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A Survey on Systems of Nonlinear Schrödinger Equations (*)

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Dedicated to the memory of Guido Stampacchia

Abstract. – *We survey some recent results dealing with some classes of systems of nonlinear Schrödinger equations.*

1. – Introduction.

In this paper we will survey some recent results dealing with systems of nonlinear Schrödinger equations like

$$\begin{cases} -u'' + \omega_1 u &= a_1 u^3 + \lambda F_u(u,v) \\ -v'' + \omega_2 v &= a_2 v^3 + \lambda F_v(u,v), \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\omega_i, a_i > 0$, $i = 1, 2$.

Systems of nonlinear Schrödinger equations (NLS) arise in Nonlinear Optics. For example, a planar light beam propagating in the z direction in a *self-focusing* medium, can be described by a NLS of the type

$$\mathrm{i}\, E_z + E_{xx} + |E|^2 E = 0, \quad (\mathrm{i} \text{ denotes the imaginary unit})$$

where $E(x,z)$ denotes the complex envelope of the electric field. Looking for stationary pulse-like solutions in the form $E(z,x) = e^{\mathrm{i}\omega z} u(x)$, one finds that the real valued function u satisfies, up to a re-scaling, the nonlinear ODE $-u'' + \omega u = u^3$, whose non-trivial even positive solution is given by

$$U_\omega(x) = \sqrt{\omega}\, U(\sqrt{\omega}\, x), \quad \text{where} \quad U(x) = \sqrt{2}\,\mathrm{sech}(x).$$

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If the propagation takes place in a dual-core coupler, instead of a single equation, one is led, up to a re-scaling, to a *linearly coupled* system of NLS equations like

$$\begin{cases} -u'' + u &= u^3 + \lambda v, \\ -v'' + v &= v^3 + \lambda u. \end{cases} \tag{1.2}$$

which is of the form (1.1) with $F(u, v) = uv$. Another class of systems as

$$\begin{cases} -u'' + \omega_1 u &= a_1^2 u^3 + \lambda v^2 u, \\ -v'' + \omega_2 v &= a_2^2 v^3 + \lambda u^2 v, \end{cases} \tag{1.3}$$

corresponding to the *nonlinear coupling* term $F(u, v) = \frac{1}{2}\, u^2 v^2$, arises when E is the sum of two right-hand, and left-hand, polarized waves. See also [3] where the case $F(u, v) = u^2 v^2 - uv$ is discussed. For further examples of systems of NLS in Nonlinear Optics, we refer to the book [1].

In the rest of this paper we will mainly survey some recent results from [3, 4, 5, 6, 7] focusing on (1.2) and (1.3), which have new interesting features. The former system has many families of solutions and bifurcation phenomena arise. Furthermore, there exist solutions with two or more bumps, a phenomenon which usually takes place dealing with a single NLS equation with external potentials. In the case of a nonlinear coupling, one main question is to find solution pairs whose components are both not trivial. Finally, we can prove a suitable *concentration-compactness* Lemma to deal with non-autonomous linearly coupled systems. The application of this Lemma requires a sharp analysis of the solutions of the autonomous system corresponding to *the problem at infinity*.

2. – Bound and ground states.

A solution pair (u, v) of (1.1) is called a *bound state* if $u, v \in H := W^{1,2}(\mathbb{R}) \times W^{1,2}(\mathbb{R})$. Hereafter, the Sobolev space $W^{1,2}(\mathbb{R})$ is endowed with the standard scalar product and norm

$$\langle u \mid v \rangle = \int_R (u'v' + uv)dx, \qquad \|u\|^2 = \langle u \mid u \rangle,$$

whereas the scalar product in H is given by

$$\langle (u, v) \mid (\widetilde{u}, \widetilde{v}) \rangle = \langle u \mid \widetilde{u} \rangle + \langle v \mid \widetilde{v} \rangle.$$

Bound states of (1.1) are the critical points of the functional

$$I_\lambda(u, v) = I_1(u) + I_2(u) - \lambda \int_{\mathbb{R}} F(u, v)dx,$$

where it is assumed that $F(u,v) \in L^1(\mathbb{R})$ for all $(u,v) \in H$ and

$$I_i(u) = \frac{1}{2}\int_{\mathbb{R}} [|u'|^2 + \omega_i u^2]dx - \frac{1}{4}\int_{\mathbb{R}} a_i u^4 dx, \qquad i = 1,2.$$

Of course, since $\omega_i > 0$, the quantity $\int_{\mathbb{R}} [|u'|^2 + \omega_i u^2]dx$ defines a norm $\|u\|_i^2$ equivalent to the usual one and we can write

$$I_i(u) = \frac{1}{2}\|u\|_i^2 - \frac{1}{4}\int_{\mathbb{R}} a_i u^4 dx, \qquad i = 1,2.$$

In the sequel we will use the same symbol I_λ to denote both the Euler functional of (1.2) and of (1.3). In the former case, $\omega_i = a_i = 1$, $i = 1,2$ and $F(u,v) = uv$.

A bound state (u,v) of (1.1) is said *non-trivial*, resp. positive, if $(u,v) \neq (0,0)$, resp. $u > 0, v > 0$. The set of non-trivial bound states of (1.1) will be denoted by $\mathfrak{B}_\lambda$. A pair $(\widetilde{u},\widetilde{v}) \in \mathfrak{B}_\lambda$, is called a *ground state* if

$$I_\lambda(\widetilde{u},\widetilde{v}) = \min\{I_\lambda(u,v) : (u,v) \in \mathfrak{B}_\lambda\}.$$

The relevance of the ground states relies on the fact that they are the natural candidates to be orbitally stable for the corresponding evolution equation, see e.g. [10, 13, 18].

In order to find ground states it is convenient to consider the *Nehari manifold*

$$M_\lambda = \{(u,v) \in H \setminus (0,0) : \langle I'_\lambda(u,v) \mid (u,v)\rangle = 0\}.$$

Obviously, for all $\lambda > 0$, $\mathfrak{B}_\lambda \subset M_\lambda$. Let us first consider the case of (1.2). One finds that $(u,v) \in H \setminus (0,0)$ belongs to M_λ whenever

$$\|u\|^2 + \|v\|^2 = \int_{\mathbb{R}} (u^4 + v^4)dx + 2\lambda \int_{\mathbb{R}} uv\,dx. \tag{2.1}$$

It follows that for every $\lambda \in (0,1)$, M_λ as the following properties:

($M1$) $\exists\, \rho_\lambda > 0$ such that $\|(u,v)\| \geq \rho_\lambda$, $\forall\,(u,v) \in M_\lambda$.

($M2$) M_λ is a smooth manifold of codimension one in H and $\forall\,(u,v) \in H \setminus (0,0)$ there exists a unique $t > 0$ such that $(tu,tv) \in M_\lambda$.

($M3$) M_λ is that it is a *natural constraint*, in the sense that $(u,v) \in \mathfrak{B}_\lambda$ whenever (u,v) is a critical point of I_λ constrained on M_λ.

Similarly, in the case of (1.3), one has that $(u,v) \in M_\lambda$ provided

$$\|u\|_1^2 + \|v\|_2^2 = \int_{\mathbb{R}} (a_1 u^4 + a_2 v^4)dx + 2\lambda \int_{\mathbb{R}} u^2 v^2\,dx. \tag{2.2}$$

In this case M_λ satisfies the properties $(M1-2-3)$ for all $\lambda > 0$.

Moreover, I_λ is bounded from below on M_λ. Actually, using (2.1), resp. (2.2), one finds

$$I_\lambda(u,v) = \frac{1}{4}(\|u\|^2 + \|v\|^2), \qquad \forall\,(u,v) \in M_\lambda,\ \forall\,\lambda \in (0,1), \tag{2.3}$$

respectively,

$$I_\lambda(u,v) = \frac{1}{4}(\|u\|_1^2 + \|v\|_2^2), \quad \forall\,(u,v) \in M_\lambda,\ \forall\,\lambda > 0. \tag{2.4}$$

Let

$$m_\lambda = \inf\{I_\lambda(u,v) : (u,v) \in M_\lambda\}.$$

From the preceding properties, we infer that in order to find a ground state of (2.1), resp. (2.2), it suffices to show that m_λ is achieved.

3. – Existence of ground states.

In this Section we discuss the existence of ground states for (1.2) and (1.3).

First we deal with the case of a linear coupling $F(u,v) = uv$, proving the existence of a ground state of (1.2) for all $\lambda \in (0,1)$. Roughly, we argue as follows. Let $(u_n, v_n) \in M_\lambda$ be such that $I_\lambda(u_n, v_n) \to m_\lambda$. Consider the pair $(|u_n|, |v_n|)$ and let $t_n \in \mathbb{R}$ be such that $(t_n|u_n|, t_n|v_n|) \in M_\lambda$. Using (2.1) we find that $t_n > 0$ and satisfies

$$\begin{aligned} t_n^2 &= \frac{\|\,|u_n|\,\|^2 + \|\,|v_n|\,\|^2 - 2\lambda \int_{\mathbb{R}} |u_n|\,|v_n| dx}{\int_{\mathbb{R}} \left[|u_n|^4 + |v_n|^4\right] dx} \\ &\le \frac{\|u_n\|^2 + \|v_n\|^2 - 2\lambda \int_{\mathbb{R}} u_n v_n dx}{\int_{\mathbb{R}} [u_n^4 + v_n^4] dx} = 1. \end{aligned}$$

Using (2.3) we get $I_\lambda(t_n|u_n|, t_n|v_n|) = t_n^2 I_\lambda(u_n, v_n) \le I_\lambda(u_n, v_n)$ and hence we can assume that u_n and v_n do not change sign. A similar argument allows us to suppose that, in addition, u_n and v_n are Steiner symmetric. Otherwise, denoting by u_n^*, v_n^* the Steiner symmetrization of u_n, resp. v_n, and letting t_n^* be such that $(t_n^* u_n^*, t_n^* v_n^*) \in M_\lambda$ one finds that $t_n^* \in (0,1]$ yielding

$$I_\lambda(t_n^* u_n^*, t_n^* v_n^*) \le I_\lambda(u_n, v_n).$$

It is now easy to check that, up to a subsequence, (u_n, v_n) converges strongly to some $(u_\lambda, v_\lambda) \in M_\lambda$, such that $I_\lambda(u_\lambda, v_\lambda) = m_\lambda$.

If we consider the nonlinear coupling $F(u,v)=\frac{1}{2}u^2v^2$, we can repeat the same arguments, by using (2.2) and (2.4). In conclusion we have:

PROPOSITION 3.1. – *For all* $\lambda \in (0,1)$*,* (1.2) *has a ground state: there exists* $(u_\lambda, v_\lambda) \in M_\lambda$ *such that* $I_\lambda(u_\lambda, v_\lambda) = m_\lambda$.

In the case of (1.3), *the same result holds for all* $\lambda > 0$.

4. – The linearly coupled system (1.2).

In this section we will discuss in more details the linearly coupled system (1.2). First of all, let us list some facts about the solutions (u,v) of (1.2).

(i_1) For all $\lambda \geq 0$, if (u,v) is a solution of (1.2), then also $(-u,-v)$ does.

(i_2) If $\lambda = 0$, (1.2) has the following four non-trivial solutions in H: $(U,0)$, $(0,U)$, (U,U), $(U,-U)$ and their antipodal pairs $(-U,0)$, $(0,-U)$, $(-U,-U)$, $(-U,U)$. By a direct inspection, one checks that the ground states are given by $(\pm U,0)$ and $(0,\pm U)$, namely $m_0 = I_0(\pm U,0) = I_0(0,\pm U)$.

(i_3) For any $\lambda > 0$, if $(u,v) \neq (0,0)$ is a solution of (1.2), then both the components have to be different from zero.

(i_4) For any $\lambda \in (0,1)$, if (u_λ, v_λ) is any ground state of (1.2) then $u_\lambda \cdot v_\lambda > 0$ and, up to a translation, u_λ and v_λ are Steiner symmetric, see [5, Lemma 3.6].

(i_5) The map $[0,1) \ni \lambda \mapsto m_\lambda$ is strictly decreasing and continuous, see [5, Lemma 6.2].

Moreover, the set $\mathfrak{B}_\lambda$ of non-trivial bound states of (1.2) has several families of solutions.

First of all, there are two families of explicit solutions: the *symmetric states* $(U_{1-\lambda}, U_{1-\lambda})$, for all $\lambda \in (0,1)$; and the *anti-symmetric states* $(U_{1+\lambda}, -U_{1+\lambda})$, for all $\lambda > 0$.

In addition, using the *Implicit Function Theorem*, one easily checks that a unique branch of non-trivial bound states of (1.2) emanates from each of the bound states for $\lambda = 0$, see (i_2) before. Of course the branch starting from (U,U), resp. $(U,-U)$, is given (locally) by the family $(U_{1-\lambda}, U_{1-\lambda})$, resp. $(U_{1+\lambda}, -U_{1+\lambda})$. It has also been proved in [4] that at $\lambda = 3/5$ there is a unique secondary bifurcation from the symmetric states $(U_{1-\lambda}, U_{1-\lambda})$, see [4, Lemma 3.2].

REMARK 4.1. – It would be interesting to show that the branch bifurcating from the family $(U_{1-\lambda}, U_{1-\lambda})$ is a curve having one endpoint at $(U,0)$, in such a way that locally it coincides with the branch emanating from $(U,0)$. □

In order to find which one, among the preceding bound states, is a ground state we first point out that, by (i_5) and (i_2) it follows that

$$\lim_{\lambda \to 0} m_\lambda = m_0 = I_0(\pm U, 0) = I_0(0, \pm U). \tag{4.1}$$

Since $m_0 < I_0(U,U) = \lim_{\lambda\to 0} I_\lambda(U_{1-\lambda},U_{1-\lambda})$, (4.1) implies that

$$m_\lambda < I_\lambda(U_{1-\lambda},U_{1-\lambda}), \quad \forall\, \lambda > 0,\ \lambda \sim 0.$$

More precisely, one can prove that there exists $\delta > 0$ such that for $\lambda \in (0,\delta)$, the branches bifurcating from $(\pm U, 0)$ or $(0, \pm U)$ are, up to translations, the ground states of (1.2), see [5, Lemma 3.8]. Moreover, one can show that there exists $\delta' > 0$ such that for $\lambda \in (1-\delta', 1)$ the ground states of (1.2) are given, up to translations, by the symmetric states $(U_{1-\lambda},U_{1-\lambda})$ or $(-U_{1-\lambda}, -U_{1-\lambda})$, see [5, Lemma 3.11]. In particular, we get $\lim_{\lambda\to 1} m_\lambda = 0$.

REMARK 4.2. – It is an open problem to show that the ground states coincide with:

- the branches emanating from $(\pm U, 0)$, $(0, \pm U)$ for all $0 < \lambda < 3/5$;
- the families $(U_{1-\lambda},U_{1-\lambda})$, $(-U_{1-\lambda}, -U_{1-\lambda})$, for all $3/5 < \lambda < 1$. □

We end this section by stating a result from [7] dealing with the existence of *multi-bump* solutions of (1.2). It is worth pointing out that, multi-bump solutions have been broadly investigated in the case of semi-classical states of a single NLS equation in the presence of an external potential. Here the new fact is that, dealing with linearly coupled systems, multi-bump solutions arise in the autonomous case.

THEOREM 4.3 ([7, Theorem 1.1]). – *There exists $\varepsilon > 0$ such that if $0 < \lambda < \varepsilon$, system (1.2) has a solution $(u^*_\lambda, v^*_\lambda) \in H$ such that $u^*_\lambda \sim U(x+\xi_\lambda) + U(x-\xi_\lambda)$, $v^*_\lambda \sim -U(x)$ as $\lambda \to 0$, where $\xi_\lambda \sim \log(1/\lambda)$.*

The existence of this solution pairs relies on a perturbation method, variational in nature, which is usually employed to find multi-bump *semiclassical states* for a single NLS equation in the presence of an external potential, see e.g. [8]. Here, the system is autonomous and, roughly, the counterpart of the external potential is played by the coupling term. The proof requires that the second component is negative. Actually, a negative coupling term is "attractive" and can be used to balance the repulsive effect of the two bumps of the first component.

REMARK 4.4. – We conjecture that the family $(u^*_\lambda, v^*_\lambda)$ can be continued for all $\lambda \in (0,1)$. The bumps ξ_λ of the first component should tend to zero and $(u^*_\lambda, v^*_\lambda)$ should converge to $(U_{1+\lambda}, -U_{1+\lambda})$ for $\lambda = 1$, in such a way that $\lambda = 1$ is a secondary bifurcation for the family of the anti-symmetric states. The existence of such a bifurcation has been shown only by means of some numerical argument, see [2], whereas a rigorous proof has not been provided, yet. □

5. – The nonlinearly coupled system (1.3).

In contrast to the linearly coupled case (see property (i_3) above), system (1.3) has solutions with one trivial component (these solutions will be called *semi-trivial*). For example, if $(u,0)$ is a semi-trivial solution, then u solves $-u''+\omega_1 u = a_1 u^3$ and hence

$$u = \widetilde{U}_1(x) = \frac{1}{\sqrt{a_1}}\; U_{\omega_1}(x) = \sqrt{\frac{2\omega_1}{a_1}}\,\mathrm{sech}(\sqrt{\omega_1}x).$$

Similarly, if $(0,v)$ is a solution of (1.3), then

$$v = \widetilde{U}_2(x) = \frac{1}{\sqrt{a_2}}\; U_{\omega_2}(x) = \sqrt{\frac{2\omega_2}{a_2}}\,\mathrm{sech}(\sqrt{\omega_2}x).$$

In order to establish if these semi-trivial solutions are the ground states found in Proposition 3.1, we follow [6]. Let γ_i, $i=1,2$ be given by

$$\gamma_1^2 = \inf_{\varphi\in W^{1,2}(\mathbb{R})\setminus\{0\}} \frac{\|\varphi\|_2^2}{\int \widetilde{U}_1^2\varphi^2}, \qquad \gamma_2^2 = \inf_{\varphi\in W^{1,2}(\mathbb{R})\setminus\{0\}} \frac{\|\varphi\|_1^2}{\int \widetilde{U}_2^2\varphi^2}.$$

The values γ_j can be estimated as follows, see [6, Lemma 5.5]

$$a_j\left(\frac{\omega_i}{\omega_j}\right)^{\frac{3}{4}} \le \gamma_j^2 \le \max\left\{a_j\left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{2}}, a_j\frac{\omega_i}{\omega_j}\right\}, \quad i,j=1,2,\ i\neq j. \tag{5.1}$$

Moreover, letting $\Lambda = \min\{\gamma_1^2,\gamma_2^2\}$ and $\Lambda' = \max\{\gamma_1^2,\gamma_2^2\}$, one shows that

LEMMA 5.1. – (*i*) $\forall\,\lambda<\Lambda$, $(\widetilde{U}_1,0)$ *and* $(0,\widetilde{U}_2)$ *are strict local minima of* I_λ *on* M_λ.

(*ii*) $\forall\,\lambda>\Lambda'$, $(\widetilde{U}_1,0)$ *and* $(0,\widetilde{U}_2)$ *are saddle points of* I_λ *on* M_λ.

The former property, jointly with Proposition 3.1, implies that I_λ has a saddle point $(\widehat{u}_\lambda,\widehat{v}_\lambda)\in\mathfrak{B}_\lambda$, which cannot be semi-trivial and this allows us to prove that $\widehat{u}_\lambda>0, \widehat{v}_\lambda>0$.

On the other hand, from (*ii*) it follows that the minimum of I_λ on M_λ is achieved by a pair (u_λ,v_λ) whose components are both different from zero, namely (u_λ,v_λ) is a ground state. Moreover, one can prove that $u_\lambda>0, v_\lambda>0$.

We collect these results in the following theorem, see [6]; for a result similar in nature see also [14, 15].

THEOREM 5.2. – (*i*) *If* $\lambda\in(0,\Lambda)$, *then* (1.3) *has a positive bound state* $(\widehat{u}_\lambda,\widehat{v}_\lambda)$.

(*ii*) *If* $\lambda>\Lambda'$ *then* (1.3) *has a positive ground state* (u_λ,v_λ).

In the preceding statement, both the components are, up to translation, Steiner symmetric. Moreover, Theorem 5.2 can be used to given an answer, though still not complete, to the question raised in Remark 4.2. Actually, let (u, v) be a solution of (1.2) and set $\phi = u + v$, $\psi = u - v$. Then ϕ, ψ satisfy

$$(5.2) \qquad \begin{cases} -\phi'' + (1-\lambda)\phi & = \dfrac{1}{4}\phi^3 + \dfrac{3}{4}\phi\psi^2 \\ -\psi'' + (1+\gamma)\psi & = \dfrac{1}{4}\psi^3 + \dfrac{3}{4}\psi\phi^2, \end{cases}$$

which is nothing but a specific case of (1.3) with $\omega_1 = 1 - \lambda$, $\omega_2 = 1 + \lambda$, $a_1 = a_2 = 1/4$ and coupling coefficient $= 3/4$. For $0 < \lambda < 1$, the semi-trivial solutions $(2U_{1-\lambda}, 0)$, correspond to the symmetric states $(U_{1-\lambda}, U_{1-\lambda})$ of (1.2). Here, the estimates (5.1) become $\Lambda \le 1/2$ and $\Lambda' \ge \lambda^*$, where $\lambda^* > 0$ is such that

$$\left(\frac{1+\lambda^*}{1-\lambda^*}\right)^{3/4} = 3, \qquad (\lambda^* \sim 0.62).$$

Using Theorem 5.2, we infer that for $0 < \lambda < 1/2$, $(2U_{1-\lambda}, 0)$, and hence $(U_{1-\lambda}, U_{1-\lambda})$, is not a ground state, whereas for $\lambda^* < \lambda < 1$, $(2U_{1-\gamma}, 0)$, and hence $(U_{1-\lambda}, U_{1-\lambda})$, is a ground state.

REMARK 5.3. – Since $\lambda^* > 3/5$, a comparison between the preceding discussion and what conjectured in Remark 4.2, highlights that the results of Theorem 5.2 are not sharp. □

6. – The PDE case.

All the results discussed in the preceding sections can be extended to the PDE case, namely to systems as

$$\begin{cases} -\Delta u + \omega_1 u & = a_1 u^p + \lambda F_u(u, v) \\ -\Delta v + \omega_2 v & = a_2 v^p + \lambda F_v(u, v), \end{cases}$$

where $(u, v) \in H := W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$, $n \ge 2$, $1 < p < 2^* - 1$, and F is such that $F(u, v) \in L^1(\mathbb{R}^n)$, for all $(u, v) \in H$. Let us remark that in the case of the nonlinear coupling $F(u, v) = \frac{1}{2}u^2 v^2$, this implies that $n = 2, 3$.

In particular, it is interesting to see what is the counterpart of Theorem 4.3 in dimension $n = 2, 3$. We will follow [7].

Consider the linearly coupled system

$$(6.1) \qquad \begin{cases} -\Delta u + u & = u^3 + \lambda v \\ -\Delta v + v & = v^3 + \lambda u, \end{cases}$$

and let $\mathcal{P}$ be any regular polygon in $\mathbb{R}^2$ or any platonic solid in $\mathbb{R}^3$, centered at the origin. Let $P_i \neq 0,\ i = 1, \dots, m$, denote the vertices of $\mathcal{P}$, and set $s = \min\{|P_i - P_j| : i \neq j\}$ and $r = |P_i|$ (we include the degenerate case of a segment $[-P, P]$).

THEOREM 6.1. – *If $r < s$, then there exists $\varepsilon > 0$ such that if $0 < \lambda < \varepsilon$, system* (6.1) *has a solution $(u_\lambda, v_\lambda) \in W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ such that $v_\lambda \sim -U(x)$, whereas u_λ is a multi-bump with maxima located near $\xi_\lambda P_i$, namely $u_\lambda(x) \sim \sum_1^m U(x + \xi_\lambda P_i) + U(x - \xi_\lambda P_i)$, where ξ_λ satisfies*

$$\xi_\lambda \sim \frac{\log(1/\lambda)}{s - r}.$$

REMARK 6.2. – The condition $r < s$ is satisfied by the regular polygons in $\mathbb{R}^2$ with less than 6 sides and by all the regular polyhedra in $\mathbb{R}^3$ with the exception of the dodecahedron. In the degenerate case of two vertices, one has $s = 2r$, and we recover the result stated in Theorem 4.3. □

7. – Non-autonomous systems.

In this final section we deal with systems in the presence of external potentials. We focus on some recent results of [4, 5].

Let us consider the system

$$\begin{cases} -\Delta u + u = (1 + a(x))|u|^{p-1}u + \lambda v, \\ -\Delta v + v = (1 + b(x))|v|^{p-1}v + \lambda u, \end{cases} \tag{7.1}$$

where $n \geq 2$, $1 < p < 2^* - 1$ and a, b satisfy

$$a, b \in L^\infty(\mathbb{R}^n), \qquad \lim_{|x|\to\infty} a(x) = \lim_{|x|\to\infty} b(x) = 0,$$

$$\inf_{\mathbb{R}^n}\{1 + a(x)\} > 0, \qquad \inf_{\mathbb{R}^n}\{1 + b(x)\} > 0.$$

If the preceding conditions hold, one can prove the following results:

THEOREM 7.1. – (*i*) *If $a(x) + b(x) \geq 0$, then for every $0 < \lambda < 1$,* (7.1) *has a positive ground state.*

(*ii*) *If $a(x) \geq 0,\ a(x) \not\equiv 0$, resp. $b(x) \geq 0,\ b(x) \not\equiv 0$, then there exists $\lambda^* \in (0, 1)$ depending only on a, resp. b, such that* (7.1) *has a positive ground state for all $\lambda \in (0, \lambda^*]$.*

(*iii*) *If $a \leq 0$, $b \leq 0$, $a + b \not\equiv 0$, then there exists $0 < \lambda_1 \leq \lambda_2 < 1$ such that* (7.1) *has a positive bound state for all $\lambda \in (0, \lambda_1) \cup (\lambda_2, 1)$, provided* $\max\{|a|_\infty, |b|_\infty\}$ *is sufficiently small.*

The proofs are based upon concentration-compactness arguments, the *problem at infinity* being (7.1) with $a = b = 0$, namely (1.2). Roughly, the Euler functional of (7.1) is given by

$$\Phi_\lambda(u,v) = I_\lambda(u,v) - \frac{1}{p+1}\int_{\mathbb{R}^n}(a(x)|u|^{p+1} + b(x)|v|^{p+1})dx, \quad (u,v) \in H,$$

where I_λ denotes the Euler functional of (1.2). Consider the *Nehari* manifold corresponding to Φ_λ,i.e.

$$N_\lambda = \{(u,v) \in H \setminus (0,0) : \langle \Phi_\lambda'(u,v) \mid (u,v)\rangle = 0\},$$

and set

$$c_\lambda = \inf\{\Phi_\lambda(u,v) : (u,v) \in N_\lambda\}.$$

The key ingredient is the following lemma (for the proof, see [5, Section 4]).

LEMMA 7.2. – *Φ_λ satisfies the* (*PS*) *condition on N_λ at any level smaller than m_λ. In particular, if $c_\lambda < m_\lambda$ then* (7.1) *has a positive ground state.*

Next, to prove (i) of Theorem 7.1, one shows that $a(x) + b(x) \geq 0 \;\Rightarrow\; c_\lambda < m_\lambda$.

To prove (ii) one takes into account a different problem at infinity, namely the system

$$\begin{cases} -\Delta u + u &= a(x)|u|^{p-1}u, \\ -\Delta v + v &= |v|^{p-1}v. \end{cases}$$

Letting m_a denote the ground state level of the preceding system, it is possible to check that $c_\lambda < m_a < m_{\lambda=0}$. Using the property ($i_5$) of m_λ, (ii) follows.

The proof of (iii) is more delicate, because if $a \leq 0$, $b \leq 0$ and $a + b \not\equiv 0$ then one proves that $c_\lambda = m_\lambda$ and that no ground state exists. As a consequence, bound states have to be searched on N_λ by a min-max procedure. To carry out this program, we need first to investigate the (*PS*) condition, proving

LEMMA 7.3. – *Assume that m_λ is an isolated critical level of I_λ and let $\widehat{m}_\lambda$ denote the smallest critical level of I_λ greater than m_λ. Then Φ_λ satisfies the* (*PS*) *condition on N_λ at any level d such that $m_\lambda < d < \min\{\widehat{m}_\lambda, 2m_\lambda\}$.*

A sharp analysis of the ground states of (1.2), based also on the discussion carried out after Remark 4.1, allows us to prove that the assumption that m_λ is an isolated critical level of I_λ, holds true for all $\lambda \sim 0$ and $\lambda \sim 1$.

Finally, to conclude the proof of (iii), one defines a min-max level $d_\lambda > m_\lambda$ using the method of *barycenter*, see [9, 11, 12] and shows

LEMMA 7.4. – *If* $\max\{|a|_\infty, |b|_\infty\} \ll 1$ *then* $m_\lambda < d_\lambda < \min\{\widehat{m}_\lambda, 2m_\lambda\}$.

REMARK 7.5. – Singularly perturbed, nonlinearly coupled, systems in $\mathbb{R}^3$ with external potentials as

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u^3 + \beta v^2 u, \\ -\varepsilon^2 \Delta v + W(x)v = v^3 + \beta u^2 v, \end{cases}$$

have been recently investigated in order to find spikes, namely solutions which concentrate at a point as $\varepsilon \to 0$. See [16] and [17], which deals with the case $\beta < 0$. The existence of spikes in the presence of a linearly coupling term, is studied in [4, Section 5]. □

We conclude this section by stating a perturbation result, see [4], dealing with the existence of solutions of

$$\begin{cases} -u'' + u &= (1 + \varepsilon a(x))u^3 + \lambda v, \\ -v'' + v &= (1 + \varepsilon b(x))v^3 + \lambda u, \end{cases} \tag{7.2}$$

near the family of symmetric states. First, one proves that for all $\lambda \in (0, 1)$, $\lambda \neq 3/5$, $(U_{1-\lambda}, U_{1-\lambda})$ are non-degenerate in an appropriate sense. Then one uses the perturbation techniques from [8] yielding

THEOREM 7.6. – *For all* $\lambda \in (0, 1)$, $\lambda \neq 3/5$, (7.2) *has a solution near* $(U_{1-\lambda}, U_{1-\lambda})$, *provided* ε *is sufficiently small.*

It is an open problem to extend the preceding result to the PDE case.

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