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Bounded Solutions for the Degasperis-Procesi Equation (*)

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Abstract. – *This paper deals with the well-posedness in $L^1 \cap L^\infty$ of the Cauchy problem for the Degasperis-Procesi equation. This is a third order nonlinear dispersive equation in one spatial variable and describes the dynamics of shallow water waves.*

1. – Introduction.

In this paper we investigate the wellposedness in $L^1 \cap L^\infty$ of the Cauchy problem for the *Degasperis-Procesi* equation

$$\partial_t u - \partial^3_{txx} u + 4u\partial_x u = 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \qquad (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \tag{1}$$

endowed with the initial condition u_0:

$$u(0,x) = u_0(x), \qquad x \in \boldsymbol{R}, \tag{2}$$

where we assume that

$$u_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R}). \tag{3}$$

Degasperis and Procesi [9] deduced (1) studying the following family of third order dispersive nonlinear equations, indexed over six constants $a, \gamma, c_0, c_1, c_2, c_3 \in \boldsymbol{R}$:

$$\partial_t u + c_0 \partial_x u + \gamma \partial^3_{xxx} u - a^2 \partial^3_{txx} u = \partial_x \big(c_1 u^2 + c_2 (\partial_x u)^2 + c_3 u \partial^2_{xx} u\big).$$

Using the method of asymptotic integrability, they found that only three equations within this family were asymptotically integrable up to the third order: the *KdV equation* ($a = c_2 = c_3 = 0$), the *Camassa-Holm equation* $\left(c_1 = -\frac{3c_3}{2a^2}, c_2 = \frac{c_3}{2}\right)$, and one new equation $\left(c_1 = -\frac{2c_3}{a^2}, c_2 = c_3\right)$, which properly scaled reads

$$\partial_t u + \partial_x u + 6u\partial_x u + \partial^3_{xxx} u - a^2 \left(\partial^3_{txx} u + \frac{9}{2}\partial_x u \partial^2_{xx} u + \frac{3}{2} u \partial^3_{xxx} u\right) = 0. \tag{4}$$

(*) Comunicazione di 30 minuti tenuta a Bari il 26 settembre 2007 in occasione del XVIII Congresso dell'Unione Matematica Italiana.

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, equation (4) can be transformed into the form (1), see [7, 8] for details.

Let us spend some words on the Korteweg-deVries (KdV) and Camassa-Holm equations. The first one models weakly nonlinear unidirectional long waves, and arises in various physical contexts. In particular, it models surface waves of small amplitude and long wavelength on shallow water: $u(t, x)$ represents the wave height above a flat bottom, with x being proportional to the distance in the propagation direction and t being proportional to the elapsed time. The KdV equation is completely integrable and possesses solitary waves that are solitons. The wellposedness of the initial value problem for the this equation is well studied, see [19] and the references cited therein. In particular, it is globally weellposed in $H^1(\boldsymbol{R})$.

Camassa and Holm [1] deduced their equation as a model for the the propagation of unidirectional shallow water waves on a flat bottom: $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [1, 20].

It is a water wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations, while the KdV equation appears at first order in this expansion [1, 20]. In another interpretation, Dai [11] derived the same equation as a model for finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The Camassa-Holm equation possesses many interesting properties, among which we highlight its bi-Hamiltonian structure (an infinite number of conservation laws) [1, 15] and that it is completely integrable [1]. Moreover, it has an infinite number of non-smooth solitary wave solutions called peakons (since at the wave peak they are continuous but not C^1), which interact like solitons and are stable. We point out that the KdV equation admits solitary waves that are solitons too, but it does not model wave breaking because its solitions are smooth. The Camassa-Holm equation is remarkable in the sense that it admits soliton solutions and at the same time allows wave breaking. For a discussion of the Camassa-Holm equation as well as other related equations and a complete list of references, see the recent paper [17]. From a mathematical point of view the Camassa-Holm equation is rather well studied. There are several results on local wellposedness, global existence for a certain class of initial data, blow up in finite time for a large class of initial data, and existence and uniqueness results for global weak solutions, a complete list of refecences can be found in [2].

Let us now turn to the Degasperis-Procesi equation (1). As mentioned before, it was singled out first in [9] by an asymptotic integrability test within a family of nonlinear third order dispersive equations. In [8] Degasperis, Holm, and Hone proved the exact integrability of (1) by constructing a Lax pair. Moreover, they displayed a relation to a negative flow in the Kaup-Kupershmidt hierarchy by a reciprocal transformation and derived two infinite sequences of conserved quantities along with a bi-Hamiltonian structure. They also showed that the Degasperis-Procesi equation possesses "non-smooth" solutions that are superpositions of multipeakons and described the integrable finite-dimensional peakon dynamics, which were compared with the multipeakon dynamics of the Camassa-Holm equation. An explicit solution was also found in the perfectly anti-symmetric peakon-antipeakon collision case. Lundmark and Szmigielski [22] presented an inverse scattering approach for computing n-peakon solutions to (1). Mustafa [24] proved that smooth solutions to (1) have infinite speed of propagation: they lose instantly the property of having compact support. Regarding wellposedness (in terms of existence, uniqueness, and stability of solutions) of the Cauchy problem for the Degasperis-Procesi equation (1), Escher, Liu, and Yin have studied this within certain functional classes in a series of recent papers [12, 13, 14, 21, 27, 28, 29, 30]. In particular, those results concern continuous solutions, that is consequence of the Sobolev regularity they assume on the initial condition.

In [4] we proved the well posedness of the so called entropy solutions, that will be defined later, within the class of the functions with bounded variation. Here we want to prove an analogous result within the class of bounded functions. One of the motivations is the uniqueness principle proved in [5] based on an Oleinik type estimate for L^∞ solutions to (1). In the present paper we sharpen the Oleinik estimate of [4], that involves the total variation of the initial datum. In this way we have the perfect equivalence between the infinite family of entropy inequality and the one-side Lipshitz inequality.

Finally, we recall that the existence of a semigroup of solutions generated by L^2 initial conditions and the convergence of numerical schemes for (1) were proved in [6, 10, 16].

Formally, problem (1), (2) is equivalent to the hyperbolic-elliptic system

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2} \right) + \partial_x P = 0, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ -\partial_{xx}^2 P + P = \dfrac{3}{2} u^2, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}. \end{cases} \tag{5}$$

Since $e^{-|\xi|}/2$ is the Green's function of the differential operator $1 - \partial_{xx}^2$ the function P has a convolution structure:

$$P(t,x) = P^u(t,x) := \frac{3}{4} \int_{\boldsymbol{R}} e^{-|x-y|} u^2(t,y) dy, \tag{6}$$

and (5) can be written as a conservation law with a nonlocal flux function:

$$
(7)\qquad \begin{cases} \partial_t u + \partial_x \left[\frac{u^2}{2} + \frac{3}{4} \int_{\boldsymbol{R}} e^{-|x-y|} u^2(t,y) dy \right] = 0, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}. \end{cases}
$$

Following [4], our starting point is that formally there is an L^2 bound on the solution in terms of the L^2 norm of the initial data u_0. Indeed, if we introduce the quantity $v := (4 - \partial^2_{xx})^{-1} u$, then formally the following conservation law can be derived:

$$
(8)\qquad \begin{aligned} &\partial_t \Big((\partial^2_{xx} v)^2 + 5(\partial_x v)^2 + 4v^2 \Big) \\ &+ \partial_x \left(\frac{2}{3} u^3 + 4v\,(1 - \partial^2_{xx})^{-1}(u^2) + \partial_x v\, \partial_x \big[(1 - \partial^2_{xx})^{-1}(u^2) \big] - 4u^2 v \right) = 0. \end{aligned}
$$

It follows from this that $v \in L^\infty(\boldsymbol{R}_+; H^2(\boldsymbol{R}))$ and thereby also $u \in L^\infty(\boldsymbol{R}_+; L^2(\boldsymbol{R}))$. The L^2 estimate on u is the key to deriving a series of other (formal) estimates, among which we highlight

$$
(9)\qquad P \in L^\infty(\boldsymbol{R}_+; W^{1,\infty}(\boldsymbol{R})), \quad \partial^2_{xx} P \in L^\infty([0,T]; L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})), \quad T > 0,
$$

and

$$
u \in L^\infty([0,T]; L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})), \qquad T > 0,
$$

where the L^∞ estimate is particularly important as it implies a one side Lipshitz inequlity on u (that is an upper bound for $\partial_x u$).

To prove existence of a global weak solution we construct a family of approximate smooth solutions for which similar bounds can be derived rigorously. To this end, we introduce the smooth solutions u_ε of the following fourth order viscous approximation of the Degasperis-Procesi equation (1):

$$
(10)\qquad \partial_t u_\varepsilon - \partial^3_{txx} u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon = 3\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon - \varepsilon \partial^4_{xxxx} u_\varepsilon.
$$

This equation can be written in the more suggestive form of a viscous conservation law with a non-local flux:

$$
(11)\qquad \partial_t u_\varepsilon + \partial_x \left[\frac{u_\varepsilon^2}{2} + \frac{3}{4} \int_{\boldsymbol{R}} e^{-|x-y|} u_\varepsilon^2(t,y) dy \right] = \varepsilon \partial^2_{xx} u_\varepsilon.
$$

Assuming that the initial condition u_0 satisfies (3), we establish a series of ε - uniform estimates that are analogous to the formal ones discussed above. For example, $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\boldsymbol{R}_+; L^2(\boldsymbol{R}))$ and $L^\infty([0,T]; L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R}))$, for any $T > 0$, which implies that a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ converges strongly in $L^p_{\mathrm{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R})$, for any $1 \le p < \infty$, and also in

$L^p(\boldsymbol{R}_+ \times \boldsymbol{R})$, for any $1 \leq p < 2$, to a limit function u that satisfies (8) and (9), which we furthermore prove is a weak solution of the Degasperis-Procesi equation. By a *weak solution* we mean a function u that belongs to $L^\infty(\boldsymbol{R}_+; L^2(\boldsymbol{R}))$ and satisfies (7) in $\mathcal{D}'([0,\infty) \times \boldsymbol{R})$. In addition to the estimates mentioned above, we also prove that the weak solution u satisfies a one-sided Lipschitz estimate: $\partial_x u(t,x) \leq \frac{1}{t} + K_T$ for a.e. $(t,x) \in (0,T) \times \boldsymbol{R}$, where K_T is a constant that depends on T and the $L^2 \cap L^\infty$ norm of u_0. An implication of this estimate is that if the weak solution u contains discontinuities (shocks) then they must be nonincreasing. To obtain the desired strong compactness we use the compensated compactness method [25].

To assert that the weak solution is unique we would need to know somehow that the chain rule holds for our weak solutions. However, since we work in spaces of discontinuous functions, this is not true. Instead we shall borrow ideas from the theory of conservation laws and replace the chain rule with an infinite family of entropy inequalities. Namely, we shall require that an admissible weak solution should satisfy the "entropy" inequality (P^u is defined in (6))

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \leq 0 \quad \text{in } \mathcal{D}'\ ([0,\infty) \times \boldsymbol{R}), \tag{12}$$

for all convex C^2 entropies $\eta : \boldsymbol{R} \to \boldsymbol{R}$ and corresponding entropy fluxes $q : \boldsymbol{R} \to \boldsymbol{R}$ defined by $q'(u) = \eta'(u)\,u$. We call a weak solution u that also satisfies (12) an *entropy weak solution*. We prove that the above mentioned weak solution, which is obtained as the limit of a sequence of viscous approximations, satisfies the entropy inequality (12), and thus is an entropy weak solution of (1), (2).

Finally, we stress that there is a strong analogy with nonlinear conservation laws (Burgers' equation). Indeed, we can view (7) as Burgers' equation perturbed by a nonlocal source term. This point of view works due to the boundednes of $\partial_x P^u$ (see (9)). This analogy makes it possible to prove L^1 stability (and thereby uniqueness) of entropy weak solutions to the Degasperis-Procesi equation by a straightforward adaption of Kruzkov's uniqueness proof [18].

The remaining part of this paper is organized as follows. In Section 2 we give the precise definitions of weak and entropy weak solution for (1), (2), and state our main wellposedness result. In Section 3 we define the viscous approximations and establish some important a priori estimates. In Section 4 we prove our main result.

2. – Definition of entropy solution and main results.

Let us be more precise about the meaning of solution for the Cauchy problem (1), (2). We begin by introducing a suitable notion of weak solution.

DEFINITION 1 (WEAK SOLUTION). – *We call a function* $u : \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}$ *a weak solution of the Cauchy problem* (1), (2) *provided* *i)* $u \in L^\infty(\boldsymbol{R}_+; L^2(\boldsymbol{R}))$;

ii) $\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \partial_x P^u = 0$ *in* $\mathcal{D}'([0,\infty)\times\boldsymbol{R})$, *that is, the following identity holds*

$$\int_{\boldsymbol{R}+}\int_{\boldsymbol{R}}\left(u\partial_t\phi + \frac{u^2}{2}\partial_x\phi - \partial_x P^u\phi\right)dx\,dt + \int_{\boldsymbol{R}} u_0(x)\phi(0,x)\,dx = 0, \tag{13}$$

for all $\phi \in C_c^\infty([0,\infty)\times\boldsymbol{R})$, *where*

$$P^u(t,x) = (1-\partial_{xx}^2)^{-1}\left(\frac{3}{2}u^2\right)(t,x) = \frac{3}{4}\int_{\boldsymbol{R}} e^{-|x-y|}(u(t,y))^2\,dy.$$

REMARK 1. – It follows from part *i)* of Definition 1 that $u \in L^1((0,T)\times\boldsymbol{R})$ for any $T>0$ and $\partial_x P^u \in L^\infty(\boldsymbol{R}_+\times\boldsymbol{R})$ (consult the proof of Corollary 2). Hence equation (13) makes sense.

We extend the definition of a weak solution by requiring the fulfillment of an entropy condition so we arrive at the notion of an entropy weak solution for the Degasperis-Procesi equation.

DEFINITION 2 (ENTROPY WEAK SOLUTION). – *We call a function* $u:\boldsymbol{R}_+\times\boldsymbol{R}\to\boldsymbol{R}$ *an entropy weak solution of the Cauchy problem* (1), (2) *provided*

i) *u is a weak solution in the sense of Definition* 1;

ii) $u \in L^\infty([0,T]\times\boldsymbol{R})$, for any $T>0$;

iii) *for any convex* C^2 *entropy* $\eta:\boldsymbol{R}\to\boldsymbol{R}$ *with corresponding entropy flux* $q:\boldsymbol{R}\to\boldsymbol{R}$ *defined by* $q'(u) = u\,\eta'(u)$ *there holds*

$$\partial_t\,\eta(u) + \partial_x\, q(u) + \eta'(u)\,\partial_x P^u \le 0 \text{ in } \mathcal{D}'([0,\infty)\times\boldsymbol{R}),$$

that is, for every $\phi \in C_c^\infty([0,\infty)\times\boldsymbol{R})$, $\phi \ge 0$,

$$\int_{\boldsymbol{R}+}\int_{\boldsymbol{R}}(\eta(u)\partial_t\phi + q(u)\partial_x\phi - \eta'(u)\partial_x P^u\phi)\,dx\,dt + \int_{\boldsymbol{R}}\eta(u_0(x))\phi(0,x)\,dx \ge 0. \tag{14}$$

REMARK 2. – It takes a standard argument to see that it suffices to verify (14) for the Kruzkov entropies/entropy fluxes

$$\eta(u) := |u-c|, \qquad q(u) := \operatorname{sign}(u-c)\left(\frac{u^2}{2}-\frac{c^2}{2}\right), \qquad c\in\boldsymbol{R}.$$

Using the Kruzkov entropies/entropy fluxes it can then be seen that the weak formulation (13) is a consequence of the entropy formulation (14).

Our main results are collected in the following theorem:

THEOREM 1 (WELL-POSEDNESS). – *Suppose condition* (3) *holds. Then there exists an entropy weak solution to the Cauchy problem* (1), (2). *Fix any* $T > 0$, *and let* $u, v : \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}$ *be two entropy weak solutions to* (1), (2) *with initial data* $u_0, v_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$, *respectively. Then for almost any* $t \in (0, T)$

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\boldsymbol{R})} \leq e^{M_T t}\|u_0 - v_0\|_{L^1(\boldsymbol{R})}, \tag{15}$$

where

$$M_T := \frac{3}{2}\left(\|u\|_{L^\infty((0,T)\times\boldsymbol{R})} + \|v\|_{L^\infty((0,T)\times\boldsymbol{R})}\right) < \infty. \tag{16}$$

Consequently, there exists at most one entropy weak solution to (1), (2).

The entropy weak solution u *satisfies the following estimates for any* $t \geq 0$:

$$\|u(t,\cdot)\|_{L^1(\boldsymbol{R})} \leq \|u_0\|_{L^1(\boldsymbol{R})} + 12t\|u_0\|^2_{L^2(\boldsymbol{R})}, \tag{17}$$

$$\|u(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \leq \|u_0\|_{L^\infty(\boldsymbol{R})} + 6t\|u_0\|^2_{L^2(\boldsymbol{R})}. \tag{18}$$

Finally, the following Oleinik type estimate holds for a.e. (t, x), $(t, y) \in (0, T] \times \boldsymbol{R}$, $x \neq y$,

$$\frac{u(t,x) - u(t,y)}{x - y} \leq \frac{1}{t} + K_T, \tag{19}$$

where

$$K_T := \left[6\|u_0\|^2_{L^2(\boldsymbol{R})} + \frac{3}{2}\left(\|u_0\|_{L^\infty(\boldsymbol{R})} + 6T\|u_0\|^2_{L^2(\boldsymbol{R})}\right)^2\right]^{1/2}. \tag{20}$$

3. – Viscous approximations and a priori estimates

We will prove existence of a solution to the Cauchy problem (1), (2) by analyzing the limiting behavior of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where each function u_ε solves the viscous problem (10) endowed with the initial condition

$$u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in \boldsymbol{R},$$

or equivalently the following parabolic-elliptic system:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x\left(\dfrac{u_\varepsilon^2}{2}\right) + \partial_x P_\varepsilon = \varepsilon\partial^2_{xx}u_\varepsilon, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ -\partial^2_{xx}P_\varepsilon + P_\varepsilon = \dfrac{3}{2}u_\varepsilon^2, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \boldsymbol{R}, \end{cases} \tag{21}$$

where we assume that

$$
(22)\qquad
\begin{aligned}
&u_{0,\varepsilon} \in H^{\ell}(\boldsymbol{R}), \ \text{for some } \ell \ \geq \ 2,\\
&\|u_{0,\varepsilon}\|_{L^2(\boldsymbol{R})} \leq \|u_0\|_{L^2(\boldsymbol{R})}, \quad \|u_{0,\varepsilon}\|_{L^\infty(\boldsymbol{R})} \leq \|u_0\|_{L^\infty(\boldsymbol{R})}, \ \text{for every } \varepsilon > 0,\\
&u_{0,\varepsilon} \to u_0 \quad \text{in } L^2(\boldsymbol{R}\ ), \ \text{as } \varepsilon \to 0.
\end{aligned}
$$

Using again the fact that $e^{-|\xi|}/2$ is the Green's function of the operator $1 - \partial_{xx}^2$ we have an explicit expression for P_ε in terms of u_ε:

$$
P_\varepsilon(t,x) = P^{u_\varepsilon}(t,x) = (1-\partial_{xx}^2)^{-1}\left(\frac{3}{2}u^2\right)(t,x) = \frac{3}{4}\int_{\boldsymbol{R}} e^{-|x-y|}u_\varepsilon^2(t,y)\,dy.
$$

The wellposedness of the viscous problem (5) in $C([0,\infty); H^{\ell}(R))$ for each fixed $\varepsilon > 0$ can be proved using an argument similar the one of [3, Theorem 2.3].

Next we have a uniform L^2 bound on the approximate solution u_ε (see [4, Lemma 2.2]). The argument is based on a preliminary H^2 estimate on the quantity $v_\varepsilon = v_\varepsilon(t,x)$ defined by (see [4, Lemma 2.3])

$$
v_\varepsilon(t,x) = \big((4-\partial_{xx}^2)^{-1}u_\varepsilon\big)(t,x) = \frac{1}{4}\int_{\boldsymbol{R}} e^{-\frac{|x-y|}{2}}u_\varepsilon(t,y)\,dy, \qquad t \geq 0,\ x \in \boldsymbol{R}.
$$

The use of the quantity v_ε is motivated by the fact that $\int_{\boldsymbol{R}} v\big(u - \partial_{xx}^2 u\big)\,dx$ is a conserved quantity of the Degasperis-Procesi equation (see (8)), where $4v - \partial_{xx}^2 v = u$ and u solves (1) (see [7]).

LEMMA 1 (ENERGY ESTIMATE). – *Let us assume* (3) *and* (22). *Then the following bounds*

$$
(23)\qquad \|u_\varepsilon(t,\cdot)\|_{L^2(\boldsymbol{R})} \leq 2\sqrt{2}\|u_0\|_{L^2(\boldsymbol{R})}, \qquad \sqrt{\varepsilon}\|\partial_x u_\varepsilon\|_{L^2(\boldsymbol{R}_+\times\boldsymbol{R})} \leq 2\|u_0\|_{L^2(\boldsymbol{R})}
$$

hold for any $\varepsilon > 0$ *and* $t \geq 0$.

Due to the integral identities

$$
(24)\qquad P_\varepsilon(t,x) = \frac{3}{4}\int_{\boldsymbol{R}} e^{-|x-y|}u_\varepsilon^2(t,y)\,dy,
$$

$$
(25)\qquad \partial_x P_\varepsilon(t,x) = \frac{3}{4}\int_{\boldsymbol{R}} e^{-|x-y|}\mathrm{sign}(y-x)u_\varepsilon^2(t,y)\,dy.
$$

We have some bounds on the nonlocal term P_ε, which all are consequences of the L^2 bound in Lemma 1 (see [4, Lemma 2.4]).

LEMMA 2. – *Assume* (3) *and* (22) *hold, and fix any* $\varepsilon > 0$. *Then*

$$P_\varepsilon \geq 0, \tag{26}$$

$$\|P_\varepsilon(t,\cdot)\|_{L^1(\boldsymbol{R})}, \|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(\boldsymbol{R})} \leq 12\|u_0\|^2_{L^2(\boldsymbol{R})}, \qquad t \geq 0, \tag{27}$$

$$\|P_\varepsilon\|_{L^\infty(\boldsymbol{R}_+\times\boldsymbol{R})}, \|\partial_x P_\varepsilon\|_{L^\infty(\boldsymbol{R}_+\times\boldsymbol{R})} \leq 6\|u_0\|^2_{L^2(\boldsymbol{R})}, \tag{28}$$

$$\|\partial^2_{xx} P_\varepsilon(t,\cdot)\|_{L^1(\boldsymbol{R})} \leq 24\|u_0\|^2_{L^2(\boldsymbol{R})}, \qquad t \geq 0. \tag{29}$$

Using the L^2 bound in Lemma 1, we can bound u_ε in L^1 (see [4, Lemma 2.5]).

LEMMA 3 (L^1-ESTIMATE). – *Assume* (3) *and* (22). *Then*

$$\|u_\varepsilon(t,\cdot)\|_{L^1(\boldsymbol{R})} \leq \|u_0\|_{L^1(\boldsymbol{R})} + 12t\|u_0\|^2_{L^2(\boldsymbol{R})}, \tag{30}$$

holds for every $t \geq 0$ *and* $\varepsilon > 0$.

We continue this section by proving some estimates that are sharper than the correspondent ones in [4].

Using the $W^{1,\infty}$ bound on $\{P_\varepsilon\}_{\varepsilon>0}$ stated in Lemma 2, we show that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in L^∞.

LEMMA 4 (L^∞-ESTIMATE). – *Assume* (3) *and* (22). *Then*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \leq \|u_0\|_{L^\infty(\boldsymbol{R})} + 6\|u_0\|^2_{L^2(\boldsymbol{R})} t, \qquad t \geq 0, \ \varepsilon > 0. \tag{31}$$

PROOF. – Due to (21) and (28) we know that

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial^2_{xx} u_\varepsilon = -\partial_x P_\varepsilon \leq 6\|u_0\|^2_{L^2(\boldsymbol{R})}.$$

Since the map

$$f(t) = \|u_0\|_{L^\infty(\boldsymbol{R})} + 6\|u_0\|^2_{L^2(\boldsymbol{R})} t, \qquad t \geq 0,$$

solves the equation

$$\frac{df}{dt} = 6\|u_0\|^2_{L^2(\boldsymbol{R})}$$

and, due to (22)

$$u_\varepsilon(0,x) \leq f(0), \qquad x \in \boldsymbol{R},$$

then the comparison principle for parabolic equations says

$$u_\varepsilon(t,x) \leq f(t), \qquad t \geq 0, \quad x \in \boldsymbol{R}.$$

□

As a consequence of (28) and (31), from the second equation in (21) we deduce the following estimate.

LEMMA 5. – *Let* $\varepsilon > 0$ *and* (3) *and* (22) *hold. Then*

$$\|\partial_{xx}^2 P_\varepsilon(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \leq 6\|u_0\|_{L^2(\boldsymbol{R})}^2 + \frac{3}{2}\left(\|u_0\|_{L^\infty(\boldsymbol{R})} + 6\|u_0\|_{L^2(\boldsymbol{R})}^2 t\right)^2, \quad t \geq 0. \tag{32}$$

We conclude this section by showing through an estimate of Oleinik type that a solution of the Degasperis-Procesi equation can only contain decreasing discontinuities (shocks), which coincides with what is known for the Burgers' equation. We point out that the Oleinik estimate proved here does not depend on the total variation of the initial condition as the one proved in [4].

LEMMA 6 (OLEINIK TYPE ESTIMATE). – *Let* $\varepsilon, T > 0$*. If* (3) *and* (22) *hold, then*

$$\partial_x u_\varepsilon(t,x) \leq \frac{1}{t} + K_T, \qquad x \in \boldsymbol{R},\ 0 < t \leq T, \tag{33}$$

where the constant K_T *is defined in* (20).

PROOF. – Setting $q_\varepsilon := \partial_x u_\varepsilon$, it follows from (21) and (32) that

$$\partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + q_\varepsilon^2 - \varepsilon \partial_{xx}^2 q_\varepsilon = -\partial_{xx}^2 P_\varepsilon \leq K_T^2. \tag{34}$$

The map

$$f(t) = \frac{1}{t} + K_T, \qquad t > 0,$$

satisfies

$$\frac{df}{dt} + f^2 = 2\frac{K_T}{t} + K_T^2 \geq K_T^2,$$

namely f is a supersolution of (34). Therefore the comparison principle for parabolic equations says

$$q_\varepsilon(t,x) \leq f(t), \qquad (t,x) \in (0,T] \times \boldsymbol{R},$$

and hence (33) follows. □

4. – Well-posedness in $L^1 \cap L^\infty$.

Relying on the a priori estimates derived in Section 3, we prove in this section existence, uniqueness, and L^1 stability of entropy weak solutions to (1), (2) under the $L^1 \cap L^\infty$ assumption (3). These claims are immediate consequence of Lemmas 7, 10, and Corollary 1 below.

We begin by proving that there exists at least one entropy weak solution to (1), (2) under assumption (3).

LEMMA 7 (EXISTENCE). – *Suppose* (3) *holds. Then there exists at least one entropy weak solution to* (1), (2).

We will construct a weak solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations, see (10) or (21). We make the standing assumption that the approximate initial data $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ are chosen such that they respect (3) and (22). Having said that, in the present context, comparing with the framework of [4], we do not have have at our disposal a uniform BV estimate. Indeed, the relevant a priori estimates are only those contained in Lemmas 1, 2, 3, and 4. We use the the compensated compactness method [25, 26] to obtain strong convergence of a subsequence of viscosity approximations.

THEOREM 2. – *Let* $\{v_\nu\}_{\nu>0}$ *be a family of functions defined on* $(0,\infty)\times\boldsymbol{R}$ *such that*

$$\|v_\nu\|_{L^\infty((0,T)\times\boldsymbol{R})}\leq M_T, \qquad T,\ \nu>0,$$

and the family

$$\{\partial_t\eta(v_\nu)+\partial_x q(v_\nu)\}_{\nu>0}$$

is compact in $H^{-1}_{loc}((0,\infty)\times\boldsymbol{R})$, *for every convex* $\eta\in C^2(\boldsymbol{R})$, *where* $q'(u)=u\eta'(u)$. *Then there exist a sequence* $\{\nu_n\}_{n\in\boldsymbol{N}}\subset(0,\infty)$, $\nu_n\to 0$, *and a map* $v\in L^\infty((0,T)\times\boldsymbol{R})$, $T>0$, *such that*

$$v_{\nu_n}\to v \quad \text{a.e. and in } L^p_{loc}((0,\infty)\times\boldsymbol{R}),\ 1\leq p<\infty.$$

Finally, the following compact enbedding of Murat [23] is useful.

THEOREM 3. – *Let* Ω *be a bounded open subset of* $\boldsymbol{R}^N$, $N\geq 2$. *Suppose the sequence* $\{\mathcal{L}_n\}_{n\in\boldsymbol{N}}$ *of distributions is bounded in* $W^{-1,\infty}(\Omega)$. *Suppose also that*

$$\mathcal{L}_n=\mathcal{L}_n^1+\mathcal{L}_n^2,$$

where $\{\mathcal{L}_n^1\}_{n\in\boldsymbol{N}}$ *lies in a compact subset of* $H^{-1}_{\rm loc}(\Omega)$ *and* $\{\mathcal{L}_n^2\}_{n\in\boldsymbol{N}}$ *lies in a bounded subset of* $L^1_{loc}(\Omega)$. *Then* $\{\mathcal{L}_n\}_{n\in\boldsymbol{N}}$ *lies in a compact subset of* $H^{-1}_{\rm loc}(\Omega)$.

We now turn to the proof of Lemma 7, which will be accomplished through two lemmas.

LEMMA 8. – *Let us suppose* (3) *holds. Then there exists a subsequence* $\{u_{\varepsilon_k}\}_{k\in\boldsymbol{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a limit function*

$$u\in L^\infty(\boldsymbol{R}_+;L^2(\boldsymbol{R}))\cap L^\infty((0,T);L^\infty(\boldsymbol{R})\cap L^1(\boldsymbol{R})),\ T>0, \tag{35}$$

such that

$$u_{\varepsilon_k}\to u \ \text{ in } L^p((0,T)\times\boldsymbol{R}),\quad T>0,\ 1\leq p<\infty . \tag{36}$$

PROOF. – Let $\eta : \boldsymbol{R} \to \boldsymbol{R}$ be any convex C^2 entropy function, and let $q : \boldsymbol{R} \to \boldsymbol{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)\,u$. By multiplying the first equation in (21) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}_\varepsilon^1} \underbrace{-\varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \eta'(u_\varepsilon)\partial_x P_\varepsilon}_{=:\mathcal{L}_\varepsilon^2}, \tag{37}$$

where $\mathcal{L}_\varepsilon^1$, $\mathcal{L}_\varepsilon^2$ are distributions. We claim that

$$\begin{aligned} &\mathcal{L}_\varepsilon^1 \to 0 \text{ in } \mathrm{H}^{-1}([0,\mathrm{T}] \times \boldsymbol{R}),\ \mathrm{T} > 0, \\ &\mathcal{L}_\varepsilon^2 \text{ is uniformly bounded in } \mathrm{L}^1([0,\mathrm{T}] \times \boldsymbol{R}),\ \mathrm{T} > 0. \end{aligned} \tag{38}$$

Indeed, (23), (31), and (27) imply

$$\|\varepsilon \partial_x \eta(u_\varepsilon)\|_{L^2(\boldsymbol{R}_+ \times \boldsymbol{R})} \le 2\sqrt{\varepsilon}\|\eta'\|_{L^\infty(\boldsymbol{R})}\|u_0\|_{L^2(\mathcal{I}_T)} \to 0, \tag{39}$$

$$\left\|\varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2\right\|_{L^1(\boldsymbol{R}_+ \times \boldsymbol{R})} \le 4\|\eta''\|_{L^\infty(\mathcal{I}_T)}\|u_0\|^2_{L^2(\boldsymbol{R})}, \tag{40}$$

$$\|\eta'(u_\varepsilon)\partial_x P_\varepsilon\|_{L^1((0,T)\times \boldsymbol{R})} \le 12T\|\eta'\|_{L^\infty(\mathcal{I}_T)}\|u_0\|^2_{L^2(\boldsymbol{R})}, \tag{41}$$

where

$$\mathcal{I}_T = \Big[-\Big(\|u_0\|_{L^\infty(\boldsymbol{R})} + 6\|u_0\|^2_{L^2(\boldsymbol{R})}T\Big), \|u_0\|_{L^\infty(\boldsymbol{R})} + 6\|u_0\|^2_{L^2(\boldsymbol{R})}T\Big].$$

Hence, (38) follows. Therefore, Theorems 3 and 2 give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\boldsymbol{N}}$ and a limit function u satisfying (35) such that as $k \to \infty$

$$\begin{aligned} u_{\varepsilon_k} &\to u \text{ in } L^p_{\text{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}) \text{ for any } p \in [1,\infty), \\ u_{\varepsilon_k} &\to u \text{ a.e. in } \boldsymbol{R}_+ \times \boldsymbol{R}. \end{aligned} \tag{42}$$

Thanks to the L^1 and L^∞ estimates (30) and (31) we can upgrade (42) to (36). □

LEMMA 9. – *Suppose* (3) *holds. Then*

$$P_{\varepsilon_k} \to P^u \text{ in } L^p([0,T]; W^{1,p}(\boldsymbol{R})),\ T > 0,\ 1 \le p < 2. \tag{43}$$

where the sequence $\{\varepsilon_k\}_{k\in\boldsymbol{N}}$ *and the function* u *are constructed in Lemma* 8.

PROOF. – Using the integral representation of P_{ε_k} stated in (25), Lemma 4, and arguing as in [4, Theorem 3.2] we have that

$$\left\|P_{\varepsilon_k} - P^u\right\|_{L^p([0,T]\times\boldsymbol{R})} \le \frac{3}{2}\Big(\|u_0\|_{L^\infty(\boldsymbol{R})} + 6T\|u_0\|^2_{L^2(\boldsymbol{R})}\Big)\left\|u_{\varepsilon_k} - u\right\|_{L^p([0,T]\times\boldsymbol{R})},$$

for every $1 \le p < \infty$ and $T > 0$. In light of Lemma 8 we get (43). □

PROOF OF LEMMA 7. Let $\varphi \in C^\infty(\boldsymbol{R}_+ \times \boldsymbol{R})$ be a compactly supported test function. Due to (21)

$$\int_{\boldsymbol{R}+}\int_{\boldsymbol{R}} \left(u_\varepsilon \partial_t \phi + \frac{u_\varepsilon^2}{2} \partial_x \phi - \partial_x P_\varepsilon \phi + \varepsilon u_\varepsilon \partial_{xx}^2 \phi \right) dx\, dt + \int_{\boldsymbol{R}} u_{0,\varepsilon}(x)\phi(0,x)\, dx = 0.$$

Therefore, (22) and Lemma 8 say that the function u constructed in Lemma 8 is a weak solution of (1), (2) in the sense of Definition 1.

Finally, we have to verify that u satisfies the entropy inequalities stated in Definition 2. Let $\eta \in C^2(\boldsymbol{R})$ be a convex entropy with flux q defined by $q'(u) = u\eta'(u)$. The convexity of η and (21) yield

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon)\partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \le \varepsilon \partial_{xx}^2 \eta(u_\varepsilon).$$

Therefore, the entropy inequalities follow from Lemmas 8 and 9. □

Using the Kruzkov's method of doubling the variables [18] we can prove the L^1 stability (and thus uniqueness) of entropy weak solutions [4, Theorem 3.3].

LEMMA 10 (L^1 STABILITY). – *Let u and v be two entropy weak solution of* (1) *with initial data $u(0,\cdot) = u_0$ and $v(0,\cdot) = v_0$ satisfying* (3). *Fix any $T > 0$. Then*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\boldsymbol{R})} \le e^{M_T t}\|u_0 - v_0\|_{L^1(\boldsymbol{R})}, \qquad a.e. \ \ t \in (0,T), \tag{44}$$

where the positive constant M_T is defined in (16).

As an immediate consequence of this lemma we have.

COROLLARY 1 (UNIQUENESS). – *Suppose condition* (3) *holds. Then the Cauchy problem* (1), (2) *admits at most one entropy weak solution.*

PROOF OF THEOREM 1. The existence, stability, and uniqueness of entropy weak solutions are stated in Lemmas 8, 10 and Corollary 1. The L^1, L^∞ and Oleinik type estimates (17) and (19) follow from Lemmas 3, 4, 6 and 8. □

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