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The Ordinary Differential Equation with non-Lipschitz Vector Fields (*)

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Abstract. – *In this note we survey some recent results on the well-posedness of the ordinary differential equation with non-Lipschitz vector fields. We introduce the notion of regular Lagrangian flow, which is the right concept of solution in this framework. We present two different approaches to the theory of regular Lagrangian flows. The first one is quite general and is based on the connection with the continuity equation, via the superposition principle. The second one exploits some quantitative a-priori estimates and provides stronger results in the case of Sobolev regularity of the vector field.*

1. – Introduction.

When $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded smooth vector field, the *flow of* b is the smooth map $X : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$
\begin{cases} \dfrac{\partial X}{\partial t}(t, x) = b(t, X(t, x)) \,, & t \in [0, T] \\ X(0, x) = x \,. \end{cases} \tag{1}
$$

Existence and uniqueness of the flow are guaranteed by the classical Cauchy-Lipschitz theorem. It turns out that additional regularity of the vector field is inherited by the flow, as for instance regularity of $x \mapsto X(t, x)$. The study of (1) out of the smooth context is of great importance (for instance, in view of the possible applications to conservation laws or to the theory of the motion of fluids) and has been studied by several authors. What can be said about the well-posedness of (1) when b is only in some class of weak differentiability? We remark from the beginning that no generic uniqueness result (i.e. for a.e. initial datum x) is presently available.

This question can be, in some sense, "relaxed" (and this relaxed problem can be solved, for example, in the Sobolev or BV framework): we look for a

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canonical selection principle, i.e. a strategy that “selects”, for a.e. initial datum x, a solution $X(\cdot, x)$ in such a way that this selection is stable with respect to smooth approximations of b. This in some sense amounts to redefine our notion of solution: we add some conditions which select a “relevant” solution of our equation. This is encoded in the concept of *regular Lagrangian flow* (see Definition 4): we consider only the flows such that there are no concentrations of the trajectories.

The plan of this note is the following. In Section 2 we introduce a related partial differential equation, the continuity equation, and for it we study measure-valued solutions; we present, in the smooth case, the connection between the continuity equation and the ordinary differential equation, which is based on the theory of characteristics. Section 3 is devoted to the notion of superposition solution and to the discussion of the superposition principle, which extends the theory of characteristics to the non-smooth case. In Section 4 we introduce the concept of regular Lagrangian flow, the good notion of solution to the ordinary differential equation in the non-smooth context, we show how uniqueness results for bounded solutions to the continuity equation imply the well-posedness of the regular Lagrangian flow and we present an outline of the theory of renormalized solutions and the main points in the well-posedness theory for the continuity equation. In Section 5 we review some recent results on the regularity of the flow with respect to the spatial variable. Finally, in Sections 6 and 7, we present the quantitative a-priori estimates for regular Lagrangian flows relative to Sobolev vector fields and we illustrate the various consequences of these estimates for the well-posedness and the regularity of the flow.

We close this introduction by presenting some notation which will be used in the sequel. Let X be a separable metric space. We denote by $\mathcal{P}(X)$ the family of the Borel probability measures on X, by $\mathcal{M}(X)$ the family of the Borel measures on X which are finite on compact sets and by $\mathcal{M}_+(X)$ the subset of $\mathcal{M}(X)$ consisting of all nonnegative Borel measures on X which are finite on compact sets. A measure $\mu \in \mathcal{M}(X)$ is *concentrated* on a Borel set $E \subset X$ if $|\mu|(X \setminus E) = 0$. If $\mu \in \mathcal{M}(X)$ and $E \subset X$ is a Borel set, the *restriction* of μ to E is the measure $\mu \llcorner E \in \mathcal{M}(X)$ defined by $(\mu \llcorner E)(A) = \mu(A \cap E)$ for every Borel set $A \subset X$. If $f : X \to Y$ is a Borel map between two separable metric spaces X and Y and $\mu \in \mathcal{M}(X)$ we denote by $f_\# \mu \in \mathcal{M}(Y)$ the *push-forward* of the measure μ, defined by

$$(f_\#\mu)(E) = \mu(f^{-1}(E)) \qquad \text{for every Borel set } E \subset Y. \tag{2}$$

We observe that the measure $f_\#\mu$ is characterized by the following relation:

$$\int_Y \varphi(y)\, d\big(f_\#\mu\big)(y) = \int_X \varphi(f(x))\, d\mu(x) \qquad \forall \varphi \in C_c(Y)\,. \tag{3}$$

We denote by $\mathscr{L}^d$ the Lebesgue measure on $\mathbb{R}^d$. We finally recall that a sequence of Borel maps $\{f_n\}$ defined on $\mathbb{R}^k$ is said to be *locally convergent in measure* to f in $\mathbb{R}^k$ if

$$\lim_{n\to\infty} \mathscr{L}^k\big(\{x \in B_R(0)\ :\ |f_n(x) - f(x)| > \delta\}\big) = 0$$

for every $R > 0$ and every $\delta > 0$. If the sequence $\{f_n\}$ is locally equibounded in $L^\infty(\mathbb{R}^k)$, then the local convergence in measure is equivalent to the strong convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^k)$.

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2. – Measure-valued solutions of the continuity equation.

In the smooth context, the ordinary differential equation (1) is related to a first-order partial differential equation, the so-called *continuity equation*

$$\partial_t \mu + \mathrm{div}\,(b\mu) = 0\,, \tag{4}$$

for which the Cauchy problem with initial data $\mu_0 = \bar{\mu} \in \mathcal{M}(\mathbb{R}^d)$ can be considered. In the above equation μ_t, for $t \in [0,T]$, is a family of locally finite signed measures on $\mathbb{R}^d$, which depends on the time parameter $t \in [0,T]$. The continuity equation is intended in the usual distributional sense: for every test function $\varphi \in C^\infty_c(]0,T[\times\mathbb{R}^d)$ we require that

$$\int_0^T \int_{\mathbb{R}^d} \big[\partial_t\varphi(t,x) + b(t,x)\cdot\nabla\varphi(t,x)\big]\, d\mu_t(x)\, dt = 0\,.$$

It is a standard result in the theory of evolutionary partial differential equations that, up to a redefinition of μ_t in a negligible set of times, the map $t\mapsto\mu_t$ is weakly* continuous with values in $\mathcal{M}(\mathbb{R}^d)$. This also gives a sense to the initial data $\mu_0 = \bar{\mu}$.

The connection between these two problems when b is smooth is based on the *theory of characteristics*: the solution μ_t of (4) at time t is given by the push-forward (according to the definition in (2)) of the initial data $\bar{\mu}$ via the flow:

$$\mu_t = X(t,\cdot)_\# \bar{\mu}\,. \tag{5}$$

This can be easily checked as follows. Notice first that we need only to check the distributional identity $\partial_t\mu + \mathrm{div}\,(b\mu) = 0$ on test functions of the form $\psi(t)\varphi(x)$, with $\psi \in C^\infty_c(]0,T[)$ and $\varphi \in C^\infty_c(\mathbb{R}^d)$, that is

$$\int_0^T \psi'(t)\langle\mu_t,\varphi\rangle\, dt + \int_0^T \psi(t)\int_{\mathbb{R}^d} b(t,x)\cdot\nabla\varphi(x)\, d\mu_t(x)\, dt = 0\,. \tag{6}$$

We notice that the map

$$t \mapsto \langle \mu_t, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\bar{\mu}(x)$$

belongs to $C^1([0,T])$, since the flow X is C^1 with respect to the time variable. In order to check (6) we need to show that the distributional derivative of this map is $\int_{\mathbb{R}^d} b(t,x) \cdot \nabla\varphi(x)\, d\mu_t(x)$, but by the C^1 regularity we only need to compute the pointwise derivative. Since the flow satisfies

$$\frac{\partial X}{\partial t}(t,x) = b(t, X(t,x))$$

for every t and x we can deduce

$$\begin{aligned} \frac{d}{dt}\langle \mu_t, \varphi \rangle &= \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\bar{\mu}(x) \\ &= \int_{\mathbb{R}^d} \nabla\varphi(X(t,x)) \cdot b(t, X(t,x))\, d\bar{\mu}(x) \\ &= \langle b(t,\cdot)\mu_t, \nabla\varphi \rangle\,, \end{aligned}$$

thus (5) holds.

3. – Superposition solutions of the continuity equation.

In order to understand better the meaning of the superposition principle we recall that, from formula (5) and from the characterization of the push-forward in (3), it follows that, when there is a unique flow $X(t,x)$ associated to the vector field b, the only solution of the continuity equation with initial data $\bar{\mu} \in \mathcal{M}(\mathbb{R}^d)$ is the measure μ_t characterized by

$$\langle \mu_t, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\bar{\mu}(x) \qquad \forall \varphi \in C_c(\mathbb{R}^d)\,. \tag{7}$$

In the following we use the notation Γ_T for the space $C([0,T];\mathbb{R}^d)$ of continuous paths in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$ let us consider a probability measure $\eta_x \in \mathcal{P}(\Gamma_T)$ concentrated on the trajectories $\gamma \in \Gamma_T$ which are absolutely continuous integral solutions of the ordinary differential equation with $\gamma(0) = x$. All the families $\{\eta_x\}_{x \in \mathbb{R}^d}$ in the following discussions are weakly measurable, i.e. for every function $\Phi \in C_b(\Gamma_T)$ the map

$$x \mapsto \langle \eta_x, \Phi \rangle = \int_{\Gamma_T} \Phi(\gamma)\, d\eta_x(\gamma)$$

is measurable.

DEFINITION 1 (Superposition solution). – *The superposition solution induced by the family $\{\eta_x\}_{x\in\mathbb{R}^d}$ is the family of measures $\mu_t^{\eta_x} \in \mathcal{M}(\mathbb{R}^d)$, for $t \in [0,T]$, defined as follows:*

$$\langle \mu_t^{\eta_x}, \varphi \rangle = \int_{\mathbb{R}^d} \left(\int_{\Gamma_T} \varphi(\gamma(t))\, d\eta_x(\gamma) \right) d\bar{\mu}(x) \qquad \forall \varphi \in C_c(\mathbb{R}^d)\,. \tag{8}$$

Using this notation we can give an alternative interpretation of (7). If for every $x \in \mathbb{R}^d$ the solution of the ODE starting from x is unique, then the only admissible measure η_x in (8) is $\eta_x = \delta_{X(\cdot,x)}$. But then we have

$$\langle \mu_t^{\delta_{X(\cdot,x)}}, \varphi \rangle = \int_{\mathbb{R}^d} \left(\int_{\Gamma_T} \varphi(\gamma(t))\, d\delta_{X(\cdot,x)}(\gamma) \right) d\bar{\mu}(x) = \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\bar{\mu}(x)\,,$$

so in this case we reduce to the "deterministic" formula (7). We can regard the superposition solution of Definition 1 as a "probabilistic" version of (7): if there is more than one solution to the ordinary differential equation, then we define our "averaged push-forward" by substituting the quantity $\varphi(X(t,x))$ with the average $\int_{\Gamma_T} \varphi(\gamma(t))\, d\eta_x(\gamma)$. It is not difficult, arguing as in the verification of (5), to check that (8) defines a solution of the continuity equation, for every family $\{\eta_x\}_{x\in\mathbb{R}^d}$ as above.

The superposition principle says that, for positive solutions, this construction can be reversed: every positive measure-valued solution μ_t of (4) can be realized as a superposition solution $\mu_t^{\eta_x}$ for some $\{\eta_x\}_{x\in\mathbb{R}^d}$ as above. For a proof of the superposition principle see for instance Section 8.2 of [9], Section 4 of [4] or Section 6.2 of [19].

THEOREM 2 (Superposition principle). – *Fix a vector field $b \in L^\infty([0,T]\times \mathbb{R}^d;\mathbb{R}^d)$ and let $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive locally finite measure-valued solution of the continuity equation. Then μ_t is a superposition solution, i.e. there exists a family $\{\eta_x\}_{x\in\mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$, with η_x concentrated on absolutely continuous integral solutions of the ODE starting from x, for $\bar{\mu}$-a.e. $x \in \mathbb{R}^d$, such that $\mu_t = \mu_t^{\eta_x}$ for any $t \in [0,T]$.*

From the superposition principle it follows a very general criterion relating the pointwise uniqueness for the ordinary differential equation with the uniqueness for positive measure-valued solutions to the continuity equation. Notice that, in order to give a meaning to the product $b\mu$ when μ is a measure, we assume b to be defined *everywhere* in $[0, T] \times \mathbb{R}^d$.

THEOREM 3. – *Let $A \subset \mathbb{R}^d$ be a Borel set. Then the following two properties are equivalent:*

(i) *solutions of the ordinary differential equation* (1) *are unique for every initial point $x \in A$;*

(ii) *positive measure-valued solutions of the continuity equation* (4) *are unique for every initial data $\bar{\mu}$ which is a positive measure concentrated on A.*

The result of Theorem 3 is very sharp and elegant, but its applicability is in fact very limited. On one hand, pointwise uniqueness for the ordinary differential equation is known only under very strong regularity assumptions on the vector field, namely under assumptions of Lipschitz regularity, one-sided Lipschitz condition, Osgood condition. On the other hand, uniqueness for the continuity equation is known only for particular classes of solutions, tipically for solutions which are bounded functions. It is reasonable that this kind of "weaker PDE uniqueness" should reflect into a weaker notion of uniqueness for the ODE: this leads to the concept of *regular Lagrangian flow*, which is presented in Definition 4.

The importance of the superposition principle also relies in the fact that it will allow, using truncations and restrictions of the measures η_x, several manipulations of solutions of the continuity equation: these constructions are not immediate at the level of the PDE, but they are extremely useful in various occasions, as we will see in the proof of Theorem 5.

4. – The regular Lagrangian flow.

As we have seen in the previous section, out of the smooth context the concept of pointwise uniqueness for the ODE is no more the appropriate one. We are going to illustrate in the sequel that well-posedness can be proved in the class of flows which do not create concentration of the trajectories. This notion is encoded in the following definition.

DEFINITION 4 (Regular Lagrangian flow). – *Let $b \in L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. We say that a map $X : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow for the vector field b if*

(i) *for a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t, x)$ is an absolutely continuous integral solution of $\dot{\gamma}(t) = b(t, \gamma(t))$ for t in $[0, T]$, with $\gamma(0) = x$;*

(ii) *there exists a constant L independent of t such that*

$$X(t,\cdot)_{\#}\mathscr{L}^d \le L\mathscr{L}^d \qquad \text{for every } t \in [0,T], \tag{9}$$

where we denote by $X(t,\cdot)_{\#}\mathscr{L}^d$ the push-forward of the Lebesgue measure via the flow, according to the definition in (2).

The constant L in (ii) will be called the compressibility constant of X.

4.1 – *"Abstract" theory of regular Lagrangian flows.*

In this subsection we present a sketch of the derivation of the results of uniqueness for the regular Lagrangian flow deduced from the well-posedness in the class of bounded solutions for the continuity equation. With the same techniques also existence and stability results can be proved. This abstract passage is due to Ambrosio; we refer to [1] for the original approach in the BV case and to [2] and [4] for the formalization of the argument in the general case. This approach is strongly based on the notion of superposition solution: starting from the "generalized flow" given by the measures $\eta_x \in \mathcal{P}(\mathbb{R}^d)$ we perform various constructions at the level of measure-valued solutions of the continuity equation; at that point the PDE well-posedness comes into play, allowing to deduce results about the measures η_x, roughly speaking obtaining that the generalized flow is in fact a regular Lagrangian flow, since η_x selects a single trajectory for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.

THEOREM 5 (Uniqueness of the regular Lagrangian flow). – Let $b \in L^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ *and assume that the continuity equation (4) has the uniqueness property in $L^\infty([0,T]\times\mathbb{R}^d)$. Then the regular Lagrangian flow associated to b, if it exists, is unique.*

This theorem is a simple consequence of the following proposition, which is an uniqueness result in the wider class of the "multivalued solutions" given by the measures $\{\eta_x\}_{x\in\mathbb{R}^d}$.

PROPOSITION 6. – *Let $b \in L^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ and assume that the continuity equation* (4) *has the uniqueness property in $L^\infty([0,T]\times\mathbb{R}^d)$. Consider a family $\{\eta_x\}_{x\in\mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$ such that η_x is concentrated on absolutely continuous integral solutions of the ordinary differential equation starting from x, for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$. Assume that the superposition solution of the continuity equation $\mu_t^{\eta_x}$ induced by this family belongs to $L^\infty([0,T]\times\mathbb{R}^d)$. Then η_x is a Dirac mass for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.*

The proof of Theorem 5 easily comes from Proposition 6. Assume that there exist two different regular Lagrangian flows $X_1(t,x)$ and $X_2(t,x)$ relative to the vector field b. We first consider for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ the measures $\eta_x^1 = \delta_{X_1(\cdot,x)}$ and $\eta_x^2 = \delta_{X_2(\cdot,x)}$. We set $\eta_x = \left(\eta_x^1 + \eta_x^2\right)/2$. It is easy to show that for the family of measures $\{\eta_x\}_{x\in\mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$ we have that

- for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ the measure η_x is concentrated on absolutely continuous integral solutions of the ordinary differential equation starting from x;
- the superposition solution $\mu_t^{\eta_x}$ induced by the family $\{\eta_x\}_{x\in\mathbb{R}^d}$ belongs to $L^\infty([0,T]\times\mathbb{R}^d)$;
- the measure η_x is not a Dirac mass for every x belonging to a set of positive Lebesgue measure, precisely for those $x \in \mathbb{R}^d$ such that $X_1(\cdot,x) \neq X_2(\cdot,x)$.

Thus the family $\{\eta_x\}_{x\in\mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$ defined above contradicts the result of Proposition 6, and the uniqueness result of Theorem 5 follows.

We now pass to the proof of Proposition 6. Assume by contradiction that η_x is not a Dirac mass for a non-negligible set of $x \in \mathbb{R}^d$. Thus we can find $\bar t \in]0,T]$, a Borel set $C \subset \mathbb{R}^d$ with $\mathscr{L}^d(C) > 0$ and a couple of disjoint Borel sets $E_1, E_2 \subset \mathbb{R}^d$ such that

$$\eta_x\big(\{\gamma \,:\, \gamma(\bar t) \in E_1\}\big)\, \eta_x\big(\{\gamma \,:\, \gamma(\bar t) \in E_2\}\big) \neq 0 \qquad \text{for every } x \in C.$$

Possibly passing to a smaller set C still having strictly positive Lebesgue measure we can assume that

$$0 < \eta_x\big(\{\gamma \,:\, \gamma(\bar t) \in E_1\}\big) \leq M\eta_x\big(\{\gamma \,:\, \gamma(\bar t) \in E_2\}\big) \qquad \text{for every } x \in C \tag{10}$$

for some constant M. We now want to localize to trajectories starting from the set C and arriving (at time $\bar t$) in the sets E_1 and E_2. We define

$$\eta_x^1 = \mathbf{1}_C(x)\eta_x \llcorner \{\gamma \,:\, \gamma(\bar t) \in E_1\} \quad \text{and} \quad \eta_x^2 = M\mathbf{1}_C(x)\eta_x \llcorner \{\gamma \,:\, \gamma(\bar t) \in E_2\}\,.$$

Now denote by μ_t^1 and μ_t^2 (for $t \in [0,\bar t]$) the superposition solutions of the continuity equation induced by the families of measures η_x^1 and η_x^2 respectively. It is easy to check that

$$\mu_0^1 = \eta_x\big(\{\gamma \,:\gamma(\bar t) \in E_1\}\big)\mathscr{L}^d \llcorner C$$

and

$$\mu_0^2 = M\eta_x\big(\{\gamma \,:\gamma(\bar t) \in E_2\}\big)\mathscr{L}^d \llcorner C\,.$$

Recalling (10) we obtain that $\mu_0^1 \leq \mu_0^2$. Now let $f : \mathbb{R}^d \to [0,1]$ be the density of μ_0^1 with respect to μ_0^2 (i.e. f satisfies $\mu_0^1 = f\mu_0^2$) and set

$$\tilde\eta_x^2 = Mf(x)\mathbf{1}_C(x)\eta_x \llcorner \{\gamma \,:\, \gamma(\bar t) \in E_2\}\,.$$

Consider the superposition solution $\tilde\mu_t^2$ (defined for $t \in [0,\bar t]$) induced by the family of measures $\tilde\eta_x^2$. We can readily check that $\mu_0^1 = \tilde\mu_0^2$ that $\mu_{\bar t}^1$ is concentrated on E_1 and $\tilde\mu_{\bar t}^2$ is concentrated on E_2. Hence μ_t^1 and $\tilde\mu_t^2$ are solutions in $L^\infty([0,T]\times\mathbb{R}^d)$

of the continuity equation with the same initial data at time $t = 0$, but they are different at time $t = \bar{t}$. We are violating the uniqueness assumption and from this contradiction we obtain the thesis of Proposition 6.

4.2 – *Well-posedness of the continuity equation.*

As we have just seen, in order to obtain existence, uniqueness and stability for the regular Lagrangian flow it is sufficient to show well-posedness of the continuity equation in the class of bounded weak solutions. The main strategy is the *theory of renormalized solutions*, which is due to DiPerna and Lions [26].

We say that a solution $\mu \in L^\infty([0,T] \times \mathbb{R}^d)$ to the continuity equation (4) is a *renormalized solution* if

$$\partial_t(\mu^2) + \operatorname{div}(b\mu^2) = -\mu^2 \operatorname{div} b \tag{11}$$

holds in the sense of distributions in $[0,T] \times \mathbb{R}^d$. Notice that (11) is trivially satisfied if μ is smooth, by an immediate application of the chain rule formula. A vector field b has the *renormalization property* if every bounded weak solution of the continuity equation with vector field b is a renormalized solution. The importance of this notion comes from the fact that, if the renormalization property holds, then the continuity equation is well-posed. The idea, at least at a formal level, is quite simple: we simply integrate (11) over $\mathbb{R}^d$ for every fixed time t, and assuming a sufficiently fast decay at infinity we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mu^2(t,x)\,dx = -\int_{\mathbb{R}^d} \mu^2(t,x)\operatorname{div} b(t,x)\,dx\,. \tag{12}$$

Assuming an L^∞ control on the divergence of b we can deduce

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mu^2(t,x)\,dx \le \|\operatorname{div} b\|_\infty \int_{\mathbb{R}^d} \mu^2(t,x)\,dx\,,$$

thus a simple application of the Gronwall inequality implies that, if $\bar{\mu} = 0$, then $\mu_t = 0$ for every t. Being the continuity equation linear, this is enough to prove uniqueness.

The main tool to show the renormalization property is a *regularization scheme* due to DiPerna and Lions. Since smooth solutions are renormalized, a natural attempt consists in convolving the continuity equation (4) with a regularization kernel ρ^ε in $\mathbb{R}^d$, using the classical chain rule for the regularized solution $\mu * \rho^\varepsilon$ and finally trying to pass to the limit in the equality. In order to do this, it turns out that it is necessary to study the behaviour of an error term, called the *commutator*, which has the form

$$R^\varepsilon = b \cdot \nabla(\mu * \rho^\varepsilon) - (b \cdot \nabla\mu) * \rho^\varepsilon\,.$$

In order to show convergence to zero in a sufficiently strong sense of the commutator R^ε it is necessary (at least at a first level) to have some control on the behaviour of the difference quotients of the vector field b. Here the weak differentiability assumptions on b come into play: the renormalization property can be proved, for instance, if b has Sobolev regularity (DiPerna and Lions [26]) or BV regularity (Ambrosio [1]) with respect to the spatial variable, and under boundedness assumptions on the divergence of the vector field. For a survey of some other renormalization results see for instance Chapter 2 of [19].

5. – Approximate differentiability of the regular Lagrangian flow.

The first result relative to the regularity of the regular Lagrangian flow with respect to the spatial variable is due to Le Bris and Lions [30]: for vector fields with Sobolev regularity, they are able to show (using an extension of the theory of renormalized solutions) the existence of measurable maps $W_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\frac{X(t,x+\varepsilon y) - X(t,x) - \varepsilon W_t(x,y)}{\varepsilon} \to 0 \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y. \tag{13}$$

However, it turns out (see [11]) that the differentiability property expressed in (13) does not imply the classical approximate differentiability. We recall that a map $f : \mathbb{R}^k \to \mathbb{R}^m$ is said to be *approximately differentiable at* $x \in \mathbb{R}^k$ if there exists a linear map $L(x) : \mathbb{R}^k \to \mathbb{R}^m$ such that

$$\frac{f(x+\varepsilon y) - f(x) - \varepsilon L(x)y}{\varepsilon} \to 0 \quad \text{locally in measure in } \mathbb{R}^d_y.$$

Notice also that this concept has a pointwise meaning, while the one in (13) is global. Moreover, it is possible to show that the map f is approximately differentiable a.e. in $\Omega \subset \mathbb{R}^k$ if and only if the following Lusin-type approximation with Lipschitz maps holds: for every $\varepsilon > 0$ it is possible to find a set $\Omega' \subset \Omega$ with $\mathscr{L}^k(\Omega \setminus \Omega') \leq \varepsilon$ such that $f|_{\Omega'}$ is Lipschitz.

Approximate differentiability for regular Lagrangian flows relative to $W^{1,p}$ vector fields, with $p > 1$, has been first proved by Ambrosio, Lecumberry and Maniglia in [10]. The need for considering only the case $p > 1$ comes from the fact that some tools from the theory of maximal functions are used, as will be explained in the next section. In [10] the strategy is no more an extension of the theory of renormalized solutions: the authors introduce some new estimates along the flow, inspired by the remark that, at a formal level, we can control the time derivative of $\log(|\nabla X(t,x)|)$ with $|\nabla b|(t,X(t,x))$. The strategy of [10] allows to make this remark rigorous: it is possible to consider some integral quantities which contain a discretization of the space gradient of the flow and prove some estimates along the flow. Then, the application of Egorov theorem allows the passage from integral estimates to pointwise estimates on big sets, and from this it is possible to recover Lipschitz regularity on big sets, and eventually one gets the approximate differentiability. However, the application of Egorov theorem implies a loss of quantitative informations: this strategy does not allow a control of the Lipschitz constant in terms of the size of the "neglected" set.

6. – Quantitative estimates for the regular Lagrangian flow.

Starting from the result of Ambrosio, Lecumberry and Maniglia [10], the main point of [21] is a modification of the estimates in such a way that quantitative informations are not lost. The main result of [21] is the following quantitative Lusin-type approximation of the regular Lagrangian flow with Lipschitz maps.

THEOREM 7 (Lipschitz quantitative estimate). – *Let $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ with $p > 1$ be a bounded vector field and let X be a regular Lagrangian flow associated to b. Fix $R > 0$. Then there exists a constant $C > 0$, which depends only on R, $\|Db\|_{L^1_t(L^p_x)}$ and the compressibility constant L of X, such that the following holds: for every $\varepsilon > 0$ it is possible to find a set $K \subset B_R(0)$ with $\mathscr{L}^d(B_R(0)\setminus K) \leq \varepsilon$ and*

$$\mathrm{Lip}(X(t,\cdot)|_K) \leq \exp\Big(C\varepsilon^{-1/p}\Big) \qquad \forall t \in [0,T]. \tag{14}$$

This means that we are able to estimate the growth of the Lipschitz constant in terms of the size of the neglected set. The proof of this result is divided in two steps, as illustrated in the remaining of this section.

6.1 – *A-priori estimate of an integral quantity.*

For every $R > 0$ we define the quantity

$$A_p(R,X) := \left\| \sup_{0\leq t\leq T} \sup_{0<r<R} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \right\|_{L^p_x(B_R(0))}.$$

It is possible to give an a-priori estimate for the functional $A_p(R,X)$ in terms of the $L^1_t(L^p_x)$ norm of Db and the compressibility constant of the flow. Trying to estimate the quantity

$$\frac{d}{dt} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \tag{15}$$

we get some difference quotients of the vector field computed along the flow. Here comes into play the theory of maximal functions.

We recall that, for $f \in L^1_{\rm loc}(\mathbb{R}^k;\mathbb{R}^m)$, we can define the *maximal function* of f as

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy\,.$$

It is well-known (see for instance [31]) that for every $p>1$ the strong estimate

$$\|Mf\|_{L^p(\mathbb{R}^k)} \leq C_{k,p}\|f\|_{L^p(\mathbb{R}^k)} \tag{16}$$

holds, while this is not true in the limit case $p=1$. Moreover, if f has Sobolev regularity, we can estimate the increments using the maximal function of the derivative: there exists a negligible set $N\subset\mathbb{R}^k$ such that for every x, $y\in\mathbb{R}^k\setminus N$ we have

$$|f(x)-f(y)| \leq C_k|x-y|\big(MDf(x)+MDf(y)\big)\,.$$

Going back to (15), we see that it is possible to estimate the difference quotients which appear in the time differentiation using the maximal function of Db, computed along the flow. After, we integrate with respect to the time, we pass to the supremums and eventually we take the L^p norm in order to reconstruct the quantity $A_p(R,X)$. Then, changing variable (and for this we just pay a factor given by the compressibility constant L) and using the strong estimate (16) in order to express the bound in term of Db, we finally get the a-priori quantitative estimate

$$A_p(R,X) \leq C\big(R, L, \|Db\|_{L^1_t(L^p_x)}\big)\,. \tag{17}$$

6.2 – *Quantitative Lipschitz property.*

From the bound (17) we can obtain the quantitative Lusin-type Lipschitz approximation of the regular Lagrangian flow stated in Theorem 7. For every fixed $\varepsilon>0$ and every $R>0$ we apply Chebyshev inequality to get a constant

$$M = M(\varepsilon) = \frac{A_p(R,X)}{\varepsilon^{1/p}}$$

and a set $K\subset B_R(0)$ with $\mathscr{L}^d(B_R(0)\setminus K)\leq\varepsilon$ such that for every $x\in K$

$$\sup_{0\leq t\leq T}\ \sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)dy \leq M\,.$$

From this pointwise control it easily follows that

$$|X(t,x)-X(t,y)| \leq \exp\left(\frac{c_d A_p(R,X)}{\varepsilon^{1/p}}\right)|x-y| \qquad \forall t\in[0,T]$$

for every x, $y\in K$, thus the conclusion follows.

7. – Well-posedness and properties of the regular Lagrangian flow.

7.1 – *Approximate differentiability.*

Recalling the equivalence stated immediately after the definition of approximate differentiability in Section 5, it follows that the regular Lagrangian flow is approximately differentiable with respect to the spatial variable $\mathscr{L}^d$-a.e. in $\mathbb{R}^d$. We notice that the quantitative result expressed in (14) is not strictly necessary for this first consequence.

7.2 – *Compactness.*

The quantitative version of the Lusin-type Lipschitz approximation can be used to show the precompactness in L^1_{loc} for the regular Lagrangian flows $\{X_n\}$ generated by a sequence $\{b_n\}$ of vector fields equibounded in L^∞ and in $L^1_t(W^{1,p}_x)$ (for $p > 1$), under the assumption that the compressibility constants of the regular Lagrangian flows are equibounded. We illustrate here the main idea to get this result.

On every ball $B_R(0)$, the regular Lagrangian flows are equibounded. We fix $\varepsilon > 0$. For every n we apply (14) to find a set K_n with $\mathscr{L}^d(B_R(0) \setminus K_n) \leq \varepsilon$ such that the Lipschitz constants of the maps $X_n|_{K_n}$ are equibounded. Then we can extend every map $X_n|_{K_n}$ to a map $\widetilde{X}_n$ defined on all $B_R(0)$ in such a way that the sequence $\{\widetilde{X}_n\}$ is equibounded and equicontinuous over $B_R(0)$. Hence we can apply Ascoli-Arzelà theorem to this sequence, getting strong compactness in L^∞. But since every map $\widetilde{X}_n$ coincides with the regular Lagrangian flow X_n out of a small set, it is simple to check that this implies strong compactness in L^1 for the regular Lagrangian flows $\{X_n\}$.

We finally remark that an extension of our strategy to the case $p = 1$ would give a positive answer to a conjecture proposed by Bressan in [15]. See also [14] for a related conjecture on mixing flows.

7.3 – *Quantitative stability.*

With similar techniques it is possible to show a result of quantitative stability for regular Lagrangian flows relative to $W^{1,p}$ vector fields (here we need again the assumption $p > 1$). The stability results in [26] and [1] were obtained using some abstract compactness arguments, hence they do not give a rate of convergence. However, we are able to show that, for the regular Lagrangian flows X_1 and X_2 relative to bounded vector fields b_1 and b_2 belonging to $L^1_t(W^{1,p}_x)$, the following estimate holds:

$$\|X_1(T,\cdot) - X_2(T,\cdot)\|_{L^1(B_r)} \leq C \left|\log\left(\|b_1 - b_2\|_{L^1([0,T]\times B_R)}\right)\right|^{-1}.$$

The constants C and R depend on the usual uniform bounds on the vector fields. We remark that this estimate also gives a new direct proof of the uniqueness of the regular Lagrangian flow.

7.4 – *An a-priori estimates purely Lagrangian approach.*

The estimates we have presented give a new possible approach to the theory of regular Lagrangian flows. In particular, we can develop (in the $W^{1,p}$ context with $p > 1$) a theory of ODEs completely independent from the associated PDE theory. The general scheme is the following:

- the compactness we have illustrated can be used to show *existence* of the regular Lagrangian flow, via regularization, for vector fields with bounded divergence;
- the *uniqueness* comes together with the *stability*, which is recovered in a new quantitative fashion;
- in addition to these results, we can show the quantitative *regularity* expressed in (14), which implies the *approximate differentiability*;
- finally, a new *compactness* result is obtained.

All this results are obtained at the Lagrangian level, with no mention to the transport equation theory.

REFERENCES

[1] L. AMBROSIO, *Transport equation and Cauchy problem for BV vector fields.* Invent. Math., **158** (2004), 227-260.

[2] L. AMBROSIO, *Transport equation and Cauchy problem for non-smooth vector fields.* Lecture Notes in Mathematics “Calculus of Variations and Non-Linear Partial Differential Equation” (CIME Series, Cetraro, 2005) **1927**, B. Dacorogna and P. Marcellini eds., 2-41, 2008.

[3] L. AMBROSIO - F. BOUCHUT - C. DE LELLIS, *Well-posedness for a class of hyperbolic systems of conservation laws in several space dimensions.* Comm. PDE, **29** (2004), 1635-1651.

[4] L. AMBROSIO - G. CRIPPA, *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields.* To appear in the series UMI Lecture Notes (available at http://cvgmt.sns.it).

[5] L. AMBROSIO - G. CRIPPA - S. MANIGLIA, *Traces and fine properties of a BD class of vector fields and applications.* Ann. Sci. Toulouse, **XIV** (4) (2005), 527-561.

[6] L. AMBROSIO - C. DE LELLIS, *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions.* International Mathematical Research Notices, **41** (2003), 2205-2220.

[7] L. AMBROSIO - C. DE LELLIS - J. MALÝ, *On the chain-rule for the divergence of BV like vector fields: applications, partial results, open problems.* To appear in the forthcoming book by the AMS series in contemporary mathematics “Perspectives in Nonlinear Partial Differential Equations: in honor of Haim Brezis” (available at http://cvgmt.sns.it).

[8] L. AMBROSIO - N. FUSCO - D. PALLARA, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, 2000.

[9] L. AMBROSIO - N. GIGLI - G. SAVARÉ, *Gradient flows in metric spaces and in the Wasserstein space of probability measures.* Lectures in Mathematics, ETH Zurich, Birkhäuser, 2005.

[10] L. AMBROSIO - M. LECUMBERRY - S. MANIGLIA, *Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow.* Rend. Sem. Mat. Univ. Padova, **114** (2005), 29-50.

[11] L. AMBROSIO - J. MALÝ, *Very weak notions of differentiability.* Proceedings of the Royal Society of Edinburgh, **137A** (2007), 447-455.

[12] F. BOUCHUT, *Renormalized solutions to the Vlasov equation with coefficients of bounded variation.* Arch. Rational Mech. Anal., **157** (2001), 75-90.

[13] F. BOUCHUT - G. CRIPPA, *Uniqueness, Renormalization, and Smooth Approximations for Linear Transport Equations.* SIAM J. Math. Anal., **38** (2006), 1316-1328.

[14] A. BRESSAN, *A lemma and a conjecture on the cost of rearrangements.* Rend. Sem. Mat. Univ. Padova, **110** (2003), 97-102.

[15] A. BRESSAN, *An ill posed Cauchy problem for a hyperbolic system in two space dimensions.* Rend. Sem. Mat. Univ. Padova, **110** (2003), 103-117.

[16] F. COLOMBINI - G. CRIPPA - J. RAUCH, *A note on two-dimensional transport with bounded divergence.* Comm. PDE, **31** (2006), 1109-1115.

[17] F. COLOMBINI - N. LERNER, *Uniqueness of L^∞ solutions for a class of conormal BV vector fields.* Contemp. Math. **368** (2005), 133-156.

[18] G. CRIPPA, *Equazione del trasporto e problema di Cauchy per campi vettoriali debolmente differenziabili.* Tesi di Laurea, Università di Pisa, 2006 (available at http://cvgmt.sns.it).

[19] G. CRIPPA, *The flow associated to weakly differentiable vector fields.* PhD Thesis, Scuola Normale Superiore and Universität Zürich, 2007 (available at http://cvgmt.sns.it).

[20] G. CRIPPA - C. DE LELLIS, *Oscillatory solutions to transport equations.* Indiana Univ. Math. J., **55** (2006), 1-13.

[21] G. CRIPPA - C. DE LELLIS, *Estimates and regularity results for the DiPerna-Lions flow.* Preprint, 2006 (available at http://cvgmt.sns.it). J. Reine Angew. Math., in press.

[22] G. CRIPPA - C. DE LELLIS, *Regularity and compactness for the DiPerna-Lions flow.* Hyperbolic Problems: Theory, Numerics, Applications. S. Benzoni-Gavage and D. Serre eds. (2008).

[23] C. DE LELLIS, *Notes on hyperbolic systems of conservation laws and transport equations.* Handbook of Differential Equations: Evolutionary Equations, vol. III. Edited by C. M. Dafermos and E. Feireisl. Elsevier/North-Holland, Amsterdam, 2006.

[24] C. DE LELLIS, *Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio (d'après Ambrosio, DiPerna, Lions).* Séminaire Bourbaki, vol. 2006/2007, n. 972.

[25] N. DEPAUW, *Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan.* C.R. Math. Sci. Acad. Paris, **337** (2003), 249-252.

[26] R. J. DIPERNA - P.-L. LIONS, *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511-547.

[27] R. J. DIPERNA - P.-L. LIONS, *On the Cauchy problem for the Boltzmann equation: global existence and weak stability.* Ann. of Math., **130** (1989), 312-366.

[28] B. L. Keyfitz - H. C. Kranzer, *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory.* Arch. Rational Mech. Anal., **72** (1980), 219-241.
[29] S. N. Kružkov, *First order quasilinear equations with several independent variables.* Mat. Sb. (N.S.), **81** (1970), 228-255.
[30] C. Le Bris - P.-L. Lions, *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications.* Annali di Matematica, **183** (2003), 97-130.
[31] E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970.

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