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When does Agglutination Arise in the Homogeneization of Ordinary Differential Equations?

ELENA BOSA - LIVIO C. PICCININI

In memory of Guido Stampacchia

Abstract. – *When dealing with Differential Equations whose coefficients are periodical, it is of interest to consider the limit when the period becomes shorter and shorter. This process is called homogeneization and leads to an equation with constant coefficients. The constants are some mean of the original coefficients, usually non trivial. We say that the mean is regular if it is increased whenever coefficients are increased on a non-zero set; on the contrary we say that agglutination arises if there are intervals of constancy. It is well known that a chessboard structure leads to agglutination. The authors give some sufficient conditions to prevent agglutination and show that some more general forms of mosaic can not save regularity.*

1. – Introduction and aim of the paper.

In the last years of his life Stampacchia renewed his interest in ordinary differential equations. The fruit was the book "Equazioni differenziali in $\mathbb{R}^n$: problemi e metodi", written with the collaboration of Piccinini and Vidossich [1], but founded upon a previous work written by himself in the fifties. He could never see his work through the press, because of his untimely death.
In this paper we wish to extend in a critical way some results about homogeneization theory that were presented in the third chapter of [6]. They are not difficult to grasp, but in the general case they show a complicated behaviour including agglutination, that is the "devil's staircase". This phenomenon was first glimpsed by Piccinini in [4] for the chessboard structure, then it deeply studied by Mortola and Peirone in [2], and in particular by Peirone in [3]. Devil's staircase is not an uncommon phenomenon, and it depends on some privileged behaviour connected with rational numbers inasmuch they allow a finite re-

[1] Later on it was translated in English ([6]). The references in this paper are with respect to the English translation

presentation. Dirichlet's function was perhaps the first example of it. A recent case was found in the generalization of Bak-Sneppen processes, as it is shown in [1]. In homogeneization theory this phenomenon arises when a first order equation periodic in both its arguments is lead to the limit. The rotation index so obtained is some form of mean of the coefficients. When homogeneization is performed with respect to a leading variable (or depends only on one variable) such mean is strictly monotone, that is if coefficients are increased on a non-zero measure domain, it increases. On the contrary, when it is performed at the same rate in both variables agglutination (or devil's staircase) may arise: it means that there are intervals of constance even when coefficients are increased on a positive measure set.
We may call regular the classes of structures for which agglutination cannot arise, and we show that there are regular classes larger than those consisting of homogeneization in a single variable. We show that in general, anyhow, a mosaic structure generates agglutination. Finally we find some (very simple) mosaic structures for which regularity is maintained.

2. – Notation and theoretical foundations.

As it is well known there exist ordinary differential equations for which Cauchy problem has infinitely many solutions (Peano's phaenomenon). The theory of G-convergence for these equations becomes more complicated, and moreover is no longer consistent with homogeneization with respect to both variables (2). Since here we are not interested in these special cases, for which homogeneization seems not to be meaningful, we can restrain ourselves to those equations for which Cauchy problem has a unique solution. In view of periodicity we can also limit ourselves to the case when solutions exist on the whole $\mathbb{R}$. We consider thus equations of the form

$$A_f[y] = y' - f(x, y) = 0,$$

where f satisfies the conditions: f is continuous with respect to y and is measurable with respect to x and there exists a constant M such that

$$|f(x, y)| \leq M(|y| + 1). \tag{2.1}$$

DEFINITION 2.1. – *Let*

$$A_n[y_n] \equiv A_{f_n}[y_n] = 0$$

(2) A general approach, even if not very elegant, was given in [5]

be a sequence of differential equations and let $A[y] \equiv A_f[y] = 0$ *be a differential equation.*

Suppose (2.1) *is satisfied uniformly and that the Cauchy problem has always a unique solution. We say that* A_n *G-converges to* A *if for every pair* (x^0, y^0) *the sequence* $\{y_n\}_n$ *consisting of the solutions of the Cauchy problems*

$$A_n[y_n] = 0$$
$$y_n(x^0) = y^0$$

converges uniformly to the solution $y(x)$ *of the Cauchy problem*

$$A[y] = 0$$
$$y(x^0) = y^0.$$

When we are dealing with periodic structures we are often interested in problems of homogeneization, that is to study a global behaviour that can be different from microlocal properties.
The general theorem on homogeneization for first order equations is the following (Theorem 2, page 190 of [6])

THEOREM 2.2. – *Let* $f(x, y)$ *be a* C^1*-function, periodic in* x *with period* L *and periodic in* y *with period* M*. Then the equations*

$$y'_k = f_k(x, y_k) = f(kx, ky_k) \tag{2.2}$$

G-converge to the equation

$$y' = p \tag{2.3}$$

where p *is a number satisfying the condition*

$$\min f(x, y) \leq p \leq \max f(x, y) \ (^3).$$

Remark that the limit can actually be given by one of the extrema as it is shown by the function $f(x, y) = \sin(x - y)$. The sequence (2.2) G-converges to $p = 1$, that is to the maximum.

A trivial case of theorem (2.2) is achieved when the function f depends only on x. In this case we get that p is given by the arithmetic mean of f on the period. Exchanging the roles of the variables one obtains a less trivial result for autonomous equations, where the function depends only on y. In this case p is given by the harmonic mean of f.

(3) If one thinks of an equation with double periodicity as a field of tangents on a thorus, it is clear that p represents the mean ratio between the number of principal rotations and secondary rotations.

In order to extend this result we recall the definition of functional mean.

Let Φ be a continuous strictly monotone function defined on some interval of $\mathbb{R}$. We say that p is the functional mean of a and b according to Φ if p is the value such that

$$\Phi(p) = \frac{\Phi(a) + \Phi(b)}{2}$$

More generally, given a normalized measure μ and a density function g on an interval $[a, b]$, its functional mean according to Φ is the unique value p such that

$$\Phi(p) = \int_a^b \Phi[g(x)] d\mu(x)$$

3. – Regular homogeneization and its extension.

We have seen that, when dealing with a class of periodic functions depending only on a unique variable, homogeneization is regular in the sense that an increase of the function causes an increase in the limit. Remark anyhow that we are allowing only changes that preserve the dependence on a unique variable, otherwise the statement would be false as we shall show in the next section. We now enlarge in a simple way this class of functions.

THEOREM 3.1. – *For* $0 \leq \theta < \frac{\pi}{2}$ *let*

$$\Phi_\theta(x) = \frac{x - \tan\theta}{1 + x\tan\theta}, \tag{3.1}$$

defined either on the interval $x > -1/\tan\theta$ *or on the interval* $x < -1/\tan\theta$. *Let* $f(x, y) = F(x\cos\theta + y\sin\theta)$, *where* F *is continuous and periodic of period* 1. *Then*

1. $f(x, y)$ *is periodic in* x *and* y *with period* $L = (\cos\theta)^{-1}$, $M = (\sin\theta)^{-1}$.
2. *If* F *does not attain the value* $-1/\tan\theta$, *the G-limit of homogeneization* (2.2) *is given by the functional mean according to function* (3.1), *and in this class homogeneization is regular.*
3. *If* F *attains at some point the characteristic value* $-1/\tan\theta$, *the G-limit is the characteristic value, and does not depend on the remaining values* [4].

PROOF. – The first statement is obvious, and the periods L and M are just the points where the first period (unitary) of F, estimated starting from $(0, 0)$, is completed. Thus theorem (2.2) holds and there exist a G-limit associated to a constant. We need just to estimate the constant.

[4] That is, homogeneization is no longer regular, as it happened for autonomous equations when the characteristic value 0 was attained

We first recall explicitely the trivial homogeneization $y' = f(x)$, where f is periodic of period 1. Since $y_n' = f(nx)$, integrating we get

$$y_n(x) - y_0 = \int_0^x f(nt)dt = \frac{1}{n}\int_0^{nx} f(s)ds$$

Let now $p = \int_0^1 f(t)dt$; we compare the behaviour of y_n with the behaviour of the limiting solution

$$y(x) = px + y^0$$

We get

$$|y(x) - y_n(x)| = \frac{1}{n}\left|(nx - [nx])p - \int_{[nx]}^{nx} f(s)ds\right| \le C\frac{nx - [nx]}{n} < \frac{C}{n}$$

what ensures the uniform convergence to the solution with constant slope p, where p is given by the arithmetic mean.
This is the case of $\theta = 0$. The general case is obtained by a positive rotation of angle θ. We interpret the differential equation as a field of directions of which we know the tangent with respect to the horizontal. We express now the new tangent with respect to the line $-x\sin\theta + y\cos\theta = 0$, getting a new direction field given by $\tan(\arctan F - \theta)$. We are thus reduced to the trivial case, where the G-limit is represented by the arithmetic mean. This will now be represented with respect to (x, y) variables, getting thus

$$p = \tan\left(\arctan\int_0^1 \tan(\arctan F(t) - \theta)dt + \theta\right) = \frac{\displaystyle\int_0^1 \frac{F(s)(1+\tan^2\theta)}{1+F(s)\tan\theta}dt}{\displaystyle\int_0^1 \frac{(1+\tan^2\theta)}{1+F(s)\tan\theta}dt},$$

that is the functional mean of the statement.
Remark that this formula for $\theta = 0$ gives the arithmetic mean, while for $\theta = \pi/2$ its limit is the harmonic mean corresponding to the case of autonomous equations.
The last part of the statement is required when the functional mean can no longer be used. Without loss of generality we may suppose that $F(0) = -1/\tan\theta$, hence we find the linear solution

$$y(x) = -x/\tan\theta.$$

Since the G-limit exists by theorem (2.2), it follows that its coefficient is actually $p = -1/\tan\theta$. □

4. – Chessboard, Mosaic and Agglutination.

By a mosaic we mean a first order differential equation $y' = f(x, y)$ where $f(x, y)$ is periodic with respect to both variables and is piecewise constant on a finite number of polygons. Existence and uniqueness are ensured when the following consistency conditions are satisfied:

1. in each polygon the associated value does not coincide with the slope of any of its sides
2. for each side the values associated to the two adjoining polygons are both either greater than the slope or less than the slope of the side.

The points of coordinates (hL, kM) are called the grid of the mosaic, while all the vertexes of the polygons are called knots. It may happen that knots do not belong to the grid, that some knots lie on the grid (it is the case of the chessboard), and even that all the knots belong to the grid.

As in any case of differential equation with doubly periodic structure, we can consider the input-output function: for any given Cauchy problem

$$\begin{aligned} y' &= f(x, y) \\ y(0) &= y^0 \end{aligned}$$

we let $Y(y^0) = y(L)$, that is the exit value at the end of one period in the independent variable. Clearly the rotation index depends only on the structure of the input-output function, and what happens in the inside does not effect the limiting value.

A useful lemma is the following:

LEMMA 4.1. – *Let a mosaic be given. Then the input-output function is piecewise linear. Changes of derivative may happen only when the solution touches a knot.*

PROOF. – Since the derivative is piecewise constant inside each polygon, it follows that the solutions are piecewise linear up to each knot. Any local (i.e. relative to a single polygon and single sides) input-output function is therefore linear in view of Talete's theorem. Since the global input-output function is built up by the local ones, and since the composition of linear functions is linear it follows the first statement. The second part follows from the fact that any change must arise from changes in the collection of local input-output functions (that otherwise are invariant), and also this fact can happen only when a knot is crossed. □

4.1. – *Generalization of the classical chessboard.*

We can make a generalization of the classical chessboard with square boxes by giving different sizes to the squares. In this case we have a big square whose side measures 2, in which we have slope a, in the two edges down on the left and aloft on the right there are two different smaller squares, in which we have slope b. If we increase the smaller squares till their sides measure 1 we have the same situation of the classical chessboard.

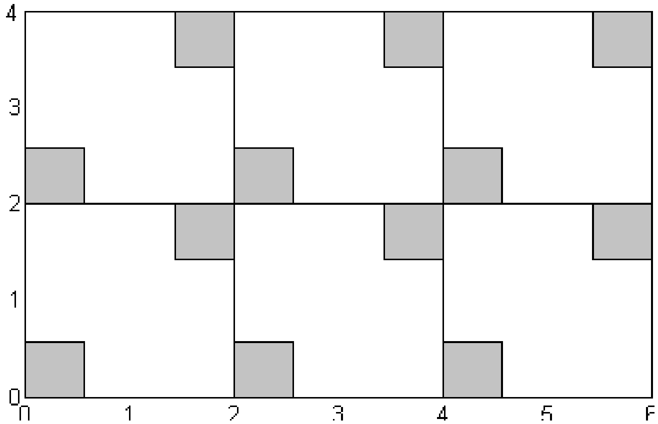

Fig. 1. – Mosaic with squares of different size

We now consider how the G-limit changes as we modify the values of the two parameters.

Remark that obviously solutions are polygonals. If the smaller squares are too big and the polygonal crosses them, then we have a situation similar to the classical chessboard. Instead, if we reduce enough these squares, the polygonal may not cross them. In this particular case whatever value we give to the parameter b, leaving the parameter a unchanged, the G-limit won't change. In this way for every rational slope $\frac{m}{n}$ we can find a particular dimension of the smallest squares for which, if we set $a = \frac{m}{n}$ and we exchange the value of the parameter b freely, the G-limit remains $\frac{m}{n}$.

Let's now consider which is the greatest dimension we can give to the smaller squares in order to maintain this feature. We examine this problem in the event that slope a is rational (in the form $\frac{m}{n}$ with $(m, n) = 1$ that is m and n are coprime) and $a < 1$. In fact the instance $a > 1$ is symmetrical. In order to be clear we may call h the dimension of the smallest squares while the dimension of the biggest ones remain fixed 2. First we can examine what happens with slope $a = \frac{1}{2}$, we have two main inequalitiess that have to be satisfied:

$$2\frac{m}{n} \geq 2h \wedge 2\frac{m}{n} \leq 2 - 2h$$

so, in this case, the greatest dimension of the smallest squares is $\frac{1}{2}$.

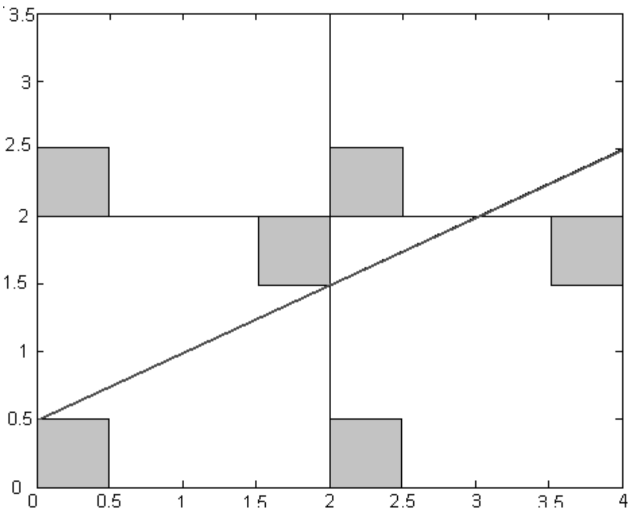

Fig. 2. – Slope $\frac{1}{2}$

Let's try to study now the general situation:

THEOREM 4.2. – *Let the main coefficient $a = \frac{m}{n}$ with m and n coprime. Then the greatest size of the secondary squares for which the couple $(\frac{m}{n}, b)$ has G-limit $\frac{m}{n}$ is $\frac{1}{n}$.*

PROOF. – First for every value m with $(m, n) = 1$ we can always find a polygonal which doesn't cross the small squares. We can consider polygonals which start from the edge aloft on the left of the first small square (as in Figure 2). In fact, if we aim to maximize the size of the squares, the polygonal somewhere will touch one of their corners.

Consider now $h = \frac{1}{n}$, the polygonal starts from $(0, h)$ and crosses n right sides of the big squares in order to arrive to $(2n, 2m + h)$. Suppose now to overlap all the intersection points between the polygonal and the vertical side of the biggest square in one unique segment 2 unites long (Figure 3). There are exactly $n + 1$ points including the starting and the arriving points. We can see that it is possible that none of this points meets up with the smaller squares. In fact if the polygonal starts from $(0, \frac{1}{n})$ and for any two units the y-value increases of a quantity $\frac{2m}{n}$ with $(m, n) = 1$ the intersection points on the segment are all of the type $h + \frac{2}{n}k$ where $k = 1, ..., n$ and they are all different. We can easily see that none of them meets up with the small squares. In this way so we have found a

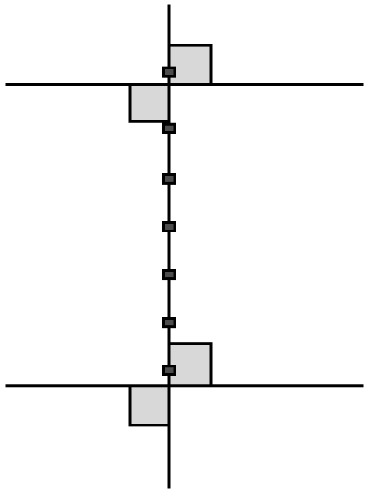

Fig. 3. – Intersection points overlapped

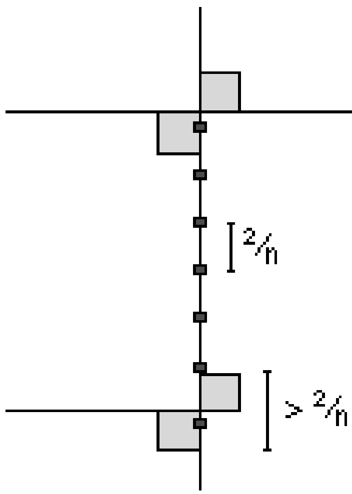

Fig. 4. – $h > \frac{1}{n}$

polygonal of slope $\frac{m}{n}$ which doesn't cross any small square of dimension $\frac{1}{n}$. Is it possible to enlarge this small squares keeping the property that there exist that particular polygonal?

It can easily be seen that this isn't possible, so the largest size of the small squares is exactly $\frac{1}{n}$. As a matter of fact if the polygonal starts from a point $(0, \frac{1}{n} + \varepsilon)$ sooner or later one of the intersection points, which are far from each other $\frac{2}{n}$ will cross one of the smaller squares. □

We come now to non regularity.

From the classical theory of the chessboards we know that agglutination arises when the G-limit is $\frac{1}{n}$ and n is odd but it does not happen when n is even. In our case, on the contrary, we shall show that for any size of the smaller squares less than 1, agglutination may arises also for even n. In particulary it is enough to show it for $n = 2$.

If $b < \frac{1}{2}$ the polygonals that touch the smaller squares, with the same coefficient a can no longer be unchanged as b varies. Hence, in order to have the same G-limit, a must change, so agglutination is found.
Else, if the smaller squares are larger than $\frac{1}{2}$ we don't have the situation described in Theorem 4.2, there isn't a singular value of the parameter in the bigger squares that gives always the same G-limit with every parameter of the smaller squares.

But, as the two parameters are $a \leq 1$ and $b \leq 1$, if we find a poligonal with starting point in $(0, 0)$ and periodic (arrives in $(4, 2)$) we know that the poligonal y^* starting from $(0, \varepsilon)$ will arrive at $(4, 2 + \varepsilon')$.

In fact, if the solution y^* would be continued with the slope a (of the smaller squares) we should attain $y^*(4) = 2 + \varepsilon$ but its actual slope is $b \neq a$ hence it attains a different value $y^*(4) = 2 + \varepsilon'$. Therefore it is possible to modify one of the values of the two parameters (for example a) in order to make this new poligonal periodic (in order to arrive at the point $(4, 2 + \varepsilon)$), this new poligonal will have slope $\frac{1}{2}$ but while the parameter b is the same of the first poligonal the parameter a has been modified. This non regularity keeps occurring till the side of the smaller squares is smaller than 1.

5. – Regular Mosaics.

We have seen that in general mosaic leads to non regularity. We wish now to investigate a class for which regularity is preserved. It holds in fact the following

THEOREM 5.1. – *The classes of mosaic in which the knots belong to the periodicity grid are regular.*

Of course the mosaic is the most elementary: a lower-left triangle versus an upper-right triangle or an upper-left triangle versus a lower-right triangle. The latter case will not be considered because it is generated by a linear transform. Remark that regularity does not imply there are simple formulas for the calculus of the rotation index. The case of Theorem 3.1 is rather an exception derived by the use of a linear transform of a single-variable periodic function.

PROOF. – Without loss of generality we consider unitary periods $L = M = 1$. First of all we need a remark about irrational G-limits. If the limit G is irrational, there cannot exist any trajectory such that $y(0) = y^0$ and $y(n) = y^0 + m$, with m and n integers. We prove now that the set of parameters for which G is attained cannot have positive measure. In fact if it were so open balls should exist for which G is also attained. Suppose that the parameters p and q lie in the interior of the ball, then there exists $\varepsilon = 1/n > 0$ such that $p + \varepsilon$ and $q + \varepsilon$ still belong to the ball. Consider the trajectory coming out from $y(0) = 0$, with parameters p and q. Let y^* be the trajectory coming out of $y^*(0) = 0$ with the increased parameters. At the point $x = n$ we get thus $y^*(n) - y(n) = 1$. Hence there exists some value $0 < \varepsilon' < \varepsilon$ for which the corresponding solution satisfies $y^{**}(n) = m$, where m is an integer. But that means that for the limit it holds

$$G(p + \varepsilon', q + \varepsilon') = \frac{m}{n},$$

that is, it is a rational number, hence, in view of monotonicity it is strictly greater than the original irrational limit $G(p, q)$. Hence $G(p + \varepsilon, q + \varepsilon) \geq G(p + \varepsilon', q + \varepsilon') > G(p, q)$, what contradicts the hypothesis ([5]).

We consider now the case when the limit G is a rational number $\frac{m}{n}$. In this case there exists a trajectory such that $y(0) = y^0$ and $y(n) = y^0 + m$. By a translation we get $y^0 = 0$. Consider now also the trajectory such that $y^*(0) = 1$ and $y^*(n) = m + 1$. In the strip $y(x) < y < y^*(x)$, $0 < x < n$ there lay exactly $(n - 1)$ knots. In view of Lemma 4.1, in order to study the n-th iterated input-output function we need only to study the trajectories that pass through those $(n - 1)$ knots. Let $y^{**}(x)$ be the trajectory that passes through a knot (n^0, m^0). In view of periodicity we get $y^{**}(x) = y(x - n^0) + m^0$ for $x > n^0$, and in particular $y^{**}(n) = y(n - n^0) + m^0$. For $x < n^0$ we get $y^{**}(x) = y(x + n - n^0) - (m - m^0)$ and in particular for $y^{**}(0) = y(n - n^0) - (m - m^0)$. So $y^{**}(n) - y^{**}(0) = m = y(n) - y(0)$. Therefore the n-th iterated input-output function is actually linear and is given by $F(y^0) = m + y^0$.
Since any increase in parameters implies a positive increase of the input-output function and of all its iterates, this means that any increase no longer allows fixed point for which $y(0) = y^0$ and $y(n) = y^0 + m$. Hence we have proved that the class is regular. □

We now want to understand which are the main differences and the main similarities between the mosaic with triangular-shaped boxes and the classical chessboards. As the knots of this new mosaic belong to the periodicity grid due to Theorem 5.1 there is regularity: if we modify one of the value of the parameters the general slope is modified itself.

([5]) Of course the argument can easily be generalized to any mosaic, and, taking the limit, to any double periodic structure.

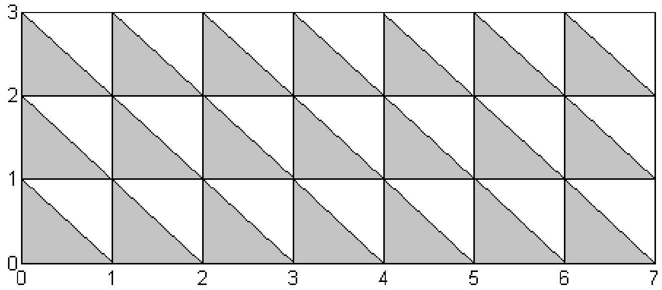

Fig. 5. – Mosaic whit triangle-shaped boxes

Let's try to find the equation that connects the two parameters p and q of the slopes in the triangular-shaped boxes. Starting from a point with coordinates $(0, y_0)$, let y_n be the y value of nth intersection with the vertical grid. We find the general relation

$$y_{n+1} = p\frac{1+q}{1+p} + y_n\frac{1+q}{1+p} \tag{5.1}$$

From (5.1), calling $A = p\frac{1+q}{1+p}$ and $B = \frac{1+q}{1+p}$ we have

$$y_n = B^n\left(y_0 - \frac{A}{1-B}\right) + \frac{A}{1-B}$$

with $y_0 = 0$

$$y_n = (1-B^n)\frac{A}{1-B} = \left(1 - \left(\frac{1+q}{1+p}\right)^n\right)\frac{1+q}{p-q}p$$

So, in the case of G-limit $\frac{1}{2}$ with the first parameter $p = 1$ (the slope in the triangles down on the left is a fixed value 1) we have $y_2 = 1$ and so the equation

$$1 = y_2 = \left(1 - \left(\frac{1+q}{2}\right)^2\right)\frac{1+q}{1-q}$$

that is

$$q^2 + 4q - 1 = 0$$

In the same way we find a similar relation in the case of G-limit $\frac{1}{3}$ and $p = 1$

$$q^3 + q^2 + 3q - 5 = 0$$

In the general case of slope $\frac{1}{n}$ and $p = 1$ we have

$$1 = \left(1 - \left(\frac{1+q}{2}\right)^n\right)\frac{1+q}{1-q}$$

that is (with $q \neq 1$)

$$2^{n+1}q - (1+q)^{n+1} = 0.$$

The next theorem allows us to enlarge to some extent the classes that are regular. It gives also the possibility of estimating the G-limit of a structure in some cases for which such a limit is known for another structure. Theorem 3.1 was a simple case of this procedure and succeeded because of the existence of explicit formulas for the single variable homogeneization.

THEOREM 5.2. – *Non singular linear transforms that are consistent with periodicity preserve the regularity or respectively the non-regularity of classes.*

The fact is obvious, since Theorem 2.2 establishes that the G-limit exists. Coefficients are transformed linearly and the values that correspond to those values that let agglutination arise will make agglutination arise in the new structure. The regularity is also preserved in view of the linearity. It is only necessary to find out which are the linear transforms that are consistent with periodicity. We must require that one point of the original grid is plotted somewhere on the y- axis, and some other point of the original grid is plotted somewhere on the x- axis, namely there must exist four integers n_1, m_1, n_2, m_2 such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_1 \\ m_1 \end{pmatrix} = \begin{pmatrix} 0 \\ M \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_2 \\ m_2 \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix}$$

Periods are obviously L and M respectively.

Remark finally that the main examples of this family of linear transforms are the rotation of $\frac{\pi}{4}$ and the shear transform that switches between the two types of triangles indicated at the beginning of this section.

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