Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.2, p. 497–505.
Unione Matematica Italiana
<http://www.bdim.eu/item?id=BUMI_2008_9_1_2_497_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2008.

Bollettino U. M. I.
(9) **I** (2008), 497-505

Decomposition Results for Functions with Bounded Variation.

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Dedicated to the memory of Guido Stampacchia

Abstract. – *Some decomposition results for functions with bounded variation are obtained by using Gagliardo's Theorem on the surjectivity of the trace operator from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. More precisely, we prove that every BV function can be written as the sum of a BV function without jumps and a BV function without Cantor part. Alternatively, it can be written as the sum of a BV function without jumps and a purely ingular BV function (i.e., a function whose gradient is singular with respect to the Lebesgue measure). It can also be decomposed as the sum of a purely singular BV function and a BV function without Cantor part. We also prove similar results for the space BD of functions with bounded deformation. In particular, we show that every BD function can be written as the sum of a BD function without jumps and a BV function without Cantor part. Therefore, every BD function without Cantor part is the sum of a function whose symmetrized gradient belongs to L^1 and a BV function without Cantor part.*

1. – Introduction.

Throughout the paper $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure in $\mathbb{R}^n$ and the $n-1$ dimensional Hausdorff measure, respectively. Unless otherwise specified, the expression *almost everywhere* (abbreviated as *a.e.*) always refers to $\mathcal{L}^n$. If $1 < r \leq \infty$ and E is a set, we use the notation $\|\cdot\|_r$ or $\|\cdot\|_{r,E}$ for the L^r norm on E with respect to $\mathcal{L}^n$ or $\mathcal{H}^{n-1}$ (or to some other measure which is clear from the context), while $\|\cdot\|_1$ denotes the norm in L^1, as well as in the space $\mathcal{M}_b$ of bounded Radon measures.

For every measure μ, the symbols μ^a, μ^s will denote the absolutely continuous and singular part of μ with respect to $\mathcal{L}^n$. The former will always be identified with its density.

Throughout the paper Ω is a bounded open set in $\mathbb{R}^n$ and $BV(\Omega)$ is the space of functions with bounded variation on Ω, i.e., the space of functions $u \in L^1(\Omega)$ whose distributional partial derivatives $D_i u$, $i = 1, \dots, n$, are bounded Radon measures on Ω. The following decomposition holds:

$$Du = D^a u + D^j u + D^c u\,, \tag{1.1}$$

where $D^a u$ is the absolutely continuous part of Du with respect to $\mathcal{L}^n$, $D^j u$ is the jump part of Du (defined as the restriction of Du to the jump set J_u), and $D^c u$ is the Cantor part of Du. We refer to [3] for the precise definitions of these measures and of J_u, as well as for other fine properties of the functions in $BV(\Omega)$ or in its subspace $SBV(\Omega)$ of special functions of bounded variation.

We consider the following subspaces of $BV(\Omega)$:

$$\begin{aligned} BV^j(\Omega) &:= \{u \in BV(\Omega) : Du = D^j u\},\\ BV^c(\Omega) &:= \{u \in BV(\Omega) : Du = D^c u\},\\ BV^a(\Omega) &:= \{u \in BV(\Omega) : Du = D^a u\} = W^{1,1}(\Omega). \end{aligned}$$

If $n = 1$ a classical result states that any function $u \in BV(\Omega)$ can be written as

$$u = v + w + z\,, \tag{1.2}$$

with $v \in BV^a(\Omega)$, $w \in BV^j(\Omega)$, and $z \in BV^c(\Omega)$ (see, e.g., [3, Corollary 3.33]). The same result cannot hold true for $n > 1$. Indeed, given a function $f \in L^1(\Omega; \mathbb{R}^n)$ with curl$f \neq 0$, Alberti [1] proved that there exists $u \in BV(\Omega)$ such that $D^a u = f$. It is clear that this function u cannot be decomposed as in (1.2), since in this case we would have $v \in W^{1,1}(\Omega)$ and $Dv = f$, hence curl$(Dv) \neq 0$. For another example see [3, Example 4.1].

We therefore introduce the following subspaces of $BV(\Omega)$:

$$\begin{aligned} BV^{aj}(\Omega) &:= \{u \in BV(\Omega) : Du = D^a u + D^j u\} = SBV(\Omega),\\ BV^{ac}(\Omega) &:= \{u \in BV(\Omega) : Du = D^a u + D^c u\},\\ BV^{jc}(\Omega) &:= \{u \in BV(\Omega) : Du = D^j u + D^c u\} \end{aligned} \tag{1.3}$$

The argument mentioned after (1.2) shows that in dimension $n > 1$ we cannot decompose $BV(\Omega)$ as $BV^a(\Omega) + BV^{jc}(\Omega)$. Moreover, we cannot decompose $BV(\Omega)$ as $BV^j(\Omega) + BV^{ac}(\Omega)$, see [3, Example 4.1], nor as $BV^c(\Omega) + BV^{aj}(\Omega)$, as the following example shows. Let $\Omega =]0,1[^{n-1} \times]-1,1[$, let $\psi: [0,1] \to [0,1]$ be the Cantor-Vitali function (see, e.g., [3, Example 1.67]), and let $u(x) = \psi(x_1)$ for $x_n > 0$, and $u(x) = 0$ for $x_n \leq 0$. If $u = v + w$ with $v \in BV^c(\Omega)$ and $w \in BV^{aj}(\Omega)$, we would have $Dv = D^c u$, hence $D_i v = 0$ for $i > 1$ and $Dv = 0$ for $x_n < 0$. These properties imply that v is constant, hence $D^c u = Dv = 0$, which contradicts the fact that $|D^c u|(\Omega) = 1$.

The main result of the paper is that any function $u \in BV(\Omega)$ can be written as the sum of two functions which belong to two of the three subspaces introduced in (1.3), namely

$$u = v + w\,,$$

with $v \in BV^{aj}(\Omega)$ and $w \in BV^{ac}(\Omega)$ (Theorem 3.1), or $v \in BV^{aj}(\Omega)$ and $w \in BV^{jc}(\Omega)$ (Theorem 3.2), or $v \in BV^{jc}(\Omega)$ and $w \in BV^{ac}(\Omega)$ (Theorem 3.3).

To this end, we first prove in Section 2 some extension results analogue to

Gagliardo's theorem, showing that, if Ω has a C^1-boundary $\partial\Omega$, then any function in $L^1(\partial\Omega)$ can be obtained as the trace of a function in any of the three subspaces $BV^a(\Omega)$, $BV^c(\Omega)$, or $BV^j(\Omega)$.

We conclude the paper by proving a similar decomposition result (Theorem 4.1) for the space $BD(\Omega)$ of functions with bounded deformation.

2. – Extension results.

Let $Q_{n-1} :=]0,1[^{n-1} \times \{0\}$, $Q :=]0,1[^{n-1} \times]0,1[$, and let $x := (\overline{x}, x_n)$, with $\overline{x} \in]0,1[^{n-1}$ denote the coordinates of a generic point in Q. Let us recall Gagliardo's extension theorem ([4], see also, e.g., [5, Proposition 2.15]).

THEOREM 2.1. – *Let $\varphi \in L^1(Q_{n-1}; \mathbb{R}^m)$ with compact support in Q_{n-1}. Then for every $\varepsilon > 0$ there exists $u \in BV^a(Q; \mathbb{R}^m) = W^{1,1}(Q; \mathbb{R}^m)$ with trace φ on Q_{n-1} such that*

$$\operatorname{supp}(u) \cap \partial Q \subset Q_{n-1}\,, \tag{2.1}$$

$$\int_Q |u|\,dx \le \varepsilon \int_{Q_{n-1}} |\varphi|\,d\mathcal{H}^{n-1}\,, \tag{2.2}$$

$$\int_Q |Du|\,dx \le (1+\varepsilon) \int_{Q_{n-1}} |\varphi|\,d\mathcal{H}^{n-1}\,. \tag{2.3}$$

We prove now the analogue of Gagliardo's result for purely Cantorian functions.

THEOREM 2.2. – *Let $\varphi \in L^1(Q_{n-1}; \mathbb{R}^m)$ with compact support in Q_{n-1}. Then for every $\varepsilon > 0$ there exists $u \in BV^c(Q; \mathbb{R}^m)$ with trace φ on Q_{n-1} such that*

$$\operatorname{supp}(u) \cap \partial Q \subset Q_{n-1}\,, \tag{2.4}$$

$$\int_Q |u|\,dx \le \varepsilon \int_{Q_{n-1}} |\varphi|\,d\mathcal{H}^{n-1}\,, \tag{2.5}$$

$$|Du|(Q) \le (1+\varepsilon) \int_{Q_{n-1}} |\varphi|\,d\mathcal{H}^{n-1}\,. \tag{2.6}$$

PROOF. – We begin by observing that, if A and B are open subsets of Q_{n-1} and $]0,1[$, $v \in BV^c(A; \mathbb{R}^m)$, $w \in BV^c(B)$, and $u(\overline{x}, x_n) = v(\overline{x})w(x_n)$, then

$u \in BV^c(A \times B; \mathbb{R}^m)$ and

$$D_i u = D_i v \otimes w \quad i = 1, \dots, n-1\,, \qquad D_n u = v \otimes Dw\,, \tag{2.7}$$

where v and w are interpreted as measures on A and B, respectively.

Following the proof of Gagliardo's extension theorem, we approximate φ with purely Cantorian functions φ_k. More precisely, given $\varepsilon \in]0,1[$ we may choose a compact set $K \subset Q_{n-1}$ and a sequence of functions $\varphi_k \in BV^c(Q_{n-1}; \mathbb{R}^m)$, such that $\mathrm{supp}(\varphi_k) \subset K$ and φ_k converges to φ strongly in $L^1(Q_{n-1}; \mathbb{R}^m)$. Moreover, we may assume that $\varphi_0 = 0$, $\|\varphi_k\|_1 \le 2\|\varphi\|_1$, and

$$\sum_{k=1}^{\infty} \|\varphi_k - \varphi_{k-1}\|_1 < \Big(1 + \frac{\varepsilon}{n}\Big) \|\varphi\|_1\,. \tag{2.8}$$

Given a decreasing sequence t_k converging to 0, we consider the stripes $t_k \le x_n \le t_{k-1}$ and we interpolate between φ_k and φ_{k-1} using the Cantor-Vitali function $\psi\colon [0,1] \to [0,1]$ instead of the linear interpolation. For $t_k \le x_n \le t_{k-1}$ we define

$$u(\overline{x}, x_n) = \varphi_k(\overline{x}) + (\varphi_{k-1}(\overline{x}) - \varphi_k(\overline{x}))\, \psi_k(x_n)\,,$$

where

$$\psi_k(x_n) := \psi\Big(\frac{x_n - t_k}{t_{k-1} - t_k}\Big)\,,$$

and we set $u(\overline{x}, x_n) = 0$ for $x_n \ge t_0$.

By (2.7) $u \in BV^c(Q_{n-1} \times]t_k, t_{k-1}[; \mathbb{R}^m)$ and $D_n u = (\varphi_{k-1} - \varphi_k) \otimes D\psi_k$ so that

$$|D_n u|(]0,1[^{n-1} \times]t_k, t_{k-1}[) = |D\psi_k|(]t_k, t_{k-1}[) \|\varphi_k - \varphi_{k-1}\|_1\,,$$

and, for $i < n$, $D_i u = D_i \varphi_{k-1} \otimes \psi_k + D_i \varphi_k \otimes (1 - \psi_k)$, so that, as $0 \le \psi_k \le 1$,

$$|D_i u|(]0,1[^{n-1} \times]t_k, t_{k-1}[) \le (t_{k-1} - t_k)(|D_i \varphi_{k-1}|(Q_{n-1}) + |D_i \varphi_k|(Q_{n-1}))\,.$$

Since there is no jump at $x_n = t_k$, we have also $|Du|(]0,1[^{n-1} \times \{t_k\}) = 0$.

As in the proof of Gagliardo's extension theorem, it suffices now to choose the decreasing sequence (t_k) so that

$$t_{k-1} - t_k \le \frac{\varepsilon}{n 2^k} \frac{\min\{1, \|\varphi\|_1\}}{1 + \|D\varphi_k\|_1 + \|D\varphi_{k-1}\|_1}\,.$$

Under these assumptions we get not only that $u \in BV^c(Q; \mathbb{R}^m)$, but also that (2.4), (2.5), and (2.6) hold.

Since, by construction, $u(\cdot, x_n) \to \varphi$ in $L^1(Q_{n-1}; \mathbb{R}^m)$, as $x_n \to 0$, the trace of u on Q_{n-1} coincides with φ. $\square$

The analogue extension result to a purely jump function holds.

THEOREM 2.3. – *Let $\varphi \in L^1(Q_{n-1};\mathbb{R}^m)$ with compact support in Q_{n-1}. Then for every $\varepsilon > 0$ there exists $u \in BV^j(Q;\mathbb{R}^m)$ with trace φ on Q_{n-1} such that*

$$\operatorname{supp}(u) \cap \partial Q \subset Q_{n-1}\,, \tag{2.9}$$

$$\int\limits_Q |u|dx \le \varepsilon \int\limits_{Q_{n-1}} |\varphi| d\mathcal{H}^{n-1}\,, \tag{2.10}$$

$$|Du|(Q) \le (1+\varepsilon) \int\limits_{Q_{n-1}} |\varphi| d\mathcal{H}^{n-1}\,. \tag{2.11}$$

PROOF. – We approximate now the boundary datum φ with piecewise constant functions $\varphi_k \in BV^j(Q_{n-1};\mathbb{R}^m)$ such that their support is contained in the same compact subset of Q_{n-1}, φ_k converges to φ in $L^1(Q_{n-1};\mathbb{R}^m)$, $\varphi_0 = 0$, $\|\varphi_k\|_1 \le 2\|\varphi\|_1$, and (2.8) holds. Let t_k be as in the previous proof, and set $u(\overline{x}, x_n) = \varphi_k(\overline{x})$ if $t_k < x_n \le t_{k-1}$, and $u(\overline{x}, x_n) = 0$ if $x_n > t_0$. Then $u \in BV^j(Q;\mathbb{R}^m)$ and (2.9), (2.10), and (2.11) hold. □

By a partition of unity argument we can extend the previous results from Q to a bounded open set Ω with C^1-boundary, preserving the constant 1 in the estimates.

THEOREM 2.4. – *Assume that Ω has a C^1-boundary and let $\varphi \in L^1(\partial\Omega;\mathbb{R}^m)$. For every $\varepsilon > 0$ there exists a function $u \in BV^a(\Omega;\mathbb{R}^m) = W^{1,1}(\Omega;\mathbb{R}^m)$ with trace φ on $\partial\Omega$ and such that*

$$\int\limits_\Omega |u|dx \le \varepsilon \int\limits_{\partial\Omega} |\varphi| d\mathcal{H}^{n-1}\,, \tag{2.12}$$

$$|Du|(\Omega) \le (1+\varepsilon) \int\limits_{\partial\Omega} |\varphi| d\mathcal{H}^{n-1}\,. \tag{2.13}$$

The same result holds for $BV^c(\Omega;\mathbb{R}^m)$ and $BV^j(\Omega;\mathbb{R}^m)$.

PROOF. – The result for $BV^a(\Omega;\mathbb{R}^m) = W^{1,1}(\Omega;\mathbb{R}^m)$ is already known, see, e.g., [5, Theorem 2.16, Remark 2.17]. We give an outline of the proof only in the case $BV^j(\Omega;\mathbb{R}^m)$.

We begin by observing that the construction of the extension theorems above can be repeated in any cube and that the estimates do not depend on the size of the cube. For every $x_0 \in \partial\Omega$, let $R_\delta(x_0)$ be an open parallelepiped centred in x_0, whose base is an $n-1$-dimensional cube of size δ parallel to the tangent hyperplane to $\partial\Omega$ at x_0, and having height 2δ. As $\partial\Omega$ is of class C^1, if δ is small enough there exists a diffeomorphism, close to an isometry in a C^1-sense, which maps

$R_\delta(x_0) \cap \Omega$ onto $]0, \delta[^n$ and $R_\delta(x_0) \cap \partial\Omega$ onto $]0, \delta[^{n-1} \times \{0\}$. By using this change of variables we obtain from Theorem 2.3 that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\varphi \in L^1(\partial\Omega; \mathbb{R}^m)$ has compact support in $\partial\Omega \cap R_\delta(x_0)$, then there exists a function $u \in BV^j(\Omega; \mathbb{R}^m)$, with trace φ on $\partial\Omega$ and $\mathrm{supp}(u) \subset R_\delta(x_0) \cap \overline{\Omega}$, satisfying (2.12) and (2.13).

Given $\varepsilon > 0$, by compactness we can cover $\partial\Omega$ with a finite number of such parallelepipeds. Let a_i be a partition of unity associated to this covering. By the previous step of the proof for every i there exists a function $u_i \in BV^j(\Omega; \mathbb{R}^m)$, with trace $a_i\, \varphi$ on $\partial\Omega$, satisfying (2.12) and (2.13) with φ replaced by $a_i\, \varphi$. Then the function $u = \sum_i u_i$ belongs to $BV^j(\Omega; \mathbb{R}^m)$, has trace φ on $\partial\Omega$, and satisfies (2.12) and (2.13).

3. – Decomposition results.

For every $u \in BV(\Omega; \mathbb{R}^m)$ the set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable. Therefore we can define a Borel function $\nu_u : J_u \to \mathbb{R}^n$ such that $\nu_u(x)$ is the approximate unit normal to J_u at x. The approximate limits $u^+(x)$ and $u^-(x)$ associated with $\nu_u(x)$ are defined according to [3, Definition 3.67].

We apply now Gagliardo's extension Theorem to prove the following decomposition result, see also [3, Theorem 4.6].

THEOREM 3.1. – *Let $u \in BV_{loc}(\Omega; \mathbb{R}^m)$ with $|Du|(\Omega) < +\infty$. For every $\varepsilon > 0$ there exists $v \in BV^{aj}(\Omega; \mathbb{R}^m) = SBV(\Omega; \mathbb{R}^m)$ such that $J_v = J_u$ and $v^+ - v^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on J_u, and*

$$\int_\Omega |v|\, dx \le \varepsilon \int_{J_u} |u^+ - u^-|\, d\mathcal{H}^{n-1}\,, \tag{3.1}$$
$$\int_\Omega |D^a v|\, dx \le (1+\varepsilon) \int_{J_u} |u^+ - u^-|\, d\mathcal{H}^{n-1}\,.$$

In particular, $u - v \in BV^{ac}_{loc}(\Omega; \mathbb{R}^m)$.

PROOF. – As J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, we may write

$$J_u = N_0 \cup \bigcup_{i=1}^{\infty} N_i\,,$$

with $\mathcal{H}^{n-1}(N_0) = 0$ and with each N_i contained in an $(n-1)$-dimensional C^1-manifold M_i for $i > 0$. We may assume without loss of generality that the sets N_i are pairwise disjoint and that each manifold M_i is the boundary of an open set U_i which lies on one side of M_i. We may choose ν_u so that $\nu_u(x)$ points towards U_i for each $x \in N_i$.

Let us fix $\varepsilon > 0$ and define $\varphi_i = u^+ - u^-$ on N_i and $\varphi_i = 0$ on $\partial U_i \setminus N_i$. By Gagliardo's extension theorem for sets with C^1-boundary (Theorem 2.4) there exists $v_i \in W^{1,1}(U_i;\mathbb{R}^m)$ such that

$$\int_{U_i} |v_i|\,dx \le \varepsilon \int_{\partial U_i} |\varphi_i| d\mathcal{H}^{n-1}\,,$$

$$\int_{U_i} |Dv_i|\,dx \le (1+\varepsilon) \int_{\partial U_i} |\varphi_i| d\mathcal{H}^{n-1}\,.$$

We set $v_i = 0$ on $\Omega \setminus U_i$. In this way $v_i \in SBV(\Omega;\mathbb{R}^m)$, $J_{v_i} = N_i$, $v_i^+ = \varphi_i$, $v_i^- = 0$, so that $v_i^+ - v_i^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on N_i. Moreover,

$$\text{(3.2)}\qquad \begin{aligned} &\int_{\Omega} |v_i|\,dx \le \varepsilon \int_{N_i} |u^+ - u^-| d\mathcal{H}^{n-1}\,,\\ &\int_{\Omega} |D^a v_i|\,dx \le (1+\varepsilon) \int_{N_i} |u^+ - u^-| d\mathcal{H}^{n-1}\,. \end{aligned}$$

Since $|Du|(\Omega) < +\infty$, we have

$$\sum_{i=1}^{\infty} \int_{N_i} |u^+ - u^-| d\mathcal{H}^{n-1} = \int_{J_u} |u^+ - u^-| d\mathcal{H}^{n-1} < +\infty\,,$$

hence the sequence $w_k = v_1 + \ldots + v_k$ converges strongly in $L^1(\Omega;\mathbb{R}^m)$ to some function v, while $D^a w_k$ converges strongly in $L^1(\Omega;\mathbb{R}^{m\times n})$ to some function ψ. Moreover, $w_k \in SBV(\Omega;\mathbb{R}^m)$ and, since the sets N_i are pairwise disjoint, we have $J_{w_k} = N_1 \cup \ldots \cup N_k$, and $w_k^+ - w_k^- = u^+ - u^-$ on $N_1 \cup \ldots \cup N_k$, hence $D^j w_k = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner \bigcup_{i=1}^{k} N_i$. This shows that $D^j w_k$ converges to $(u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$ strongly in the space of bounded Radon measures on Ω and thus Dw_k converges to the measure $\mu := \psi + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$. We conclude that $Dv = \mu$, hence $v \in SBV(\Omega;\mathbb{R}^m)$, $D^a v = \psi$, $D^j v = (u^+ - u^-)\mathcal{H}^{n-1} \llcorner J_u$, which implies $J_v = J_u$ and $v^+ - v^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on J_u, and thus $D^j(u - v) = 0$. This gives $u - v \in BV^{ac}_{loc}(\Omega)$. Inequalities (3.1) follow from (3.2). □

THEOREM 3.2. – *Let $u \in BV_{loc}(\Omega;\mathbb{R}^m)$ with $|Du|(\Omega) < +\infty$. For every $\varepsilon > 0$ there exists $v \in BV^{jc}(\Omega;\mathbb{R}^m)$ such that $J_v = J_u$ and $v^+ - v^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on J_u, and*

$$\text{(3.3)}\qquad \begin{aligned} &\int_{\Omega} |v|\,dx \le \varepsilon \int_{J_u} |u^+ - u^-|\, d\mathcal{H}^{n-1}\,,\\ &|D^c v|(\Omega) \le (1+\varepsilon) \int_{J_u} |u^+ - u^-|\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

In particular, $u - v \in BV^{ac}_{loc}(\Omega;\mathbb{R}^m)$.

PROOF. – In the previous proof it is enough to replace the extension to $W^{1,1}(U_i;\mathbb{R}^m)$ by the extension to $BV^c(U_i;\mathbb{R}^m)$, also provided by Theorem 2.4. $\square$

The last decomposition result of this type is just Alberti's Theorem [1] for BV functions.

THEOREM 3.3. – *For every $u \in BV_{loc}(\Omega;\mathbb{R}^m)$ with $|Du|(\Omega) < +\infty$ there exists $v \in BV^{aj}(\Omega;\mathbb{R}^m)$ such that $D^a v = D^a u$ a.e. in Ω, and hence $u - v \in BV^{jc}_{loc}(\Omega;\mathbb{R}^m)$.*

4. – Other decomposition results.

Let us consider now the space $BD(\Omega)$ of functions of bounded deformation (see [6]), i.e., the functions $u \in L^1(\Omega;\mathbb{R}^n)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a matrix-valued bounded Radon measure. As for functions in $BV(\Omega)$ we have the decomposition $Eu = E^a u + E^c u + E^j u$. We refer to [2] for the precise definition of $E^a u$, $E^c u$, $E^j u$, as well as for other fine properties of the functions in $BD(\Omega)$ and in its subspace $SBD(\Omega)$. As for $BV(\Omega)$ we introduce the following subspaces of $BD(\Omega)$:

$$BD^{aj}(\Omega) := \{u \in BD(\Omega) : Eu = E^a u + E^j u\} = SBD(\Omega),$$
$$BD^{ac}(\Omega) := \{u \in BD(\Omega) : Eu = E^a u + E^c u\},$$
$$BD^{jc}(\Omega) := \{u \in BD(\Omega) : Eu = E^j u + E^c u\}.$$

The following result can be obtained by adapting the proof of Theorem 3.1.

THEOREM 4.1. – *Let $u \in BD_{loc}(\Omega)$ with $|Eu|(\Omega) < +\infty$. For every $\varepsilon > 0$ there exists $v \in BV^{aj}(\Omega;\mathbb{R}^n) = SBV(\Omega;\mathbb{R}^n)$ such that $J_v = J_u$, $v^+ - v^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on J_u, and* (3.1) *holds. In particular, $u - v \in BD^{ac}_{loc}(\Omega)$.*

PROOF. – Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and $u^+ - u^-$ is $\mathcal{H}^{n-1}$-integrable on J_u (see [2]), as in the proof of Theorem 3.1, we find a function $v \in BV^{aj}(\Omega;\mathbb{R}^n)$ such that $J_v = J_u$, $v^+ - v^- = u^+ - u^-$ $\mathcal{H}^{n-1}$-a.e. on J_u, and (3.1) holds. Then $E^j(u - v) = 0$, so that $u - v \in BD^{ac}_{loc}(\Omega)$. $\square$

We recall that $LD(\Omega) = \{u \in BD(\Omega) : Eu = E^a u\}$. The following corollary can be easily deduced from Theorem 4.1.

COROLLARY 4.2. – *Given any $u \in SBD(\Omega)$ there exist $v \in LD(\Omega)$ and $w \in SBV(\Omega;\mathbb{R}^n)$ such that $u = v + w$.*

Acknowledgments. This work is part of the Project "Variational problems with multiple scales" 2006, supported by the Italian Ministry of University and Research.

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Received February 29, 2008 and in revised form March 5, 2008