Bonn.

2. – Stochastic Differential Games with Discount Control.

In this section, for convenience of the reader, we present the formalism on how to derive the Bellman system from a stochastic differential game. We follow [BF84].

Consider N players $1, \ldots, N$, who can modify the evolution of a dynamic system

$$dy = g(y, v_1, \ldots, v_N)\, dt + \sigma(y)\, dw\,, \qquad y(0) = x\,. \tag{4}$$

Equation (4) is a stochastic differential equation ; g represents the drift term, σ the diffusion term, $w(t)$ is a standardized Wiener process in $\mathbb{R}^n$ $\big(y(t) \in \mathbb{R}^n\big)$. In (4)

$y(t)$ represents the state of the system at time t. The controls $v_1(t), \dots, v_N(t)$ are stochastic processes.

Let $\mathcal{O}$ be a bounded smooth domain of $\mathbb{R}^n$, and τ the exit time of $y(t)$ from the domain. Then player i is interested in minimizing its own cost functional

$$J_i\big(x, v(\,\cdot\,)\big) = E_i\Big[\int_0^\tau l_i\big(y(t); v(t)\big) \exp\Big(-\int_0^t c_i\big(y(s), v(s)\big)\,dx\Big) + \phi_i\big(y(\tau)\big) \exp\Big(-\int_0^\tau c_i(t), v(t)\big)\,dt\Big], \qquad i = 1, \dots, v_N\,, \tag{5}$$

Here $v(\,\cdot\,)$ stands for $\big(v_1(\,\cdot\,), \dots v_N(\,\cdot\,)\big)$, and $l_i(x, v_1, \dots, v_N)$, $c_i(x, v_1, \dots, v_N)$, $\phi_i(x)$ are given functions.

The concept of a Nash point is the following. Find $\hat{v}_1(\,\cdot\,), \dots, \hat{v}_N(\,\cdot\,)$ such that

$$J_i\big(x, \hat{v}_1(\,\cdot\,), \dots, \hat{v}_i(\,\cdot\,), \dots, \hat{v}_N(\,\cdot\,)\big) \le J_i\big(x, \hat{v}_1, \dots, v(\,\cdot\,), \dots \hat{v}_N(\,\cdot\,)\big) \tag{6}$$

for all $v(\,\cdot\,)$ which are admissible for the i-th player. For one player ($N = 1$), the problem reduces to the classical stochastic control problem. The factor

$$\exp\Big(-\int_0^t c_i\big(y(s), v(s)\big)\,dx\Big)$$

is the *discount factor* of the i-th player which can be influenced by him/her.

As it is well known for stochastic control Dynamic programming leads to an analytic problem (a non linear P.D.E., called the Hamilton-Jacobi-Bellman equation) whose solution allows to derive an optimal stochastic control.

For Nash points, the situation is similar. However, the non-linear P.D.E. must be replaced by a system of non-linear P.D.E.

Let $\lambda_i, p_i \in \mathbb{R} \times \mathbb{R}^n$ be parameters and

$$L_i(x, \lambda_i, p_i, v) = l_i(x, v) + p_i\, g(x, v) - \lambda_i c_i(x, v)\,. \tag{7}$$

In the sequel, we will assume a simple form, in which the coupling occurs from the state only, namely

$$l_i(x, v) = f_i(x) + \phi_i(v_i)$$

Fixing x, λ_i, p_i, we look for a Nash point $\hat{V}_1(x, \lambda, p), \dots, \hat{V}_N(x, \lambda, p)$ for the functionals L_i. Here λ stands for $(\lambda_i, \dots, \lambda_N)$ and p for $(p_1, \dots, p_N)$. We then define

$$H_i(x, \lambda, p) = L_i\big(x, \lambda_i, p_i, \hat{V}(x, \lambda, p)\big)\,. \tag{8}$$

The non-linear system of P.D.E. is the following

$$Au_i = H_i(x, u, Du)\,, \qquad u_i = \phi_i \text{ on } \partial\mathcal{O} \tag{9}$$

where

$$A = -\sum_{i,j} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$$

and the matrix $a = \frac{1}{2}\sigma\sigma^*$.

Once we have found a regular solution of (9), say $u_i \in W^{2,p}(\mathcal{O}), p > n$, we may set

$$\hat{v}_i(x) = \hat{V}_i\big(x, u(x), Du(x)\big) \tag{10}$$

and obtain an optimal feedback for the player i, in the sense that

$$\hat{v}_i(t) = \hat{v}_i\big(y(t)\big) \tag{11}$$

is a solution to (6). Moreover the left hand side is nothing other than $u_i(x)$.

Therefore the problem of finding a regular solution of the system (9) is the tool to obtain Nash equilibrium points for the stochastic differential game.

Such results are available if H_i has a growth (in Du), which is less than quadratic.

Note that we have limited the presentation to stochastic processes which are killed at the exit of a domain. This leads to Dirichlet boundary value problems. Our theory treats also Neumann boundary or mixed boundary value problems. Neumann conditions require processes which are reflected at the boundary, and mixed Dirichlet Neumann require processes which are killed at the part of the boundary where the Dirichlet condition is given, and reflected in the other part.

3. – A Standard Example for Lagrangians Modelling Discount Control.

Let B_ν, C_ν^i positively symmetric definite real $m \times m$ matrices and A_ν be $m \times n$ matrices ($A_\nu \in Hom(\mathbb{R}^m \to \mathbb{R}^n)$, $\nu = 1, \dots, N$). The coefficients may depend Lipschitz-continuously on $x \in \Omega$; for the sake of simplicity we do not elaborate this. For $i = 1, \dots, N$

$$f_i \in L^\infty(\Omega) \tag{12}$$

A reasonably simple class of Lagrangians L_i is

$$L_i(x, \lambda_i, p_i, v) = \frac{1}{2} v_i \cdot B_i v_i + p_i \sum_{\nu=1}^N A_\nu v_\nu - \frac{1}{2}\lambda_i \Big(\sum_{\nu=1}^N v_\nu \cdot C_\nu^i v_\nu\Big) + f_i(x)\,. \tag{13}$$

A Nash point v^* of L_i satisfies

$$(14)\qquad B_i v_i^* + A_i^T p_i - \lambda_i C_i^i v_i^* = 0 \quad \text{and} \quad v_i^* = -(B_i - \lambda_i C_i^i)^{-1} A_i^T p_i =: E_i p_i \,.$$

Since λ_i corresponds to the unknown function u_i, one has to arrange via maximum principle arguments that u_i ranges in an interval such that $B_i - \lambda_i C_i$ is positively semi definite and bounded (see § 8).

Thus we obtain for the Hamiltonian

$$(15)\qquad \begin{aligned} H_i(x,u,\nabla u) &= \frac{1}{2} E_i \nabla u_i \cdot B_i E_i \nabla u_i \\ &\quad - \nabla u_i \cdot \sum_{\nu=1}^{N} A_\nu E_\nu \nabla u_\nu - \frac{1}{2} u_i \sum_{\nu=1}^{N} E_\nu \nabla u_\nu \cdot C_\nu^i E_\nu \nabla u_\nu + f_i(x) \\ &=: H_{0i}(x,u,\nabla u_i) - \nabla u_i \cdot L(x,u,\nabla u) - u_i F_i(x,u,\nabla u) \end{aligned}$$

where

$$E_i = -(B_i - u_i C_i)^{-1} A_i^T \,.$$

4. – Discussion of the Hamiltonian with Respect to PDE-Theory.

The Hamiltonian (15) is of the form

$$(16)\qquad H_i(x,u,\nabla u) = H_{0i}(x,u,\nabla u_i) - \nabla u_i L(x,u,\nabla u) - u_i F_i(x,u,\nabla u) + f_i \,.$$

We emphasize that H_{0i} does not depend on ∇u_ν, $\nu \neq i$, and that L does not depend on the index i (which belongs to the i-th equation of the diagonal system).

Under the a priori condition

$$(17)\qquad \lambda_0 I \le B_i - u_i C_i^i \le \Lambda_0 I$$

with $\lambda_0 > 0$, $\lambda_0 \in \mathbb{R}$, and the assumption that A_i, B_i, C_ν^i are Lipschitz continuous and the positivity assumptions on C_ν^i, we have

(18) *H_{0i}, L, F_i are Lipschitz continuous on compact subsets of $\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{nm}$*

$$(19)\qquad |H_{0i}(x,u,\eta_i)| \le K|\eta_i|^2 + K_0$$

$$(20)\qquad 0 \le F_i(x,u,\eta) \le K|\eta|^2 + K_0$$

$$(21)\qquad |L(x,u,\eta)| \le K|\eta| + K_0$$

with some constant $K, K_0, x \in \bar{\Omega}, \eta_i \in \mathbb{R}^n, \eta \in \mathbb{R}^{nN}$, u satisfying (17). In the case of our Lagrangian

$$(22)\qquad K_0 = 0 \,.$$

For diagonal systems

$$-\Delta u_i + a_i u_i = H_i(x, u, \nabla u) \tag{23}$$

with Dirichlet periodic or Neumann boundary *without* discount control ($F_i = 0$) there is a satisfactory theory concerning global solvability and regularity of solutions, even for more general Hamiltonians including some couplings $v_i v_k$ of the controls of the i-th and k-th player ([BF95], [BF02], [BF94]). Alternatively there is Wiegner's theorem ([Wie81]) which covers the case $H_{0i} = 0$, $F_i = F_0 \neq 0$, i.e. the players control their functionals only via the discount factor and the stochastic one. There should be a combination of the theory of Wiegner and the authors which would cover the structures (16), (19)–(21), but this is not available. Thus, it is of interest that we are at least able to treat the case of *two* space dimensions.

In (23), the Laplacian Δ may be replaced by a uniformly elliptic scalar operator

$$L_0 = -\sum_{i,k=1}^{n} D_i\big(a_{ik}(x) D_k\big)$$

with Lipschitz coefficients and ellipticity constant ν_0.

The techniques we use remind to those in [Fre79], where a logarithmic Morrey condition for ∇u is proved. However, due to the special situation here more complicated test functions are used. We present a simplified approach concerning the derivation of a logarithmic Morrey condition; this could be applied also in [Fre79]. The key is always to prove H^1 and C^a estimates for approximations of the system. This works even in the case of measurable a_{ik}. The step thereafter, obtaining $H^{2,p}$-estimates, is well known ([LU68], [Fre81]).

5. – Approximation.

Let μ_i be numbers such that for the matrices B_i, C_i from § 3 we have that

$$B_i - \mu_i C_i^i \textit{ is uniformly positively definite}$$

Recall that B_i, C_i^i are positively definite.

Let $[u_i]_{\mu_i} = \min(\mu_i, u_i)$. Then the matrix $B_i - [u_i]_{\mu_i} C_i^i$ is uniformly positively definite and the matrices

$$E_i^\mu = -\big(B_i - [u_i]_{\mu_i} C_i^i\big)^{-1} A_i^T \tag{25}$$

are uniformly bounded and Lipschitz continuous in x.

In the definition of H_{i0}, F_i and L in (15), we replace E_i by E_i^μ and denote the corresponding functions which arise by $H_{i0}^\mu, F_0^\mu, L^\mu$.

We now start the approximation procedure. This concerns also the general case (16), (18)–(21). Approximate the whole right hand side H_i by

$$H_i^{\mu\delta} = \left[H_{0i}^{\mu} - \nabla u_i L^{\mu} - u_i F_i^{\mu}(x,u,\nabla u)\right]\left(1+\delta|u|^2+\delta|\nabla u|^2\right)^{-1} + f\,. \tag{26}$$

(The function f is put outside of the normalization in case maximum principle based inequalities are of importance in condition (24).)

Now, $H_i^{\mu\delta}$ is bounded and the theory of elliptic equations gives the existence of a solution $u \in H^{2,p}$ of the diagonal system

$$-\Delta u_i + a_i u_i = H_i^{\mu\delta}(x,u,\nabla u) \tag{27}$$

with Dirichlet or Neumann boundary conditions at $\partial\Omega$ if the data is smooth. For more general boundary $\partial\Omega$ we may confine ourselves to estimates in $H^1 \cap L^\infty \cap H^{2,p}_{loc}$, however for the final proof it is convenient to have Hölder continuity up to the boundary. For this, we assume a Wiener type condition for $\partial\Omega$ or simply "condition A" of Ladyzenskaya-Uralzeva

$$\mu\left(B_R(x_0) \cap \partial\Omega\right) \geq c_0 R^n\,, \qquad x_0 \in \partial\Omega \tag{28}$$

or, in the case of Neumann or mixed boundary,

$$\partial\Omega \textit{ is Lipschitz continuous.} \tag{29}$$

In the case of mixed boundary we need in addition a Wiener type condition for the Dirichlet part $\Gamma_D \subset \partial\Omega$ of the boundary

$$cap\left(\Gamma_D \cap B_R\,;\; B_{2R}\right) \geq c_0 R^{n-2} \tag{30}$$

for all balls $B_0(x_0)$, $x_0 \in \Gamma_D$, $R \leq 1$. cap denotes the relative capacity.

In order to justify the passage to the limit as $\delta \to 0$, we establish several estimates in § 7 - § 10. In § 7 we derive L^∞-estimates based on the maximum principle (which also allow to get rid of the index μ in (25)). In § 8 we give H^1-estimates, in § 9 a logarithmic Morrey estimate and, finally, in § 10 an uniform C^a-estimate which allows the passage to the limit $\delta \to 0$ and $H^{2,p}$-regularity.

The following chapter is devoted to the discussion of the test functions we use.

6. – Iterated Exponential Test Functions.

Similar as in [BF84], we shall work with the iterated exponential functions

$$\psi_i = \tau(e^{\beta u_i} - e^{-\beta u_i}) \exp\left(c\sum_{\nu=1}^{N}\left(e^{\beta u_\nu} + e^{-\beta u_\nu}\right)\right) \tag{31}$$

where $\tau \geq 0$ may be a localization function, or $\tau = 1$ or τ is a singular weight. The parameters c and μ will be chosen large.

We shall use (31) for proving a) maximum principle type estimates and b) an H^1-bound for the solution from a L^∞-bound; c) a logarithmic Morrey estimate.

Effect of the test function on the right hand side of the equation (16). This is valid for any dimension.

We have the identity

$$\sum_{\nu=1}^{N}(\nabla u_\nu, \nabla\psi_\nu) + a\sum_{\nu=1}^{N}\int u_\nu\psi_\nu dx = T_1 + T_2 + T_3 + T_4 \tag{32}$$

where

$$T_1 = \beta\sum_{\nu=1}^{N}\int_\Omega |\nabla u_\nu|^2(e^{\beta u_\nu} + e^{-\beta u_\nu})\exp(.)\tau\, dx$$

$$T_2 = c\beta^{-1}\int_\Omega \Big|\nabla\sum_{\nu=1}^{N}(e^{\beta u_\nu} + e^{-\beta u_\nu})\Big|^2 \exp(.)\tau\, dx$$

$$T_3 = +\beta^{-1}\int_\Omega \sum_{\nu=1}^{N}\nabla(e^{\beta u_\nu} + e^{-\beta u_\nu})\exp(.)\nabla\tau\, dx$$

$$T_4 = \sum_{\nu=1}^{N}\int a_\nu u_\nu(e^{\beta u_\nu} - e^{-\beta u_\nu})\exp(.)\tau\, dx\,.$$

If we have to deal with a general diagonal elliptic operator then (32) becomes an inequality and $(\nabla u_\nu, \nabla\psi_\nu)$ and T_3 are modified respectively.

For the right hand side of the equation in the approximate case or the limiting case we have

$$\sum_{\nu=1}^{N}\Big[(H_{0\nu}, \psi_\nu) - (\nabla u_\nu, L\psi_\nu) - (u_\nu F_\nu, \psi_\nu) + (f_\nu, \psi_\nu)\Big] = S_1 + S_2 + S_3 + S_4 \tag{33}$$

and estimate (in the case $K_0 = 0$; if $K_0 \neq 0$, then some pollution terms would occur)

$$|S_1| = |\sum_{\nu=1}^{N}(H_{0\nu}, \psi_\nu)| \le K\sum_{\nu=1}^{N}\int |\nabla u_\nu|^2(e^{\lambda u_\nu} + e^{-\lambda u_\nu})\exp(.)\tau\, dx \tag{34}$$

$$\begin{aligned}|S_2| &= \Big|\sum_{\nu=1}^{N}\int \beta^{-1}\nabla(e^{\beta u_\nu} + e^{-\beta u_\nu})\cdot L\exp(.)\tau\, dx\Big| \\ &\le \delta_0\int L^2\exp(.)\, dx + \beta^{-2}\delta_0^{-1}\int\Big|\nabla\sum_{\nu=1}^{N}(e^{\beta u_\nu} + e^{-\beta u_\nu})\Big|^2\exp(.)\tau\, dx\end{aligned} \tag{35}$$

$$S_3 \le 0 \tag{36}$$

since $F_\nu \ge 0$ and $u_\nu(e^{\beta u_\nu} - e^{-\beta u_\nu}) \ge 0$.

Recall $\exp(.) = \exp\left(c \sum_{\nu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu})\right)$.

Note, that in all this procedures the Dirichlet boundary condition for ψ_2 remains respected. One may also replace u_ν by $u_\nu - c_\nu$, c_ν constant, provided the factor τ vanishes at the Dirichlet boundary.

From the above calculations we obtain the following estimate for *bounded weak solutions* of the system

$$-\Delta u_i + a_i u_i = H_{0i}(x, u, \nabla u_i) - \nabla u_i L(x, u, \nabla u) - u_i F_i(x, u, \nabla u) + f_i(x)\,. \tag{37}$$

LEMMA 6.1. – *Let $u \in L^\infty \cap H^1(\Omega, \mathbb{R}^N)$ be a weak solution of* (37) *with respect to one of the boundary conditions Dirichlet, Neumann, mixed or periodic. Suppose that the growth and sign conditions* (19), (20), (21) *hold with $K_0 = 0$ and let $a_i \geq 0$. Set $c = \beta = 2K + 1$ where K comes from the growth condition. Then*

$$\begin{aligned} \sum_{\nu=1}^{N} \int \left[|\nabla u_\nu|^2 (e^{\beta u_\nu} + e^{-\beta u_\nu}) + (a_\nu u_\nu - f_\nu)(e^{\beta u_\nu} - e^{-\beta u_\nu}) \right] \\ \times \exp\left(c \sum_{\mu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu})\right) \tau \, dx \\ \leq - \sum_{\nu=1}^{N} \int \nabla u_\nu (e^{\beta u_\nu} - e^{-\beta u_\nu}) \exp\left(c \sum_{\nu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu})\right) \nabla \tau \, dx\,. \end{aligned} \tag{38}$$

REMARK. – Clearly, the statement of the lemma holds with $-\Delta$ replaced by L_0, with an additional factor $C_0 > 0$ in front of $|\nabla u_\nu|^2$.

COROLLARY 6.1. – *Inequality* (38) *holds if u_ν is replaced by $(u_\nu - c_\nu)$ for any constant c_ν, provided that $c_\nu \tau = 0$ at $\partial\Omega$.*

PROOF OF LEMMA 6.1. – In (34) we choose $\delta_0 = (K+1)^{-1}$. From the PDE we have the equation $\sum_{i=1}^{4} T_i = \sum_{i=1}^{4} S_i$. Inspecting the above inequalities and representation formulas for T_i, S_i, we see that T_1 dominates S_1 and also the first summand at the right hand side of (35).

The second summand in (35) is dominated by T_2. The term S_3 is dropped due to the sign situation. The terms T_3, T_4, S_4 remain untouched. Thus we obtain (38). □

7. – H^1-Estimates.

Choosing $\tau = 1$ in Lemma 6.1 we immediately obtain H^1-bounds for the solutions of the system provided that we have an L^∞-bound. This holds in all dimensions.

THEOREM 7.1. – *Let* $u \in L^\infty \cap H^1$ *be a solution of the system* (37) *and assume the conditions of Lemma* 6.1. *Then the* H^1*-norm of* u *is bounded by a constant, depending only on* $\|u\|_\infty$*, the growth constants,* $\|f\|_{L^1}$ *and* a.

REMARK. – 1. The explicit dependence of $\|u\|_{H^1}$ with respect to $\|u\|_\infty$ with use of Lemma 6.1 is dramatic due to the iterated exponentials.

2. Obviously, the solutions u_μ of the approximate problem (27) are uniformly bounded in H^1 for once an uniform L^∞-bound is established.

3. In the setting of Theorem 7.1 the H^1-estimate does not depend on Ω.

PROOF of THEOREM 7.1. – It is an obvious consequence of Lemma 6.1 using

$$e^{\beta u_\nu} + e^{-\beta u_\nu} \geq 2 \quad \text{and} \quad u_\nu(e^{\beta u_\nu} - e^{-\beta u_\nu}) \geq 2\beta u_\nu^2 e^{-\beta|u_\nu|} \geq 0\,.$$

□

Other devices for obtaining H^1*-bounds.*

The Hamiltonians arising from discount control as treated in this paper have the property

$$H_{0i}(x, u, \nabla u) \sim K(1 + |u_i|)^{-2}|\nabla u|^2$$

if a setting with an adequate sign situation is arranged. This has the consequence that one may work also with test functions $u_i e^{\beta|u|^2}$ if certain smallness conditions for the matrices $A_\nu, \beta_\nu, C_\nu^i$ *not depending* on $\|u\|_\infty$ are assumed.

8. – L^∞-Estimates.

In this section, we deal again with the diagonal system (37) and consider one of the boundary conditions Dirichlet, Neumann, mixed or periodic. For obtaining L^∞-estimates for the solution u which covers also Neumann and mixed boundary conditions we use a weak maximum principle which is proved via a truncation method (as we learned it long time ago from Guido Stampacchia see also [KS80]). We consider the following scalar equation:

$$-\Delta w + (a_0 + g_0)w + g\nabla w = f \qquad \text{in } \Omega \tag{39}$$

and assume the following conditions

$$a_0 \in \mathbb{R}\,, \qquad a_0 > 0 \tag{40}$$

$$g_0 \in L^1(\Omega)\,, \qquad g_0 \geq 0 \tag{41}$$

$$g \in L^2(\Omega) \tag{42}$$

$$f \in L^\infty(\Omega) \tag{43}$$

There exist $q > n$ such that for any neighbourhood $U(\Gamma_D)$ of the Dirichlet boundary

$$g \in L^q\big(\Omega \backslash U(\Gamma_D)\big)\,. \tag{44}$$

The reason why we need (44) is that the lemma below is applied to the case $g \sim |\nabla u|$ and ∇u need not be regular enough in $U(\bar{\Gamma}_D \cap \bar{\Gamma}_N)$, even if we approximate Ω by smoother domains.

In equation (39) we may replace Δ by a more general uniformly elliptic operation in a divergence form.

LEMMA 8.1. – *Let $w \in C(\bar{\Omega}) \cap H^1_{\Gamma_D}(\Omega)$ be a weak solution of* (39) *and assume* (40)–(44). *Then*

$$a_0^{-1}\text{ess}\ inf\,[f]_- \le w \le a_0^{-1}\text{ess}\sup\,[f]_+\,. \tag{45}$$

Notation: $[f]_- = \min\{f, 0\}$, $[f]_+ = \max\{f, 0\}$.

PROOF. – Assume that

$$\max_{\Omega} w =: M > \varepsilon + a_0^{-1}\text{ess}\sup\,[f]_+\,.$$

Then $M > \varepsilon$ and $w - (M - \delta) < 0$ on Γ_D for $0 < \delta < \varepsilon$. Thus we have for the truncated function

$$\big(w - (M - \delta)\big)_+ := \max\big(w - (M - \delta)\,, 0\big) \in H^1_{\Gamma_D} \cap L^\infty$$

and we use it as a test function in equation (39). Since we have assumed that $w \in C(\bar{\Omega})$ we know that $\big(w - (M - \delta)\big)_+ = 0$ in $U(\bar{\Gamma}_D)$.

By simple calculations, using also (41), we obtain

$$\frac{1}{2}\int_+ |\nabla w|^2\,dx + \int (a_0 w - f)\big(w - (M - \delta)\big)_+\,dx \le \frac{1}{2}\int |g|^2\big(w - (M - \delta\big)^2_+\,dx\,. \tag{46}$$

Here $\int_+$ denotes integration over the set $\big(w - (M - \delta) \ge 0\big)$.

From (46) we see that w cannot be $w = const = M$ since then (46) would imply in inequality

$$\delta \le K\delta^2$$

which cannot be true as $\delta \to 0$. Hence we have that the measure of the set of zeros of $\big(w - (M - \delta)\big)_+$ remains > 0 as $\delta \to 0$.

Since the integration $\int_+$ avoids $U(\bar{\Gamma}_D)$ we may use (44) and estimate by Hölder's inequality

$$\int_+ |g|^2\big(w - (M - \delta)\big)^2_+\,dx \le K\Big(\int_+ \big(w - (M - \delta)\big)^{q^*}_+\,dx\Big)^{2/q} \tag{47}$$

where $q^* = \dfrac{2n}{n-2} + \delta_0$ with some $\delta_0 > 0$ if $n \geq 3$, or q^* is some large number, if $n = 2$. Since the set of zeros of $\big(w - (M-\delta)\big)_+$ has positive measure as $\delta \to 0$, we know from Guido Stampacchia's work ([Sta66]) that

$$K \int |\nabla \big(w - (M-\delta)\big)_+|^r \, dx \geq \Big(\int \big(w - (M-\delta)\big)_+^{q^*} \, dx \Big)^{r/q^*}$$

with some $r = r(q^*) < 2$.

From Hölder's inequality we then obtain

$$\Big(\int_+ \big(w - (M-\delta)\big)_+^{q^*} \, dx \Big)^{2/q^*} \leq \big[\mu(M - \delta < w < M)\big]^{\delta_1} \int_+ |\nabla w|^2 \, dx \tag{48}$$

with some $\delta_1 = \delta_1(r) > 0$. Here μ denotes the Lebesgue measure. It is important to observe that in the argument of μ we have w strictly $< M$. This implies

$$\mu(M - \delta < w < M) \to 0 \ \text{ as } \delta \to 0$$

and from (48) and (47) we see that the term $\int_+ |\nabla w|^2 \, dx$ dominates the term $\int_+ |g|^2 \big(w - (M-\delta)\big)_+^2 \, dx$ for small $\delta > 0$. Thus we arrive at the inequality

$$\int (a_0 w - f)\big(w - (M-\delta)\big)_+ \, dx \leq 0$$

which is a contradiction since $a_0 w - f > 0$ on $\big(w \geq (M-\delta)\big)$. This proves the upper bound claimed in Lemma 8.1. The lower bound is proven analogously. □

We apply Lemma 8.1 to the approximation (27). In the case of Dirichlet boundary condition we need only assume "condition A" (28) at the boundary; for pure Neumann condition we assume either

$$\partial\Omega \in H^{2,\infty} \tag{49}$$

or could apply an additional approximation procedure concerning $\partial\Omega$ such that (49) holds and the approximate boundaries are uniformly Lipschitz.

In the case of mixed boundary we assume, for simplicity,

$$\partial\Omega \backslash U(\bar{\Gamma}_D) \in H^{2,\infty} \tag{50}$$

and an uniform Wiener condition for Γ_D, i.e.

$$cap(B_R \cap \Gamma_D\, ;\ B_{2R}) \geq c_0 R^{n-2}\,, \qquad R \to 0\,,\ c_0 > 0\,.$$

In this setting, we obtain

THEOREM 8.1. – *Let $u \in H^1_{\Gamma_D}(\Omega)$ be a solution of the approximate system* (27). *Assume the structure conditions* (18)–(21) *with $K_0 = 0$; let $a_i > 0$, $f \in L^\infty$, and* *let the above regularity assumptions on $\partial\Omega$, Γ_D, Γ_N be satisfied. Then*

$$a_i^{-1}\text{ess min}\,[f_i]_- \le u_i \le a_i^{-1}\text{ess max}\,(f_i)_+ \,.$$

PROOF. – The statement is an obvious consequence of Lemma 8.1 applied to each component u_i □

Let us mention a simple proof in the case of Dirichlet boundary condition and smooth functions H_{0i}, F_i, L, f. Due to regularity theory we have $u \in C^2$ in the interior of Ω. Since $u \in C(\bar{\Omega})$ and $u|_{\partial\Omega} = 0$, a positive maximum or negative minimum cannot be at the boundary. If x^* is an interior point where a positive maximum of u_i attained, then

$$-\Delta u_i(x^*) \ge 0\,, \qquad H_{0i}\big(x^*, \nabla u_i(x^*)\big) = 0\,, \qquad \nabla u_i(x^*) \cdot L = 0\,, \qquad u_i F_i \ge 0\,.$$

Hence

$$a_i u_i(x^*) \le f_i(x^*)\,.$$

The inequality from below is proven analogously.

Now we are able to give a criterium such that condition (17) holds.

Since C_i^i is positively definite, the inequality (17)

$$\lambda_0 I \le B_i - u_i C_i^i$$

with $\lambda_0 > 0$ holds if the matrix

$$B_i - \max\,(0, \text{ess sup}\, f_i)\, C_i^i \tag{51}$$

is positively definite; this is the case if the lowest eigen value σ_i of B_i satisfies

$$\gamma_i \max\,(0, \text{ess sup}\, f_i) < \sigma_i\,, \tag{52}$$

and γ_i is the largest eigen value of C_I^i.

9. – A Logarithmic Morrey Estimate for ∇u.

The following estimate works in n-dimensions but has only a consequence for $n = 2$.

LEMMA 9.1. – *Let $u \in L^\infty \cap H^1$ be a solution of the system* (37) *and assume the growth and sign conditions* (18)–(21). *Let $a_i \ge 0$, $i = 1, \dots, N$, and $f \in L^q$, with some $q > 1$. Then, for every $a \in (0,1)$ there is a constant K_a such that*

$$\int\limits_{B_{1/2}(x_0)\cap\Omega} |\nabla u|^2 \big|ln|x - x_0|\big|^a\, dx \le K_a\,, \qquad x_0 \in \Omega\,. \tag{53}$$

The constant K_a depends only on the growth constants in (19)–(21), the ellipticity quotient, if a general elliptic scalar operator is assumed, on the L^∞-bound for u, on diam Ω, the dimension and on a bound for $\sup_{x_0\in\Omega} \int_\Omega |f|\big|ln|x-x_0|\big|^a\,dx$.

REMARK. – For $n \geq 3$ one can achieve $a = n-2$ choosing a Green function for τ in Lemma (6.1) assuming, say, $f \in L^{n/2+\delta}$. Of one chooses such a function τ for $n = 2$ one obtains Lemma 9.1 even with $a = 1$. However the approach here is simpler and the method of the proof here works in some *non-diagonal settings*.

PROOF OF LEMMA 9.1. – In Lemma 6.1 we set $\tau = \big|ln\big(a|x-x_0|\big)\big|^a$ where $a = a$ (diam Ω) is chosen such that $a|x-x_0| < \frac{1}{2}$, $x_0 \in \Omega$. Then Lemma 6.1 yields

$$\int |\nabla u|^2 |ln(a|x-x_0|)|^a\,dx \leq \int K|\nabla u||ln(a|x-x_0|)|^{a-1}|x-x_0|^{-1}\,dx + K$$

$$\leq \frac{1}{2}\int |\nabla u|^2 |ln(a|x-x_0|)|^a\,dx + K\int |x-x_0|^{-2}|ln(a|x-x_0|)|^{a-2}\,dx + K\,.$$

This proves the Lemma since $2-a > 1$ and

$$\int_{B_{1/2}(x_0)} |x-x_0|^{-2}\big|ln|x-x_0|\big|^{-r}\,dx \leq K_r$$

if $r > 1$. □

10. – C^α-Estimates.

This section treats only the case of two space dimensions. We consider a diagonal system

$$-\Delta u_i + a_i u_i = H_i(x, u, \nabla u) + f_i \tag{54}$$

with boundary conditions and a growth condition

$$|H(x, u, \nabla u)| \leq K|\nabla u|^2 + K\,, \qquad f \in L^\infty\,. \tag{55}$$

Furthermore, we assume

(56) $H(x, \mu, v, \eta)$ *satisfies the Caratheodory conditions.*

In fact we need not that the system is diagonal.

We consider solutions

$$u \in L^\infty \cap H^{1,2}(\Omega) \cap C^\alpha(\bar{\Omega})$$

with the corresponding boundary condition and given bounds for $\|u\|_\infty$ and $\|u\|_{H^{1,2}}$ and the quantity $\sup_{x_0} \left| \int_{\Omega \cap B_{1/2}(x_0)} \ln|x-x_0|\right|^\beta |\nabla u|^2\, dx$ and prove an a priori bound for the C^{a_0}-norm of u with some $a_0 \in (0,a)$, a_0 and the C^{a_0} depend only on $\|u\|_\infty$, $\|u\|_{H^{1,2}}$ and the data, i.e. the constants in the growth condition and the domain Ω.

The fact that this works in this setting is well known, see [LU68], [Fre81]. The proof works via a global version of the hole filling technique. For the sake of completeness, we present the proof below in the periodic case. The treatment of other boundary conditions gives slightly more technical difficulties, but it is well known how to proceed.

We need the following version of Morrey's Lemma [Mor66]:

$$[u]^2_{C^a(B_{R_0}(x_0))} \le K_a^2 \sup\left\{ R^{-2a} \int_{B_R} |\nabla u|^2\, dx \,\middle|\, B_R \subset B_{2R_0}(x_0)\right\}. \tag{57}$$

$B_{R_0}(x_0)$ is the ball with center x_0 and radius R_0, $B_R = B_R(y)$. As usual the C^a-semi-norm is defined by

$$[u]_{C^a(W)} = \sup\left\{ |x-y|^{-a} |u(x)-u(y)| \,\middle|\, x \neq y, x, y \in W\right\}.$$

Note that in (57) the constant K_a does not depend on R_0 due to the special setting with the C^a-semi-norm extended over $B_{R_0}(x_0)$ on the left hand side and the balls $B_R \subset B_{2R_0}(x_0)$ at the right hand side. This can be seen from a scaling argument or just looking on Morrey's old Lemma.

Next, we establish Caccioppoli's inequality. In (54) we choose the function

$$(u - \bar u_R)\tau_R^2$$

as a test function where $\bar u_R$ is the mean value of the function u taken over the annulus $B_{2R}(y) - B_R(y)$ and τ_R is a Lipschitz-continuous function such that

$$\operatorname{supp}\tau_R = B_{2R}(y)\,, \qquad \tau_R = 1 \text{ on } B_R(y)\,, \qquad |\nabla \tau_R| \le R^{-1}\,.$$

With standard simple arguments we obtain

$$\begin{aligned}\int |\nabla u|^2 \tau_R^2\, dx \le K \int |u - \bar u_R|^2 |\nabla \tau_R|^2\, dx \\ + K \int |\nabla u|^2 |u - u_R| \tau_R^2\, dx + KR^2 =: A + B + KR^2\,.\end{aligned} \tag{58}$$

Since $|u - \bar u_R| \le R^a [u]_{C^a}$ we immediately see from (58) that ∇u satisfies a Morrey condition if $[u]_{C^a} < \infty$, i.e.

$$S_{R_0} := \sup\left\{ R^{-2a} \int_{B_R(z)} |\nabla u|^2\, dx \,\middle|\, z \in Q\,, R \le R_0 \right\}^{1/2} < \infty\,.$$

In fact, for B one estimates

$$B \leq K(2R)^{\alpha} \int |\nabla u|^2 \tau_R^2 \, dx \,,$$

and the latter term is dominated by the right hand side of (58) for small $R \leq r_0$ and we conclude $S_{r_0} \leq K$. Thereafter, one estimates $S_{R_0} \leq K(R_0, r_0) S_{r_0}$. Clearly, this estimate for S_{R_0} is not yet uniform, but at least we have $S_{R_0} < \infty$.

We now establish the uniformity of the Morrey estimate.

From Morrey's Lemma we have

$$[u]_{C^{\alpha}(B_{2R}(x_0))} \leq S_{4R_0} \leq K S_{R_0} \,.$$

Thus we conclude from (58)

$$\int_{B_R(x_0)} |\nabla u|^2 \, dx \leq K R^{-2} \int_{B_{2R}(x_0) - B_R(x_0)} |u - \bar{u}_R|^2 \, dx + K \int_{B_{2R(x_0)}} |\nabla u|^2 \, dx R^{2\alpha} S_{R_0}^2 + K R^2 \,.$$

We have estimated $|\nabla u|^2 S_{R_0} \tau_R^2 \leq \varepsilon |\nabla u|^2 \tau_R^2 + K_{\varepsilon} |\nabla u|^2 S_{R_0}^2$. (Simplifies the latter representation.)

We now estimate $\int |u - \bar{u}_R|$ on $B_{2R} - B_R$ via Poincaré's inequality and arrive at the *hole filling inequality*

$$\int_{B_R(x_0)} |\nabla u|^2 \, dx \leq K \int_{B_{2R}(x_0) - B_R(x_0)} |\nabla u|^2 \, dx + K \int_{B_{2R}(x_0)} |\nabla u|^2 \, dx R^{2\alpha} S_{R_0}^2 + K R^2 \,. \tag{59}$$

We add $\int\limits_{B_R(x_0)} |\nabla u|^2 \, dx$ to both sides of the hole filling inequality and obtain with $\theta = \dfrac{K}{K+1} < 1$

$$\int_{B_R(x_0)} |\nabla u|^2 \, dx \leq \theta \int_{B_{2R}(x_0)} |\nabla u|^2 \, dx + K \int_{B_{2R}} |\nabla u|^2 \, dx R^{2\alpha} S_{R_0}^2 + K R^2 \,. \tag{60}$$

Now choose $\beta \in (0,1)$ such that $\beta \leq \alpha$ and

$$\theta 2^{2\beta} = \theta_1 < 1 \,.$$

We divide (60) by $R^{2\beta}$ and pass to the supremum $R \leq R_0$, $x_0 \in Q$. Then we obtain

$$S_{R_0}^2 \leq \theta_1 S_{2R_0}^2 + K_0 S_{R_0}^2 \sup \left\{ \int_{B_{r(x_0)}} |\nabla u|^2 \, dx \middle| x_0 \in Q \,, r \leq 2R_0 \right\} + K \,.$$

Since we have the logarithmic Morrey condition from the last chapter we know that

$$\sup\left\{ \int_{B_R} |\nabla u|^2\, dx \middle| R \leq 2R_0 \right\} \leq K|(ln(2R_0)|^{-a_0} < \frac{1}{2K_0}(1-\theta_1)$$

if $R_0 = R_0(K, a_0)$ is small enough.

Hence we arrive at

$$\frac{1+\theta_1}{2} S_{R_0}^2 \leq \theta_1 S_{2R_0}^2 + K\,. \tag{61}$$

Now we have the two possibilities

$$S_{2R_0} = \sup_{0<R\leq R_0} \left\{ R^{-2a} \int_{B_R} |\nabla u|^2\, dx \right\}^{1/2} = S_{R_0}$$

or

$$S_{2R_0} = \sup_{R_0\leq R\leq 2R_0} \left\{ R^{-2a} \int_{B_R} |\nabla u|^2\, dx \right\}^{1/2}.$$

In the first case we conclude from (61)

$$\frac{(1-\theta_1)}{2} S_R^2 \leq K\,;$$

in the second case we conclude

$$S_{R_0} \leq K_{R_0} \int_Q |\nabla u|^2\, dx + K \leq K_{R_0}\,,$$

i.e. in both cases we arrive at an uniform estimate for S_{R_0}. □

Thus we have proven the a priori estimate

THEOREM 10.1. – *Let $u \in C^a(\bar{\Omega}) \cap H^1(\Omega)$ be a solution of* (54) *and assume the growth condition* (57), *the Caratheodory condition* (56) *and the inequality* (53) *from Lemma* 9.1. *Let $n = 2$. Then there is a number $\beta \leq a$*

$$[u]_{C^\beta} \leq C$$

with β and C depending on K, $\|u\|_\infty$, $\|u\|_{H^1}$, $\|f\|_{1+\delta}$ and Ω and the constant K_a in (53).

COROLLARY 10.1. – *Theorem* 10.1 *holds also in the case of Dirichlet, Neumann or mixed boundary conditions if the regularity assumptions on $\partial\Omega$* (28), (30) (49), (50) *are assumed.*

11. – Passage to the Limit $\delta \to 0$ and Main Theorem.

Once having an uniform C^a-bound for approximations (26) it is well known how to prove $H^{2,p}(\Omega_0)$-estimates, uniformly as $\delta \to 0$, for fixed $\Omega_0 \subset\subset \Omega$. This holds for diagonal systems, the elliptic principal must have Lipschitz coefficients. For a proof see the book of Ladyzenskaya-Uralzewa [LU68] or the survey [Fre81].

By Rellich's Theorem we thus may pass to the limit in the non-linearity $H(u_\delta, \nabla u_\delta)$. For Neumann's problem and mixed boundary we need also compactness of $H(u_\delta, \nabla u_\delta)$ in L^1 near the boundary; this is also well known since uniform C^a-estimates are available up to the boundary. Thus the final theorem reads:

THEOREM 11.1. – *Let the functions* H_{0i}, L, F_i *in* (16) *satisfy* (18)–(21) *with* $K_0 = 0$. *Let* $f \in L^\infty$ *and* $a_i > 0$. *Let* $\partial\Omega$ *satisfy condition "A"* (28), *in the case of Dirichlet problem, or the smoothness assumptions described in chapter* 8 *in the case of Neumann or mixed boundary. Then there is a solution*

$$u \in C^a(\bar{\Omega}) \cap H^{1,2}(\Omega) \cap H^{2,p}_{loc}(\Omega)$$

of the diagonal system (1) *with the corresponding boundary conditions, with some* $a \in (0,1)$ *and any* $p < \infty$.

FINAL REMARKS. – There are many open problems left.

1. Extend the theorem of this paper to the parabolic case in two space variables. The difficulty is to derive a logarithmic Morrey estimate as in section 9, or a substitute.
2. Extend the theorem diagonal the elliptic in n-dimensions, at least for the case $F_i = F_0$.
3. Solve a game which leads to super-quadratic Hamiltonians, i.e. $H(x, \nabla u) \sim |\nabla u|^q$, $q > 2$.

(A first result in this direction has been proposed by the authors in [BF08].)

Acknowledgment. The authors want to thank Mrs. Inge Schmelzer for her dedication in typing this article, including working at night.

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Received April 26, 2008