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Connected Components of Hurwitz Spaces of Coverings with One Special Fiber and Monodromy Groups Contained in a Weyl Group of Type B_d

FRANCESCA VETRO

Sunto. – *In questo articolo vengono studiati rivestimenti* $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ *dove* X, X', Y *sono curve proiettive complesse non singolari e* f *è un rivestimento di grado* $d \geq 3$*, con gruppo di monodromia* S_d*, ramificato in* $n_2 + 1$ *punti uno dei quali è un punto speciale* c *la cui monodromia locale ha struttura ciclica data dalla partizione* $\underline{e} = (e_1, \ldots, e_r)$ *di* d*. Inoltre* π *è un rivestimento ramificato di grado 2 con luogo discriminante contenuto in* $f^{-1}(c)$*. Se si suppone* $n_2 + |\underline{e}| \geq 2d$*, dove* $|\underline{e}| = \sum_{i=1}^{r} (e_i - 1)$*, questi rivestimenti hanno come gruppo di monodromia* G *un gruppo di Weyl di tipo* D_d *oppure* B_d*. In questo articolo viene dimostrato che quando* $G = W(D_d)$ *e* $n_2 + |\underline{e}| \geq 2d$ *gli spazi di Hurwitz che parametrizzano rivestimenti come sopra sono irriducibili, mentre quando* $G = W(B_d)$ *non lo sono e, in quest'ultimo caso, ne vengono determinate le componenti connesse. In questo modo viene completato lo studio dell'irriducibilità degli spazi di Hurwitz che parametrizzano rivestimenti con una fibra speciale e con gruppo di monodromia un gruppo di Weyl di tipo* $W(B_d)$ *iniziato in* [22].

Abstract. – *Let* X, X', Y *be smooth projective complex curves with* Y *curve of genus* ≥ 1*. Let* d *be an integer* ≥ 3*, let* $\underline{e} = (e_1, \ldots, e_r)$ *be a partition of* d *and let* $|\underline{e}| = \sum_{i=1}^{r} (e_i - 1)$*. Let* $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ *be a sequence of coverings where* π *is a degree 2 branched covering and* f *is a degree* d *covering, with monodromy group* S_d*, branched in* $n_2 + 1$ *points, one of which is special point* c *whose local monodromy has cycle type given by* $\underline{e}$*. Moreover the branch locus of the covering* π *is contained in* $f^{-1}(c)$*. In this paper we prove the irreducibility of the Hurwitz spaces that parameterize sequences of coverings as above with monodromy group a Weyl group of type* D_d *when* $n_2 + |\underline{e}| \geq 2d$*. Besides we determine the connected components of the Hurwitz spaces that parameterize sequences of coverings as above but with monodromy group a Weyl group of type* B_d*.*

Introduction.

A classical result of Hurwitz based on an earlier work of Clebsch and Lüroth states the irreducibility of Hurwitz spaces $H_{d,n}(\mathbb{P}^1)$ which parametrize simple coverings of $\mathbb{P}^1$ of degree d branched in n points [11]. In [20] Severi used this result in order to prove the irreducibility of the moduli space M_g of curves of genus g. Natanzon and Kluitmann considered coverings of $\mathbb{P}^1$ with one special fiber and proved independently the irreducibility of the corresponding Hurwitz spaces (see resp. [18], [16]). Natanzon's paper refers to some applications of this result to the theory of integrable systems. The problem of irreducibility of Hurwitz spaces was studied by Fried and Völklein with regard to applications to inverse Galois theory [7]. The Hurwitz spaces of simple coverings of curves of positive genus were studied by Harris, Graber and Starr in [9]. They proved an irreducibility result for spaces which parametrize degree d coverings branched in n points and having full monodromy group when $n \geq 2d$. This was used in their famous theorem for existence of sections of one-parameter family of complex rationally connected varieties [10]. The Hurwitz spaces $H^o_{d,n,\underline{e}}(Y)$ which parametrize degree d coverings of curves Y of positive genus with one special fiber and full monodromy group were studied in [14] and [21] where their irreducibility was proved when the number n of simple branch points is sufficiently large. The best estimate is established in [21] where the irreducibility of the spaces $H^o_{d,n,\underline{e}}(Y)$ is proved under the hypothesis $n + |\underline{e}| \geq 2d$ where $\underline{e} = (e_1, \dots, e_r)$ is the partition of d that gives the local monodromy of the special fiber and $|\underline{e}| = \sum_{i=1}^{r} (e_i - 1)$.

The symmetric group S_d is the Weyl group of the root system of type A_{d-1} and it is natural to study coverings whose monodromy group is contained in an arbitrary Weyl group. We refer the reader to the papers [5], [12], [13] where coverings of this type and their connection with spectral curves are studied. Biggers and Fried in [1] proved the irreducibility of Hurwitz spaces of coverings of $\mathbb{P}^1$, whose monodromy group is a Weyl group of type D_d, which have simple branching in the sense that each local monodromy is a reflection. Kanev in [15] generalized the result to Hurwitz spaces parameterizing Galois coverings of $\mathbb{P}^1$ whose Galois group is an arbitrary Weyl group. The author studied in [22], coverings with monodromy group the Weyl group $W(B_d)$. Such coverings may be decomposed as follows: $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where X, X', Y are smooth, connected, projective complex curves, π is a degree 2 covering with $n_1 > 0$ branch points and f is a degree $d \geq 3$ covering, with monodromy group S_d, branched in $n_2 + 1$ points, $n_2 > 0$ of which are points of simple branching while one is a special point c whose local monodromy has cycle type $\underline{e}$. Denote by D_π and D_f respectively the branch locus of π and f. In particular in [22] it was studied sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ as above such that either $f(D_\pi) \cap D_f = \emptyset$ or $D_\pi \cap f^{-1}(c) \neq \emptyset$ but D_π is not contained in $f^{-1}(c)$ and it was proved the irreducibility of the corresponding Hurwitz spaces when $Y \simeq \mathbb{P}^1$ and then it was extended the result to curves of genus ≥ 1 under the hypothesis $n_2 + |\underline{e}| \geq 2d$. In the end it was proved the irreducibility of Hurwitz spaces that parameterize coverings $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$ with $D_\pi \subset f^{-1}(c)$.

In this paper we complete the study of coverings of the type $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ with ramified π and with one special fiber, considering the remaining case: $D_\pi \subset f^{-1}(c)$ and $g(Y) \geq 1$. The arguments of [22] does not work in this case and in fact it turns out surprisingly that the corresponding Hurwitz spaces are not always irreducible. Namely, there are two possible cases under the hypothesis $n_2 + |\underline{e}| \geq 2d$. The monodromy group G of the coverings is either $W(D_d)$ or $W(B_d)$. The two main results in the paper are the following: in Theorem 1 we prove that when the monodromy group is $W(D_d)$ the corresponding Hurwitz spaces are irreducible. In Theorem 2 we prove that when $G = W(B_d)$ the corresponding Hurwitz spaces have $2^{2g} - 1$ connected components where $g = g(Y)$.

CONVENTIONS. Two degree d branched coverings $h_1 : X_1 \to Y$ and $h_2 : X_2 \to Y$ are called *equivalent* if there exists a biholomorphic map $p : X_1 \to X_2$ such that $h_2 \circ p = h_1$. We denote by $[h_1]$ the equivalence class containing h_1. Moreover here the natural action of S_d on $\{1, \ldots, d\}$ is on the *right*. We denote the action of $\sigma \in S_d$ on i with i^σ.

1. – Weyl groups of type B_d.

In this section we recall some facts about Weyl groups of type B_d. We use the notation of [3] and use some definitions of [4].

1.1 – Let $d \geq 3$ be an integer. Let $\{\varepsilon_1, \ldots, \varepsilon_d\}$ be the standard base of $\mathbb{R}^d$ and let R be the root system $\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : 1 \leq i,j \leq d\}$. The Weyl group of type B_d, $W(B_d)$, is generated by the reflections s_{ε_i}, $i = 1, \ldots, d$, and $s_{\varepsilon_i - \varepsilon_j}$, $1 \leq i < j \leq d$. The reflection s_{ε_i} interchanges ε_i and $-\varepsilon_i$ while unchanging each ε_h with $h \neq i$. The reflection $s_{\varepsilon_i - \varepsilon_j}$ interchanges ε_i and ε_j, $-\varepsilon_i$ and $-\varepsilon_j$, leaving unchanged ε_h for each $h \neq i,j$. If we identify $\{\pm\varepsilon_i : i = 1, \ldots, d\}$ with $\{-d, \ldots, -1, 1, \ldots, d\}$ by the map $\pm\varepsilon_i \to \pm i$, the action of $W(B_d)$ over $\{\pm\varepsilon_i : i = 1, \ldots, d\}$ allows us to define an injective homomorphism τ from $W(B_d)$ into S_{2d} that sends $s_{\varepsilon_i - \varepsilon_j}$ to $(i\ j)(-i\ \ -j)$, s_{ε_i} to $(i\ \ -i)$ and $s_{\varepsilon_i + \varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}$ to $(i\ \ -j)(-i\ j)$.

In particular, if one ignores the sign-changes, each element $w \in W(B_d)$ determines a permutation of the indexes $1, \ldots, d$ that can be expressed in the usual way as a product of disjoint cycles. Let $(i_1 i_2 \ldots i_e)$ be a such cycle. Then w sends $\pm\varepsilon_{i_j}$ to $\pm\varepsilon_{i_{j+1}}$, $j = 1, \ldots, e-1$, and $\pm\varepsilon_{i_e}$ to $\pm\varepsilon_{i_1}$. The cycle $(i_1 \ldots i_e)$ is called positive if $w^e(\varepsilon_{i_1}) = \varepsilon_{i_1}$, and negative if $w^e(\varepsilon_{i_1}) = -\varepsilon_{i_1}$. The lengths of these cycles together with their signs give a set of positive or negative integers called the signed cycle-type of w.

DEFINITION 1. – *We call* positive (negative) cycle of the form $(i_1 \ldots i_e)$ *each element w belonging to $W(B_d)$ satisfying the following: w sends $\pm\varepsilon_{i_j}$ to $\pm\varepsilon_{i_{j+1}}$,*

$j = 1, \ldots, e$ where $i_{e+1} := i_1$, leaving unchanged ε_h for each $h \notin \{i_1, \ldots, i_e\}$, $1 \leq h \leq d$ and moreover $w^e(\varepsilon_{i_1}) = \varepsilon_{i_1}$ (resp. $w^e(\varepsilon_{i_1}) = -\varepsilon_{i_1}$). The integer e is called the length *of the cycle w. Two cycles in $W(B_d)$ of the form $(i_1 \ldots i_e)$ and $(h_1 \ldots h_l)$ are* disjoint *if $(i_1 \ldots i_e)$ and $(h_1 \ldots h_l)$ are disjoint cycles of S_d.*

A positive cycle of the form $(i_1 \ldots i_e)$ corresponds in S_{2d} to a product of two disjoint e-cycles, ss', which move the indexes $\{\pm i_1, \ldots, \pm i_e\}$ and are such that if s sends i_j to i_{j+1} (i_j to $-i_{j+1}$) then s' sends $-i_j$ to $-i_{j+1}$ (resp. $-i_j$ to i_{j+1}), where $\pm i_{e+1} := \pm i_1$. Instead a negative cycle of the form $(i_1 \ldots i_e)$ corresponds in S_{2d} to a $2e$-cycle of type $(i_1 \ \pm i_2 \ \ldots \ \pm i_e \ \ -i_1 \ \ \mp i_2 \ \ldots \ \mp i_e)$.

Each element of $W(B_d)$ can be expressed as a product of disjoint positive and negative cycles. So it is easy to see that two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle-type.

Let $(\boldsymbol{Z}_2)^d$ be the set of the functions from $\{1, \ldots, d\}$ into $\boldsymbol{Z}_2$. Here we consider $(\boldsymbol{Z}_2)^d$ endowed with the sum operation and we denote by z_{ij} the function of $(\boldsymbol{Z}_2)^d$ so defined

$$z_{ij}(i) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(h) = \bar{0} \quad \text{for each} \quad h \neq i, j \quad \text{and } z \in \boldsymbol{Z}_2.$$

Moreover we denote by $\bar{1}_{i \ldots h}$ the function of $(\boldsymbol{Z}_2)^d$ that sends to $\bar{1}$ only the indexes $i, \ldots, h$. Let Φ be the homomorphism from S_d in $Aut((\boldsymbol{Z}_2)^d)$ which assigns to $t \in S_d$ $\Phi(t) \in Aut((\boldsymbol{Z}_2)^d)$ where

$$[\Phi(t)\ z']\ (j) \ := z'(j^t) \quad \text{for each} \quad z' \in (\boldsymbol{Z}_2)^d.$$

Let $(\boldsymbol{Z}_2)^d \times^s S_d$ be the semidirect product of $(\boldsymbol{Z}_2)^d$ and S_d through the homomorphism Φ. Given $(z'; t_1), (z''; t_2) \in (\boldsymbol{Z}_2)^d \times^s S_d$ we let

$$(z'; t_1)\ (z''; t_2) := (z' + \Phi(t_1) z'';\ t_1 t_2).$$

It is easy to check that it is possible to define an isomorphism Ψ from $W(B_d)$ into $(\boldsymbol{Z}_2)^d \times^s S_d$ which sends $s_{\varepsilon_i - \varepsilon_j}$ to $(0; (i\ j))$, s_{ε_i} to $(\bar{1}_i; id)$ and $s_{\varepsilon_i + \varepsilon_j}$ to $(\bar{1}_{ij}; (i\ j))$.

In particular the isomorphism Ψ sends a positive (negative) cycle of the form $(i_1 i_2 \ldots i_e)$ to an element of type $(\bar{1}_{i_h \ldots i_k}; (i_1 i_2 \ldots i_e))$ where $\{i_h, \ldots, i_k\} \subseteq \{i_1, i_2, \ldots, i_e\}$ and $\sharp\{i_h, \ldots, i_k\}$ is even (resp. odd).

From now on we will denote by $W(D_d)$ the subgroup of $W(B_d)$ generated by the reflections with respect to the long roots $\varepsilon_1 - \varepsilon_i$ and $\varepsilon_1 + \varepsilon_i$ with $2 \leq i \leq d$. So $W(D_d)$ is isomorphic to the subgroup of $(\boldsymbol{Z}_2)^d \times^s S_d$ generated by the elements $(0; (1\ i))$ and $(\bar{1}_{1i}; (1\ i))$, $2 \leq i \leq d$. Note that positive cycles and products of an even number of negative cycles are elements of $W(D_d)$.

1.2 – Let $\underline{e} = (e_1, \ldots, e_r)$ be a partition of d where $e_1 \geq \cdots \geq e_r$. Let w be an element belonging to $W(B_d)$ that determines on the indexes $1, \ldots, d$ a permutation with cycle type given by $\underline{e}$. So w is product of r disjoint positive and negative cycles which have lengths $e_1, \dots, e_r$. Let l be the number of the negative cycles. Since disjoint cycles commute we can place positive and negative cycles in decrescent order of length. Moreover if $e_1 = \cdots = e_{r_1}$ and f_1 negative cycles have length e_1, we can place to the first f_1 places the negative cycles. The same thing we can do if $e_{r_i+1} = \cdots = e_{r_{i+1}}$ and there are f_i negative cycles of length e_{r_i+1}. Let us denote by $\{j_1, \dots, j_l\}$ the subset of $\{1, \dots, r\}$ such that

$$j_1 = 1, \dots, j_{f_1} = f_1, j_{f_1+1} = r_1 + 1, \dots, j_{f_2} = r_1 + f_2, \dots, j_l = r_h + f_{h+1}.$$

In this way we associated to w one partition $\underline{e}$ of d and a subset $\{j_1, \dots, j_l\}$ of $\{1, \dots, r\}$. Now we use $\underline{e}^n$ to denote the partition $(e_{j_1}, \dots, e_{j_l})$ and use $\underline{e}^p$ to denote the partition $(e_{f_1+1}, \dots, e_{r_1}, \dots, e_{j_l+1}, \dots, e_r)$ determined by the lengths of the positive cycles.

DEFINITION 2. – *A* double partition of size d, $(\underline{a}, \underline{b})$, *is an ordered pair of partitions* $\underline{a} = (a_1, \dots, a_l)$ *and* $\underline{b} = (b_1, \dots, b_s)$ *such that* $a_1 + \cdots + a_l + b_1 + \cdots + b_s = d$. *$l$ and s are called respectively the* length of $\underline{a}$ and the length of $\underline{b}$.

The ordered pair $(\underline{e}^n, \underline{e}^p)$ is the double partition of size d that gives the signed cycle-type of w.

Conversely, let $(\underline{e}^n, \underline{e}^p)$ be a double partition of size d such that $\underline{e}^n = (a_1, \dots, a_l)$ and $\underline{e}^p = (b_1, \dots, b_s)$. Let $l + s = r$. If we place the a_i and the b_j in decrescent order and every time $a_i = b_j$ we first place a_i, we determine a partition $\underline{e} = (e_1, \dots, e_r)$ of d and a subset $\{j_1, \dots, j_l\}$ of $\{1, \dots, r\}$ such that $e_1 \geq \cdots \geq e_r$, $\underline{e}^n = (e_{j_1}, \dots, e_{j_l})$, $\underline{e}^p = (e_1, \dots, \widehat{e}_{j_1}, \dots, \widehat{e}_{j_l}, \dots, e_r)$ and if $(j_i - 1) \notin \{j_1, \dots, j_l\}$ then $e_{j_i - 1} > e_{j_i}$. So an element of $W(B_d)$ with signed cycle-type given by the double partition $(\underline{e}^n, \underline{e}^p)$ induces on the indexes $1, \dots, d$ a permutation whose cycle type is given by the partition $\underline{e}$.

We call $\underline{e}$ and $\{j_1, \dots, j_l\}$ *respectively the partition of d and the subset of* $\{1, \dots, r\}$ *associated to* $(\underline{e}^n, \underline{e}^p)$.

DEFINITION 3. – *We denote by C the conjugate class of $(\boldsymbol{Z}_2)^d \times^s S_d$ containing elements of type $(z_{ij}; (ij))$ and by $C_{(\underline{e}^n, \underline{e}^p)}$ the conjugate class of $(\boldsymbol{Z}_2)^d \times^s S_d \cong W(B_d)$ containing elements with signed cycle-type given by the double partition of size d $(\underline{e}^n, \underline{e}^p)$. We denote by $(a'; \xi)$ an element of $(\boldsymbol{Z}_2)^d \times^s S_d$ belonging to $C_{(\underline{e}^n, \underline{e}^p)}$.*

Note that ξ is the permutation that $(a'; \xi)$ determines on the indexes $1, \dots, d$. Moreover if the partition $\underline{e}^n$ has length l, the function a' sends to $\bar{1}$ an odd number of indexes moved by each of the l negative cycles of which is product ξ.

From now on every time we will write an element of $W(B_d)$ as product of disjoint positive and negative cycles, we will suppose these cycles placed in decrescent order of length and so that if a positive cycle has same length of a negative cycle, the negative cycle comes first positive one.

2. – The Hurwitz spaces $H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$.

In this section we define the Hurwitz spaces that will be object of our study. Let $d \geq 3$ be an integer.

DEFINITION 4. – *An ordered sequence* $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ *of elements of* $(\boldsymbol{Z}_2)^d \times^s S_d \simeq W(B_d)$ *such that* $t_i \neq (0; id)$ *for each* $i = 1, \ldots, n$ *and* $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$ *is called a* Hurwitz system with values in $(\boldsymbol{Z}_2)^d \times^s S_d$. *The subgroup of* $(\boldsymbol{Z}_2)^d \times^s S_d$ *generated by* $t_i,\ \lambda_k,\ \mu_k$ *with* $i = 1, \ldots, n$ *and* $k = 1, \ldots, g$ *is called the* monodromy group *of the Hurwitz system.*

Note that if $g = 0$ the Hurwitz systems $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ are of the form $(t_1, \ldots, t_n)$ and $t_1 \cdots t_n = (0; id)$.

DEFINITION 5. – *Two Hurwitz systems with values in* $(\boldsymbol{Z}_2)^d \times^s S_d \simeq W(B_d)$, $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ *and* $(\tilde{t}_1, \ldots, \tilde{t}_n; \tilde{\lambda}_1, \tilde{\mu}_1, \ldots, \tilde{\lambda}_g, \tilde{\mu}_g)$, *are called* equivalent *if there exists* $s \in (\boldsymbol{Z}_2)^d \times^s S_d$ *such that* $\tilde{t}_i = s^{-1}\, t_i\, s$, $\tilde{\lambda}_k = s^{-1}\, \lambda_k\, s$ *and* $\tilde{\mu}_k = s^{-1}\, \mu_k\, s$ *for each* $i = 1, \ldots, n$, $k = 1, \ldots, g$. *The equivalence class containing* $(t_1, \ldots, \mu_g)$ *is denoted by* $[t_1, \ldots, \mu_g]$.

Let X, X' and Y be smooth, connected, projective complex curves of genus ≥ 0. Let n_1 and n_2 be positive integers and let $\underline{e} = (e_1, \ldots, e_r)$ be a partition of d where $e_1 \geq \ldots \geq e_r$. In this paper we will work with sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ satisfying the following:

($\star$) *π is a degree 2 branched covering with n_1 branch points and f is a degree d covering, with monodromy group S_d, branched in $n_2 + 1$ points, n_2 of which are points of simple branching while one is a special point c whose local monodromy has cycle type $\underline{e}$. Moreover the branch locus D_π of π is contained in $f^{-1}(c)$.*

OBSERVATION 1. – From Hurwitz formula it follows that n_1 is even so $n_1 \geq 2$ and consequently $r = \sharp f^{-1}(c) \geq 2$.

DEFINITION 6. – *Two sequences of coverings* $X_1 \xrightarrow{\pi_1} X_1' \xrightarrow{f_1} Y$ *and* $X_2 \xrightarrow{\pi_2} X_2' \xrightarrow{f_2} Y$ *are called* equivalent *if there exist two biholomorphic maps* $p : X_1 \to X_2$ *and* $p' : X_1' \to X_2'$ *such that* $p' \circ \pi_1 = \pi_2 \circ p$ *and* $f_2 \circ p' = f_1$. *The equivalence class containing the covering* $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ *is denoted by* $[X \xrightarrow{\pi} X' \xrightarrow{f} Y]$.

A sequence of coverings, $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, that satisfies the conditions ($\star$) is a degree $2d$ branched covering of Y with branch locus $D = D_f$ where D_f denotes the branch locus of f. Let $b_0 \in Y - D$ and let $m : \pi_1(Y - D, b_0) \to S_{2d}$ be the monodromy homomorphism associated to $X \xrightarrow{\pi} X' \xrightarrow{f} Y$. The images via the monodromy homomorphism m of a standard generating system for $\pi_1(Y - D, b_0)$ determine an equivalence class $[t_1, \dots, t_{n_2+1};\ \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ of Hurwitz systems with values in $(\boldsymbol{Z}_2)^d \times^s S_d$ satisfying the following:

($\star\star$) *n_2 among the t_j belong to C and one belongs to a class of type $C_{(\underline{e}^n, \underline{e}^p)}$ (see Definition 3) where $(\underline{e}^n, \underline{e}^p)$ is a double partition of size d such that $\underline{e}^n$ has length n_1 and the partition of d associated to $(\underline{e}^n, \underline{e}^p)$ is $\underline{e}$. Moreover if $t_j = (*; t'_j)$, $\lambda_k = (*; \lambda'_k)$, $\mu_k = (*; \mu'_j)$, with $j = 1, \dots, n_2 + 1$ and $k = 1, \dots, g$, the group generated by $t'_j,\ \lambda'_k,\ \mu'_k$ is all S_d.*

Let $(\underline{e}^n, \underline{e}^p)$ be a double partition of size d such that the partition $\underline{e}^n$ has length given by an even integer ≥ 2. Let us denote by $H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$ the Hurwitz space that parameterizes equivalence classes of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ satisfying the conditions ($\star$), with monodromy group conjugated to G and having one branch point whose local monodromy belongs to $C_{(\underline{e}^n, \underline{e}^p)}$. Let us denote by $A_{G, n_2, (\underline{e}^n, \underline{e}^p), g}$ the set of all the equivalence classes, $[t_1, \dots, t_{n_2+1}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$, of Hurwitz systems with value in $(\boldsymbol{Z}_2)^d \times^s S_d$, satisfying the conditions ($\star\ \star$), whose monodromy group is conjugated to G.

Let $Y^{(n)}$ be the n-fold symmetric product of Y and let $\varDelta$ be the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors. Let

$$\delta : H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y) \quad \rightarrow \quad Y^{(n_2+1)} - \varDelta$$

be the map which assigns to each equivalence class $[X \xrightarrow{\pi} X' \xrightarrow{f} Y]$ the branch locus D of $X \xrightarrow{\pi} X' \xrightarrow{f} Y$. By Riemann's existence theorem we can identify the fiber of δ over D with $A_{G, n_2, (\underline{e}^n, \underline{e}^p), g}$. There is an unique topology on $H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$ such that δ is a topological covering map (see [8]). Therefore the braid group $\pi_1(Y^{(n_2+1)} - \varDelta, D)$ acts on $A_{G, n_2, (\underline{e}^n, \underline{e}^p), g}$. The orbits of this action are in one-to-one correspondence with the connected components of $H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$. So in order to determine the connected components of $H_{G,\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$ it is sufficient to find the orbits of the action of $\pi_1(Y^{(n_2+1)} - \varDelta, D)$ on $A_{G, n_2, (\underline{e}^n, \underline{e}^p), g}$.

REMARK 2.1. – Let Y be a smooth, projective complex curve of genus ≥ 1. The generators of the braid group $\pi_1(Y^{(n)} - \varDelta, D)$ are the elementary braids σ_j with $j = 1, \dots, n-1$ and the braids $\rho_{ik},\ \tau_{ik}$ with $1 \leq i \leq n$ and $1 \leq k \leq g$ (see [2], [6], [19]). The elementary move σ'_j, relative to the elementary braid σ_j, bring (see [11]) $[t_1, \dots, t_{j-1}, t_j, t_{j+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ to

$$[t_1, \dots, t_{j-1}, t_j t_{j+1} t_j^{-1}, t_j, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g].$$

Therefore its inverse σ''_j bring $[t_1, \dots, t_{j-1}, t_j, t_{j+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ to $[t_1, \dots, t_{j-1}, t_{j+1}, t_{j+1}^{-1} t_j t_{j+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$.

Here as in [14], we associate to each generator ρ_{ik}, τ_{ik} a pair of braid moves ρ'_{ik}, $\rho''_{ik} = (\rho'_{ik})^{-1}$ and τ'_{ik}, $\tau''_{ik} = (\tau'_{ik})^{-1}$ respectively.
We use the following result.

PROPOSITION 1 ([14] Theorem 1.8). – *Let* $[t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g]$ *be an equivalence class of Hurwitz systems. Let* $u_k = [\lambda_1, \mu_1] \cdots [\lambda_k, \mu_k]$ *for* $k = 1, \ldots, g$ *and let* $u_0 = id$. *The following formulae hold:*

i) *For* ρ'_{ik} *where* $1 \leq i \leq n,\ 1 \leq k \leq g$:

$t'_j = t_j$ *for each* $j \neq i$, $\lambda'_l = \lambda_l$ *for each* l, $\mu'_l = \mu_l$ *for each* $l \neq k$

$$(t_i, \mu_k) \ \rightarrow \ (t'_i, \mu'_k) = (a_1^{-1} t_i a_1,\ (b_1^{-1} t_i^{-1} b_1)\mu_k) \quad \text{where}$$

$$a_1 = (t_1 \cdots t_{i-1})^{-1}\ u_{k-1}\lambda_k(u_k^{-1}u_g)(t_{i+1} \cdots t_n)^{-1}, \quad b_1 = (t_1 \cdots t_{i-1})^{-1} u_{k-1}\lambda_k.$$

ii) *For* ρ''_{ik} *where* $1 \leq i \leq n,\ 1 \leq k \leq g$:

$t''_j = t_j$ *for each* $j \neq i$, $\lambda''_l = \lambda_l$ *for each* l, $\mu''_l = \mu_l$ *for each* $l \neq k$

$$(t_i, \mu_k) \rightarrow (t''_i, \mu''_k) = (a_2^{-1} t_i a_2,\ (b_2^{-1} t_i b_2)\mu_k) \quad \text{where}$$

$$a_2 = t_{i+1} \cdots t_n (u_k^{-1}u_g)^{-1}\lambda_k^{-1}(u_{k-1})^{-1} t_1 \cdots t_{i-1},\ b_2 = t_{i+1} \cdots t_n (u_k^{-1}u_g)^{-1}.$$

iii) *For* τ'_{ik} *where* $1 \leq i \leq n,\ 1 \leq k \leq g$:

$t'_j = t_j$ *for each* $j \neq i$, $\lambda'_l = \lambda_l$ *for each* $l \neq k$, $\mu'_l = \mu_l$ *for each* l

$$(t_i, \lambda_k) \rightarrow (t'_i, \lambda'_k) = (c_1^{-1} t_i c_1,\ (d_1^{-1}\ t_i d_1)\lambda_k) \quad \text{where}$$

$$c_1 = t_{i+1} \cdots t_n (u_k^{-1}u_g)^{-1}\mu_k(u_{k-1})^{-1} t_1 \cdots t_{i-1},\ d_1 = t_{i+1} \cdots t_n (u_k^{-1}u_g)^{-1}\mu_k.$$

iv) *For* τ''_{ik} *where* $1 \leq i \leq n,\ 1 \leq k \leq g$:

$t''_j = t_j$ *for each* $j \neq i$, $\lambda''_l = \lambda_l$ *for each* $l \neq k$, $\mu''_l = \mu_l$ *for each* l

$$(t_i, \lambda_k) \rightarrow (t''_i, \lambda''_k) = (c_2^{-1} t_i c_2,\ (d_2^{-1} t_i^{-1} d_2)\lambda_k) \quad \text{where}$$

$$c_2 = (t_1 \cdots t_{i-1})^{-1}\ u_{k-1}\mu_k^{-1}\ (u_k^{-1}u_g)(t_{i+1} \cdots t_n)^{-1}, \quad d_2 = (t_1 \cdots t_{i-1})^{-1} u_{k-1}.$$

3. – Irreducibility of $H_{W(D_d),\, n_2,\, (\underline{e}^n, \underline{e}^p)}\ (Y)$.

Let $(\underline{e}^n, \underline{e}^p)$ be a double partition of size d such that the partition $\underline{e}^n$ has even length ≥ 2. Let us denote by n_1 the length of $\underline{e}^n$. Let $\underline{e} = (e_1, \ldots, e_r)$ and $\{j_1, \ldots, j_{n_1}\}$ be the partition of d and the subset of $\{1, \ldots, r\}$ associated to $(\underline{e}^n, \underline{e}^p)$ (see Section 1, paragraph 1.2). Here we associate to $\underline{e}$ the following element in S_d

having cycle type $\underline{e}$

(1) $$(12\dots e_1)(e_1+1\dots e_1+e_2)\cdots((e_1+\cdots+e_{r-1})+1\dots d)$$

and by $|\underline{e}|$ we denote $\sum_{i=1}^{r}(e_i-1)$.

DEFINITION 7. – *We call two Hurwitz systems with values in* $(\boldsymbol{Z}_2)^d\times^s S_d\simeq W(B_d)$ braid-equivalent *if one is obtained from the other by a finite sequence of braid moves* $\sigma'_j,\ \rho'_{ik},\ \tau'_{ik},\ \sigma''_j,\ \rho''_{ik},\ \tau''_{ik}$ *where* $1\le j\le n-1$, $1\le i\le n$ *and* $1\le k\le g$. *We denote the braid equivalence by* $\sim$.

DEFINITION 8. – *Two ordered n-tuples (or sequences) of elements in* $(\boldsymbol{Z}_2)^d\times^s S_d\simeq W(B_d)$, $(t_1,\dots,t_n)$ *and* $(\tilde t_1,\dots,\tilde t_n)$, *are called* braid-equivalent *if* $(\tilde t_1,\dots,\tilde t_n)$ *is obtained from* $(t_1,\dots,t_n)$ *by a finite sequence of braid moves of type* σ'_j, σ''_j. *Note that if* $t_1\cdots t_n=s$ *then* $\tilde t_1\cdots\tilde t_n=s$.

LEMMA 1. – *Let* $(t_1,\dots,t_i,t_{i+1},\dots,t_n)$ *be a sequence of elements in* $(\boldsymbol{Z}_2)^d\times^s S_d$ *such that* $t_{i+1}=t_i^{-1}$. *Then, acting with elementary moves* σ'_j *and their inverses, we can move to the left and to the right the pair* (t_i,t_{i+1}) *leaving unchanged the other elements of the sequence.*

PROOF. – The lemma follows from the braid equivalences $(t,t_i,t_{i+1})\sim(t_i,t_i^{-1}t\,t_i,t_{i+1})\sim(t_i,t_{i+1},t)$ and $(t_i,t_{i+1},t)\sim(t_i,t_{i+1}t\,t_{i+1}^{-1},t_{i+1})\sim(t,t_i,t_{i+1})$. □

From now on let us denote the permutation (1) by

$$\varepsilon\ =\ (1_1\ 2_1\dots(e_1)_1)(1_2\ 2_2\dots(e_2)_2)\cdots(1_r\ 2_r\dots(e_r)_r)$$

and by q_i we denote the cycle $(1_i2_i\dots(e_i)_i)$. Let n_2 and g be positive integers.

PROPOSITION 2. – *If* $n_2+|\underline{e}|\ \ge\ 2d$ *each equivalence class* $[t_1,\dots,t_{n_2+1};\lambda_1,\mu_1,\dots,\lambda_g,\mu_g]$ *belonging to* $A_{G,n_2,(\underline{e}^n,\underline{e}^p),g}$ *is braid-equivalent to a class of the form*

$$\begin{aligned}&[(0;(1_12_1)),\ (0;(1_13_1)),\dots,\ (0;(1_1(e_1)_1)),\ (0;(1_22_2)),\dots,\ (0;(1_2(e_2)_2)),\dots,\\&(0;(1_r\ 2_r)),\dots,(0;(1_r\ (e_r)_r)),(z^2_{1_11_2};(1_1\ 1_2)),((z^2)^1_{1_1\ 1_2};(1_11_2)),\dots,(z^{r-1}_{1_11_{r-1}};(1_11_{r-1})),\\&((z^{r-1})^1_{1_11_{r-1}};\ (1_1\ 1_{r-1})),\ (z^r_{1_11_r};\ (1_1\ 1_r)),\ \dots,\ ((z^r)^l_{1_11_r};\ (1_1\ 1_r)),\ (\bar 1_{1_{j_1}\dots 1_{j_{n_1}}};\varepsilon^{-1});\\&(c_1;id),(d_1;id),\dots,(c_g;id),(d_g;id)]\end{aligned}$$

where if $j\in\{j_1,\dots,j_{n_1}\}$

(2) $$((z^j_{1_11_j};(1_11_j)),((z^j)^1_{1_11_j};(1_11_j)))=((\bar 1_{1_11_j};(1_11_j)),(0;(1_11_j)))$$

if instead $j\notin\{j_1,\dots,j_{n_1}\}$, $2\le j\le r$,

(3) $$((z^j_{1_11_j};(1_11_j)),((z^j)^1_{1_11_j};(1_11_j)))=((0;(1_11_j)),(0;(1_11_j))).$$

Moreover $((z^r)^m_{1_1 1_r}; (1_1 1_r)) = (0; (1_1 1_r))$ *for each* $m = 2, \dots, l$ *and the* c_k *and the* d_k *are equal to either* 0 *or* $\bar{1}_1$ *depending on wether* λ_k *and* μ_k *belong to the subgroup of* $(\boldsymbol{Z}_2)^d \times^s S_d$ *isomorphic to* $W(D_d)$ *or not.*

PROOF. – Step 1. Let $\lambda_k = (a_k; \lambda'_k)$, $\mu_k = (b_k; \mu'_k)$, $t_l = (a'; t'_l = \xi)$ for some l and $t_j = (*; t'_j)$ for each $j \neq l$. Let Y be a smooth, connected, projective complex curve of genus ≥ 1. By Riemann's existence theorem the equivalence class of Hurwitz systems $[t'_1, \dots, t'_{n_2+1}; \lambda'_1, \dots, \mu'_g]$ corresponds to an equivalence class of degree $d \geq 3$ coverings of Y, with monodromy group S_d, branched in $n_2 + 1$ points, n_2 of which are points of simple branching while one is a special point whose local monodromy has cycle type $\underline{e}$. Since $n_2 + |\underline{e}| \ \geq\ 2d$ the Hurwitz space $H^o_{d, n_2, \underline{e}}(Y)$, parameterizing equivalence classes of coverings as above, is irreducible (see [21], Theorem 1). So it is possible, acting by braid moves σ'_j, ρ'_{ik}, τ'_{ik} and their inverses, to replace $[t'_1, \dots, \mu'_g]$ with $[t''_1, \dots, t''_{n_2}, \varepsilon^{-1}; id, \dots, id]$. In this way our class $[t_1, \dots, t_{n_2+1}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ results braid-equivalent to a class of the form $[\tilde{t}_1, \dots, \tilde{t}_{n_2}, (b'; \varepsilon^{-1}); (\tilde{a}_1; id), (\tilde{b}_1; id), \dots, (\tilde{a}_g; id), (\tilde{b}_g; id)]$.

Note that

$$\tilde{t}_1 \cdots \tilde{t}_{n_2}\ (b'; \varepsilon^{-1}) \ = \ [(\tilde{a}_1; id), (\tilde{b}_1; id)] \cdots [(\tilde{a}_g; id), (\tilde{b}_g; id)] \ = \ (0; id),$$

and thus $(b'; \varepsilon^{-1})$ belongs to the group generated by $\tilde{t}_1, \dots, \tilde{t}_{n_2}$.

$[\tilde{t}_1, \dots, \tilde{t}_{n_2}, (b'; \varepsilon^{-1})]$ is a equivalence class of Hurwitz systems satisfying the following: n_2 among the $\tilde{t}_j$ belong to C, one belongs to $C_{(\underline{e}^n, \underline{e}^p)}$ and $\langle t''_1, \dots, t''_{n_2} \rangle = S_d$. Therefore the equivalence class $[\tilde{t}_1, \dots, \tilde{t}_{n_2}, (b'; \varepsilon^{-1})]$ corresponds to an equivalence class of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$ satisfying the conditions $(\star)$ and having one branch point whose local monodromy belongs to $C_{(\underline{e}^n, \underline{e}^p)}$. Since the Hurwitz spaces that parameterize sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$ as above are irreducible (see [22], Theorem 8), it is possible, conjugating by elements of type $(\bar{1}_{i \dots h}; id)$ and using elementary moves σ'_j, σ''_j with $1 \leq j \leq n_2 - 1$ (see [22], Proof of Theorem 8), to transform our class into the form

$$\begin{aligned} (4)\quad & [(0; (1_1 2_1)),\ (0; (1_1 3_1)), \dots, (0; (1_1 (e_1)_1)),\ (0; (1_2 2_2)), \dots,\ (0; (1_2 (e_2)_2)), \dots, \\ & (0; (1_r 2_r)), \dots, (0; (1_r (e_r)_r)),\ (z^2_{1_1 1_2}; (1_1 1_2)),\ ((z^2)^1_{1_1 1_2};\ (1_1 1_2)),\ (z^3_{1_1 1_3}; (1_1 1_3)), \\ & ((z^3)^1_{1_1 1_3};\ (1_1 1_3)), \dots,\ (z^r_{1_1 1_r};\ (1_1 1_r)), \dots,\ ((z^r)^l_{1_1 1_r}; (1_1 1_r)),\ (\bar{1}_{1_{j_1} \dots 1_{j_{n_1}}}; \varepsilon^{-1}); \\ & (\tilde{a}_1; id), (\tilde{b}_1; id), \dots, (\tilde{a}_g; id), (\tilde{b}_g; id)] \end{aligned}$$

where the pairs $((z^j_{1_1 1_j}; (1_1 1_j)), ((z^j)^1_{1_1 1_j}; (1_1 1_j)))$, $2 \leq j \leq r$, satisfy either (2) or (3) depending on whether j belongs to $\{j_1, \dots, j_{n_1}\}$ or not. Moreover $((z^r)^m_{1_1 1_r}; (1_1 1_r)) = (0; (1_1 1_r))$ for each $m = 2, \dots, l$.

Step 2. By Step 1 we can transform $[t_1, \dots, t_{n_2+1}; \lambda_1,\ \mu_1, \dots, \lambda_g, \mu_g]$ into (4). So if the $\tilde{a}_k$ and the $\tilde{b}_k$ are equal to either 0 or $\bar{1}_1$, for each k, the proposition is proved. Let $\tilde{a}_1$ be a function different from 0 and $\bar{1}_1$.

Let i and j be two indexes sent to $\bar{1}$ by $\tilde{a}_1$. Observe that if, acting by braid moves of type $\sigma'_l,\ \sigma''_l, 1 \leq l \leq n_2 - 1$, we can obtain a class braid-equivalent to ours in which there are both $(\bar{1}_{ij};(ij))$ and $(0;(ij))$ then our class is braid-equivalent to a class of the form $[\hat{t}_1,\ \ldots,\ \hat{t}_{n_2},\ (\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});\ (\hat{a}_1;id),(\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),\ (\tilde{b}_g;id)]$ where $\hat{a}_1$ is a function which sends to $\bar{1}$ the same indexes sent to $\bar{1}$ by $\tilde{a}_1$ except i and j. In fact, using elementary moves σ''_l we can bring to the first place one of two elements of type $(z_{ij};(ij))$ and then we apply the move τ''_{11} that transforms $(\tilde{a}_1;id)$ in $(z_{ij};(ij))(\tilde{a}_1;id)$. Now we move to the first place the other element of type $(z'_{ij};(ij))$, where $z' = \bar{1}$ if $z = \bar{0}$ and $z' = \bar{0}$ if $z = \bar{1}$ and we again act by τ''_{11}. In this way we replace $(z_{ij};(ij))(\tilde{a}_1;id)$ with $(z'_{ij};(ij))\,(z_{ij};(ij))\,(\tilde{a}_1;id) = (\bar{1}_{ij};id)\,(\tilde{a}_1;id) = (\bar{1}_{ij} + \tilde{a}_1;id)$ where $\hat{a}_1 = \bar{1}_{ij} + \tilde{a}_1$ is a function which sends i and j in $\bar{0}$.

At first we verify that (4) is braid-equivalent to a class of the form $[\hat{t}_1,\ldots,\hat{t}_{n_2},(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});\ (\hat{a}_1;id),\ (\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),(\tilde{b}_g;id)]$. If in (4) there is already the pair $((\bar{1}_{ij};(ij)),(0;(ij)))$, it is sufficient to proceed as above to bring our class to the required form. Then we analyze the case $i = 1_1$ and $j \notin \{1_{j_1},\ldots,1_{j_{n_1}}\}$ (in a similar manner one affronts the case $i = 1_*$ for some $* \in \{j_1,\ldots,j_{n_1}\}$, $* \neq 1$, and $j \neq 1_1$). If j is an index moved by some cycle q_* with $* \in \{j_1,\ldots,j_{n_1}\}$, $* \neq 1$, since $j \neq 1_*$ in (4) there is the element $(0;(1_*j))$. Let h and $h+1$ be the places occupied by the elements of the pair $((\bar{1}_{1_1 1_*};(1_1 1_*)),(0;(1_1 1_*)))$. Using moves of type σ'_l we move $(0;(1_*j))$ to the place $h-1$ and we apply the moves $\sigma'_{h-1},\ \sigma'_h$ obtaining

$$((0;(1_*j)),(\bar{1}_{1_1 1_*};(1_1 1_*)),(0;(1_1 1_*))) \sim ((\bar{1}_{1_1 j};(1_1 j)),(0;(1_1 j)),(0;(1_* j))).$$

In this way we obtained a class braid-equivalent to (4) determined by a system in which there is the pair $((\bar{1}_{1_1 j};(1_1 j)),(0;(1_1 j)))$, so (4) can be transformed into the required form. If instead j is an index moved by q_1 in (4) there is already $(0;(1_1 j))$. Using moves of type σ''_l we move $(0;(1_1 j))$ to the first place and then we apply τ''_{11} which transforms $(\tilde{a}_1;id)$ in $(0;(1_1 j))(\tilde{a}_1;id)$. Note that the move τ''_{11} also acts on $(0;(1_1 j))$ conjugating it with $c_2 = (\tilde{b}_1;id)(0;(1_1 j))$ (see Proposition 1). If $\tilde{b}_1$ sends the indexes 1_1 and j both either to $\bar{1}$ or to $\bar{0}$, $c_2^{-1}\,(0;(1_1 j))\,c_2 = (0;(1_1 j))$. If instead $\tilde{b}_1$ sends one of two indexes to $\bar{0}$ and another to $\bar{1}$, $c_2^{-1}\,(0;(1_1 j))\,c_2 = (\bar{1}_{1_1 j};(1_1 j))$. In the second case it is sufficient to apply τ''_{11} to replace $(\tilde{a}_1;(1_1 j))$ with $(\hat{a}_1;id)$. In the first case instead we move $(0;(1_1 j))$ to the left of one pair of type $((\bar{1}_{1_1 1_*};(1_1 1_*)),(0;(1_1 1_*)))$. If h and $h+1$ are the places occupied by the elements of this pair, we apply $\sigma''_{h-1},\ \sigma'_h$ to obtain

$$\begin{aligned}&((0;(1_1 j)),(\bar{1}_{1_1 1_*};(1_1 1_*)),(0;(1_1 1_*)))\\ &\sim ((\bar{1}_{1_1 1_*};\ (1_1 1_*)),(\bar{1}_{1_1 j};(1_1 j)),\ (\bar{1}_{1_* j};(1_* j))).\end{aligned}$$

Now it is sufficient to move $(\bar{1}_{1_1 j};(1_1 j))$ to the first place and then to apply τ''_{11} to transform $(\tilde{a}_1;(1_1 j))$ in $(\hat{a}_1;id)$.

Note that if j is an index moved by a cycle q_a with $a \neq 1$ and $a \notin \{j_1, \ldots, j_{n_1}\}$, then in (4) there are both $(0; (1_a\ j))$ and $((0; (1_1 1_a)),\ (0; (1_1 1_a)))$. By Lemma 1 we can move the pair $((0; (1_1 1_a)), (0; (1_1 1_a)))$ to the right of $(0; (1_a\ j))$ leaving unchanged each other element of (4). Acting with one suitable σ'_l we replace $((0; (1_a\ j)),\ (0; (1_1 1_a)),\ (0; (1_1 1_a)))$ with $((0; (1_1\ j)),\ (0; (1_a\ j)), (0; (1_1 1_a)))$. In this way we obtain a class braid-equivalent to (4) in which there is the element $(0; (1_1\ j))$ and one pair of type $((\bar{1}_{1_1 1_*}; (1_1 1_*)), (0; (1_1 1_*)))$. So from what we observed above it follows that our class is braid-equivalent to a class of the requested type.

In the end we analyze the case in which the indexes i, j do not belong to $\{1_{j_1}, \ldots, 1_{j_{n_1}}, 1_1\}$. If i is one index moved by q_1 and j one index moved by some cycle q_* with $* \in \{j_1, \ldots, j_{n_1}\}$, $* \neq 1$, in addition to the pair $((\bar{1}_{1_1 1_*}; (1_1 1_*)),\ (0; (1_1 1_*)))$, in (4) there are $(0; (1_1 i))$ and $(0; (1_* j))$. With elementary moves of type σ'_l we bring $(0; (1_1 i))$ and $(0; (1_* j))$ to the left of the pair $((\bar{1}_{1_1 1_*}; (1_1 1_*)), (0; (1_1 1_*)))$. If now $(0; (1_1 i))$ and $(0; (1_* j))$ occupy respectively the place $h-2$ and $h-1$, we act by σ''_{h-1}, σ'_{h-2}, σ'_{h-1}, σ''_{h-2}, σ'_h, σ'_{h-1} obtaining

$$((0; (1_1\ i)),\ (0; (1_*\ j)),\ (\bar{1}_{1_1 1_*}; (1_1\ 1_*)),\ (0; (1_1\ 1_*)))$$
$$\sim ((\bar{1}_{ij}; (i\ j)), (0; (i\ j)), (0; (j\ 1_*)), (0; (1_1\ i))).$$

In this way we replace our class by a class in which there is the pair $((\bar{1}_{ij}; (i\ j)), (0; (i\ j)))$, then (4) is braid-equivalent to a class of the requested type. Observe that one proceeds in the same way when i and j are indexes moved by two different cycle q_a and q_b such that either $a, b \in \{j_1, \ldots, j_{n_1}\} - \{1\}$ or $a \in \{j_1, \ldots, j_{n_1}\}$ and $b \notin \{1, j_1, \ldots, j_{n_1}\}$.

If instead the indexes i, j are moved both by q_1 (analogously one reasons when i and j are moved both by a cycle q_* with $* \in \{j_1, \ldots, j_{n_1}\}$, $* \neq 1$) in (4) there are both $(0; (1_1\ i))$ and $(0; (1_1\ j))$. Let $i < j$. Acting by inverses of elementary moves σ'_l we move $(0; (1_1\ j))$ to the right of $(0; (1_1\ i))$ and then we use a suitable σ'_l to replace $((0; (1_1 i)), (0; (1_1 j)))$ with $((0; (ij)), (0; (1_1 i)))$. We move $(0; (ij))$ to the first place and then we apply τ''_{11} which transforms $(\tilde{a}_1; id)$ in $(\tilde{a}_1; (ij))$. If τ''_{11} transforms $(0; (ij))$ into $(\bar{1}_{ij}; (ij))$ applying again τ''_{11} we conclude. Otherwise acting by moves of type σ'_l we place $(0; (ij))$ near to $(0; (1_1 i))$ and then we move both to the left of one pair of type $((\bar{1}_{1_1\ 1_*}; (1_1\ 1_*)), (0; (1_1\ 1_*)))$. If now $(0; (ij))$ and $(0; (1_1 i))$ occupy respectively the place $h-2$ and $h-1$, it is sufficient to apply σ'_{h-1}, σ'_{h-2}, σ'_h, σ''_{h-1}, σ'_{h-2} to obtain that

$$((0; (i\ j)), (0; (1_1\ i)), (\bar{1}_{1_1\ 1_*}; (1_1\ 1_*)), (0; (1_1\ 1_*)))$$
$$\sim ((\bar{1}_{ij}; (i\ j)), (\bar{1}_{j1_*}; (j\ 1_*)), (0; (j\ 1_*)), (0; (1_1\ i))).$$

We move $(\bar{1}_{ij}; (ij))$ to the first place and then we apply τ''_{11} to transform $(\tilde{a}_1; (ij))$ in $(\hat{a}_1; id)$.

Observe that if i and j are indexes moved by the same cycle q_a with

$a \notin \{1, j_1, \ldots, j_{n_1}\}$, in addition to the pair $((0;(1_1 1_a)),(0;(1_1 1_a)))$, in (4) there are the elements $(0;(1_a\ i))$ and $(0;(1_a\ j))$. Acting by elementary moves σ'_l we place $(0;(1_a\ i))$ to the left of $(0;(1_a\ j))$. Now by Lemma 1 we can move on the left the pair $((0;(1_1 1_a)),(0;(1_1 1_a)))$ leaving unchanged each other element of the system and obtaining $[\ldots,(0;(1_a\ i)),(0;(1_1 1_a)),(0;(1_1 1_a)),(0;(1_a\ j)),\ldots]$. If now the element $(0;(1_a\ i))$ occupies the place $h-1$, acting with $\sigma'_{h-1},\ \sigma'_{h+1}$ one has

$$((0;(1_a\ i)),(0;(1_1\ 1_a)),(0;(1_1\ 1_a)),(0;(1_a\ j)))$$
$$\sim ((0;(1_1\ i)),(0;(1_a\ i)),(0;(1_1\ j)),(0;(1_1\ 1_a))).$$

We obtained a class braid-equivalent to (4) in which there are the elements $(0;\ (1_1\ i))$, $(0;\ (1_1\ j))$ and one pair of type $((\bar{1}_{1_1 1_*};\ (1_1 1_*)),(0;(1_1 1_*)))$, so we can conclude that our class is braid-equivalent to a class of the required type. At the same conclusion one arrives when the indexes i and j are moved by two different cycles q_a and q_b so that both a and b do not belong to $\{1, j_1, \ldots, j_{n_1}\}$.

We verified that (4) is braid-equivalent to a class of the form $[\hat{t}_1,\ldots,\ \hat{t}_{n_2},(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});(\hat{a}_1;id),\ (\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),(\tilde{b}_g;id)]$. Now by Step 1 we can bring this class to the form

(5) $[(0;(1_1 2_1)),\ (0;(1_1 3_1)),\ldots,\ (0;(1_1(e_1)_1)),\ (0;(1_2 2_2)),\ldots,(0;(1_2(e_2)_2)),\ldots,$ $(0;\ (1_r 2_r)),\ldots,(0;\ (1_r(e_r)_r)),\ (z^2_{1_1 1_2};\ (1_1 1_2)),\ ((z^2)^1_{1_1 1_2};\ (1_1 1_2)),\ (z^3_{1_1 1_3};\ (1_1 1_3)),$ $((z^3)^1_{1_1 1_3};\ (1_1 1_3)),\ldots,\ (z^r_{1_1 1_r};\ (1_1 1_r)),\ldots,((z^r)^l_{1_1 1_r};\ (1_1 1_r)),\ (\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});$ $(\hat{a}_1;id),\ (\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),(\tilde{b}_g;id)]$

where the pairs $((z^j_{1_1 1_j};(1_1 1_j)),((z^j)^1_{1_1 1_j};(1_1 1_j)))$, $2 \le j \le r$, satisfy either (2) or (3) depending on whether j belongs to $\{j_1,\ldots,j_{n_1}\}$ or not. Moreover $((z^r)^m_{1_1 1_r};(1_1 1_r)) = (0;(1_1 1_r))$ for each $m = 2,\ldots,l$.

Note that (5) is different from (4) only by the element of place n_2+2. So one can proceed for each pair of indexes which $\hat{a}_1$ sends to $\bar{1}$ as by the pair (i,j). In this way, after a finite number of steps, we are able to replace (5) with a class of the form either $[\check{t}_1,\ldots,\check{t}_{n_2},(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});(0;id),\ (\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),(\tilde{b}_g;id)]$ or $[\check{t}_1,\ldots,\check{t}_{n_2},(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});(\bar{1}_{1\ h};id)\ (\bar{1}_h;id),\ (\tilde{b}_1;id),\ldots,(\tilde{a}_g;id),(\tilde{b}_g;id)]$ depending on whether λ_1 belongs to $W(D_d)$ or not.

If $\tilde{a}_1$ is equal to either 0 or $\bar{1}_1$ and $\tilde{b}_1$ is a function different from 0 and $\bar{1}_1$ to replace $(\tilde{b}_1;id)$ with $(d_1;id)$ one proceeds in the same way but using the braid move ρ'_{11}. Analogously one reasons when $\tilde{a}_k$ is different from 0 and $\bar{1}_1$ and $\tilde{a}_l,\ \tilde{b}_l$ are equal to 0 or $\bar{1}_1$ for each $l \le k-1$, but one use the braid move τ''_{1k}. In the end if $\tilde{b}_k$ is different from 0 and $\bar{1}_1$ and $\tilde{a}_l,\ \tilde{b}_l,\ \tilde{a}_k,\ l \le k-1$, are equal to 0 or $\bar{1}_1$, to replace $(\tilde{b}_k;id)$ with $(d_k;id)$ one applies the braid moves ρ'_{1k}.

Once checking that (4) is braid-equivalent to a class of type $[\bar{t}_1,\ldots,\bar{t}_{n_2},(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1});\ (c_1;id),\ldots,(d_g;id)]$, by Step 1 the proposition is proved. □

Let $n_2 + |\underline{e}| \geq 2d$ and let $(\bar{1}_{1_{j_1}\ldots 1_{j_{n_1}}};\ \varepsilon^{-1}) \in C_{(\underline{e}^n,\underline{e}^p)}$. From now on we will denote by Z the Hurwitz system with values in $(\boldsymbol{Z}_2)^d \times^s S_d$

$$((0;(1_1 2_1)),\ldots,(0;(1_1(e_1)_1)),\ldots,(0;(1_r 2_r)),\ldots,(0;(1_r(e_r)_r)),\ (z^2_{1_1 1_2};(1_1 1_2)),\ ((z^2)^1_{1_1 1_2};\ (1_1 1_2)),\ (z^3_{1_1 1_3};\ (1_1 1_3)),\ ((z^3)^1_{1_1 1_3};\ (1_1 1_3)),\ldots,(z^r_{1_1 1_r};\ (1_1 1_r)),\ \ldots,\ ((z^r)^l_{1_1 1_r};(1_1 1_r)),\ (\bar{1}_{1_{j_1}\ldots 1_{j_{n_1}}};\ \varepsilon^{-1})) \tag{6}$$

where the pairs $((z^j_{1_1 1_j};(1_1 1_j)),((z^j)^1_{1_1 1_j};(1_1 1_j)))$, $2 \leq j \leq r$, satisfy either (2) or (3) depending on whether j belongs to $\{j_1,\ldots,j_{n_1}\}$ or not. Moreover $((z^r)^m_{1_1 1_r};(1_1 1_r)) = (0;(1_1 1_r))$ for each $m = 2,\ldots,l$.

OBSERVATION 2. – The monodromy group H of the Hurwitz system Z is isomorphic to $W(D_d)$. In fact, since the transpositions $(1_i\ 2_i)$, $(1_i\ 3_i),\ldots,(1_i\ (e_i)_i)$, $(1_1\ 1_j)$, $i = 1,\ldots,r, j = 2,\ldots,r$, generate all S_d and H contains at least one pair of type $((\bar{1}_{1_1 1_j};(1_1\ 1_j)),(0;(1_1\ 1_j)))$, H contains both the elements $((\bar{1}_{hk};(h\ k))$ and $(0;(h\ k))$, for each $h,k \in \{1,\ldots,d\}$, $h \neq k$. Therefore H contains all the generators of $W(D_d)$. Since C and $C_{(\underline{e}^n,\underline{e}^p)}$ are contained in $W(D_d)$, H is isomorphic to $W(D_d)$.

COROLLARY 1. – *Let $n_2 + |\underline{e}| \geq 2d$ and let $[t_1,\ldots,t_{n_2+1};\lambda_1,\ \mu_1,\ldots,\lambda_g,\mu_g]$ belonging to $A_{G,n_2,(\underline{e}^n,\underline{e}^p),g}$. Then G is isomorphic either to $W(D_d)$ or to $W(B_d)$.*

PROOF. – Sine $n_2 + |\underline{e}| \geq 2d$, by Proposition 2, $[t_1,\ldots,t_{n_2+1};\lambda_1,\ \mu_1,\ldots,\lambda_g,\mu_g]$ is braid-equivalent to a class of the form $[Z;(c_1;id),(d_1;id),\ldots,(c_g;id),(d_g;id)]$ where the c_k and the d_k are equal to either 0 or $\bar{1}_1$ depending on wether λ_k and μ_k belong to the subgroup of $(\boldsymbol{Z}_2)^d \times^s S_d$ isomorphic to $W(D_d)$ or not. Moreover the group generated by elements of Z is isomorphic to $W(D_d)$ (see Observation 2). Therefore if all the c_k and the d_k are equal to 0 the Hurwitz systems belonging to $[t_1,\ldots,t_{n_2+1};\lambda_1,\ \mu_1,\ldots,\lambda_g,\mu_g]$ have monodromy group isomorphic to $W(D_d)$ and so G is isomorphic to $W(D_d)$. If instead one λ_k or μ_k is equal to $\bar{1}_1$, the Hurwitz systems in our class have monodromy group isomorphic to $W(B_d)$ and so G is isomorphic to $W(B_d)$. $\square$

THEOREM 1. – *Let Y be a smooth, connected, projective complex curve of genus ≥ 1. If $n_2 + |\underline{e}| \geq 2d$, the Hurwitz space $H_{W(D_d),\ n_2,(\underline{e}^n,\underline{e}^p)}\ (Y)$ is irreducible.*

PROOF. – Since the Hurwitz space $H_{W(D_d),\ n_2,(\underline{e}^n,\underline{e}^p)}\ (Y)$ is smooth to prove its irreducibility it is sufficient to prove that it is connected. In Section 2 we observed that the connected components of $H_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p)}(Y)$ are in one-to-one correspondence with the orbits of the action of $\pi_1(Y^{(n_2+1)} - \Delta, D)$ on $A_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p),g}$. So if this action is transitive $H_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p)}(Y)$ is connected. In order to prove the transitivity of the action of $\pi_1(Y^{(n_2+1)} - \Delta, D)$ on $A_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p),g}$ it is sufficient to verify that, acting with braid moves of type $\sigma'_j,\ \rho'_{ik},\ \tau'_{ik}$ and their inverses, it is possible to transform each equivalence class in $A_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p),g}$ into a given class.

By Proposition 2, under the hypothesis $n_2 + |\underline{e}| \geq 2d$, each equivalence class in $A_{W(D_d),n_2,(\underline{e}^n,\underline{e}^p),g}$ is braid equivalent to the class $[Z;\ (0; id), \ldots, (0; id)]$. So the theorem is proved. $\square$

4. – The connected components of $H_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p)}(Y)$.

Let $(\underline{e}^n, \underline{e}^p)$ be a double partition of size d such that the partition $\underline{e}^n$ has length even. Let us denote by $\underline{e} = (e_1, \ldots, e_r)$ the partition of d associated to $(\underline{e}^n, \underline{e}^p)$. Let Y be a smooth, connected, projective complex curve of genus ≥ 1. In the previous section we proved the irreducibility of the Hurwitz space $H_{W(D_d),\ n_2,(\underline{e}^n,\underline{e}^p)}\ (Y)$ for $n_2 + |\underline{e}| \geq 2d$. Now we fix our attention on $H_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p)}\ (Y)$. In Section 2 we observed that to determine the connected components of $H_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p)}\ (Y)$ it is sufficient to find the orbits of action $\pi_1(Y^{(n_2+1)} - \Delta, D)$ on $A_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p),g}$.

Let $n_2 + |\underline{e}| \geq 2d$. Note that:

the following equivalence classes

$[Z;\ \bar{\lambda}_1 = (\bar{1}_1; id),\ \bar{\mu}_1 = (0; id), (0; id), (0; id), \ldots, (0; id), (0; id)],$

$[Z;\ \bar{\bar{\lambda}}_1 = (0; id),\ \bar{\bar{\mu}}_1 = (\bar{1}_1; id),\ (0; id), (0; id), \ldots, (0; id), (0; id)]$

where Z is the Hurwitz system defined in (6)*, belong to $A_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p),g}$ and they are not braid-equivalent.*

In fact the only braid moves which change $\bar{\lambda}_1,\ \bar{\bar{\lambda}}_1$ are $\tau''_{i1},\ \tau'_{i1}$ and sequences of braid moves of this type. Acting by the braid moves $\tau'_{i1},\ \tau''_{i1}$ we replace $\bar{\lambda}_1$ and $\bar{\bar{\lambda}}_1$ respectively by $(d_1^{-1}t_i d_1)\bar{\lambda}_1,\ (d_2^{-1}t_i^{-1}d_2)\bar{\lambda}_1$ and $(d_1^{-1}t_i d_1)\bar{\bar{\lambda}}_1,\ (d_2^{-1}t_i^{-1}d_2)\bar{\bar{\lambda}}_1$ (see Proposition 1). Note that $(d_1^{-1}t_i d_1)$ and $(d_2^{-1}t_i^{-1}d_2)$ belong to the same conjugate class of t_i, i.e, they belong either to C or to $C_{(\underline{e}^n,\underline{e}^p)}$ and so they are elements of the subgroup of $(\boldsymbol{Z}_2)^d \times^s S_d$ isomorphic to $W(D_d)$ (recall that the partition $\underline{e}^n$ has even length). So acting by $\tau''_{i1},\ \tau'_{i1}$ one cannot replace $(0; id)$ by $(\bar{1}_1; id)$. Observe that at the same conclusion one arrives if one reasons on $\bar{\mu}_1$ and $\bar{\bar{\mu}}_1$.

Let $\{h_1, \ldots, h_s\}$ and $\{k_1, \ldots, k_l\}$ be two subsets of $\{1, \ldots, g\}$ such that at least one among s and l is major of 0. Let us denote by $[\ \hat{t}\ ]_{\{h_1,\ldots,h_s\},\{k_1,\ldots,k_l\}}$ the equivalence class

$$[Z; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g]$$

where the λ_h and the μ_k with $h \in \{h_1, \ldots, h_s\}$ and $k \in \{k_1, \ldots, k_s\}$ are equal to $(\bar{1}_1; id)$ while all others are equal to $(0; id)$. Note that there are $2^{2g} - 1$ equivalence classes of the form $[\ \hat{t}\ ]_{\{h_1,\ldots,h_s\},\{k_1,\ldots,k_l\}}$. Reasoning as above one deduces that the equivalence classes $[\ \hat{t}\ ]_{\{h_1,\ldots,h_s\},\{k_1,\ldots,k_l\}}$ belong to different orbits of the action of $\pi_1(Y^{(n_2+1)} - \Delta, D)$ on $A_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p),g}$. Since if $n_2 + |\underline{e}| \geq 2d$, by Proposition 2, each equivalence class in $A_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p),g}$ is braid-equivalent to one class of the form $[\ \hat{t}\ ]_{\{h_1,\ldots,h_s\},\{k_1,\ldots,k_l\}}$, we can enounce the following theorem:

THEOREM 2. – *Let* $n_2 + |\underline{e}| \geq 2d$. *The number of the connected components of* $H_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p)}(Y)$ *is* $2^{2g} - 1$. *The connected components of* $H_{W(B_d),n_2,(\underline{e}^n,\underline{e}^p)}(Y)$ *are in one-to-one correspondence with the orbits of the equivalence classes* $[\ \hat{t}\ ]_{\{h_1,\dots,h_s\},\{k_1,\dots,k_l\}}$.

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