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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.1, p. 79–85.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_1_79_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2008.

Bollettino U. M. I.
(9) **I** (2008), 79-85

A Note on Sectorial and R-Sectorial Operators

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Sunto. – *Si dimostra che: (i) se* $a, \beta \in \mathbf{R}^+$ *e* A *è un operatore settoriale, allora l'insieme* $\{t^a A^\beta (t+A)^{-a-\beta}; t>0\}$ *è limitato; (ii) che lo stesso insieme di operatori è R-limitato se* A *è R-settoriale.*

Abstract. – *The following results are proved: (i) if* $a, \beta \in \mathbf{R}^+$ *and* A *is a sectorial operator, then the set* $\{t^a A^\beta (t+A)^{-a-\beta}; t>0\}$ *is bounded; (ii) the same set of operators is R-bounded if* A *is R-sectorial.*

The aim of this note is to prove the following

THEOREM 1. – *Let* X *be a complex Banach space and let* A *be a R-sectorial operator acting in* X*. If* a, β *are positive real numbers, then the set* $\{t^a A^\beta (t+A)^{-a-\beta}; t>0\}$ *is R-bounded.*

This answers a question put to me by A. Favini and Ya. Yakubov (see [3]). For a comparison, I will also prove the following “folk result”:

THEOREM 2. – *Let* X *be a complex Banach space and let* A *be a sectorial operator acting in* X*. If* a, β *are positive real numbers, then the set* $\{t^a A^\beta (t+A)^{-a-\beta}; t>0\}$ *is bounded in the operator norm.*

* * *

First, I deal with Theorem 2. I recall that a sectorial operator is a (possibly unbounded) linear operator A acting in a complex Banach space X, satisfying the following conditions:

(i) the resolvent set $\rho(A)$ of A contains $\mathbf{R}^- := \{t \in \mathbf{R}; t<0\}$;
(ii) $\sup_{t>0} \|t(t+A)^{-1}\| = M < \infty$.

For a sectorial operator A, the powers of A with arbitrary complex exponent are defined. Recent literature on the subject is [2], [4], [5]. If one is interested in powers with negative exponent (but this is not our case) it is suitable to ask that A be also injective.

The usual expansion of the resolvent operators implies that if A is a sectorial operator, then there exists a sector about the negative real half-line on which the estimate $\sup_\lambda \|\lambda(\lambda - A)^{-1}\| < \infty$ holds. This means that for some $\theta \in\ ]0, \pi[$ the spectrum $\sigma(A)$ of A is contained in the closure of the open sector $\Sigma_\theta := \{re^{i\phi}; r > 0,\ -\theta < \phi < \theta\}$, while $\lambda(\lambda - A)^{-1}$ is bounded on $\mathbf{C} \setminus \overline{\Sigma}_\theta$ (that is, $\lambda(\lambda + A)^{-1}$ is bounded on $\Sigma_{\pi-\theta}$). The g.l.b. of such θ's is called the *spectral angle* of A.

LEMMA 1. – *Let A be a sectorial operator, $\lambda \in \mathbf{C}$ and $t > 0$. Then:*

(i) $(\lambda + (t + A)^{-1})^{-1} = \lambda^{-1}(t + A)(\lambda^{-1} + t + A)^{-1}$, if $\lambda \neq 0$ and $\lambda^{-1} + t \in \rho(-A)$;

(ii) $(\lambda + t(t + A)^{-1})^{-1} = \lambda^{-1}(t + A)(t + t\lambda^{-1} + A)^{-1}$, if $\lambda \neq 0$ and $t + t\lambda^{-1} \in \rho(-A)$;

(iii) $(\lambda + A(t + A)^{-1})^{-1} = \dfrac{1}{\lambda + 1}(t + A)\Big(\dfrac{\lambda t}{\lambda + 1} + A\Big)^{-1}$, if $\lambda + 1 \neq 0$ and $\dfrac{\lambda t}{\lambda + 1} \in \rho(-A)$.

PROOF. – Indeed one has

$$\lambda + (t + A)^{-1} = (\lambda(t + A) + 1)(t + A)^{-1} = \lambda(\lambda^{-1} + t + A)(t + A)^{-1};$$

$$\lambda + t(t + A)^{-1} = (\lambda(t + A) + t)(t + A)^{-1} = \lambda(t + t\lambda^{-1} + A)(t + A)^{-1};$$

$$\lambda + A(t + A)^{-1} = (\lambda(t + A) + A)(t + A)^{-1} = (\lambda + 1)\left(\frac{\lambda t}{\lambda + 1} + A\right)(t + A)^{-1}.$$

If θ is the spectral angle of A, then $\Sigma_{\pi-\theta} := \{re^{i\phi}; r > 0,\ \theta - \pi < \phi < \pi - \theta\}$ is contained in $\rho(-A)$; on the other hand $\lambda^{-1} + t$, $t + t\lambda^{-1}$ and $\dfrac{\lambda t}{\lambda + 1} = \dfrac{t}{1 + \lambda^{-1}}$ are the results of the application to λ of functions (the inversion, the multiplication by a positive real number, the addition with a positive real number) that apply $\Sigma_{\pi-\theta}$ to itself. Therefore it follows from lemma 1 that $\Sigma_{\pi-\theta}$ is also contained in the resolvent sets of the operators $-(t + A)^{-1}$, $-t(t + A)^{-1}$ and $-A(t + A)^{-1}$. Next, we have

LEMMA 2. – *Let $\theta \in\ ]0, \pi[$, $\Sigma_\theta := \{re^{i\phi}; r > 0,\ -\theta < \phi < \theta\}$. Then $\forall\, \lambda \in \Sigma_\theta$ $\left|\dfrac{\lambda + 1}{\lambda}\right| \geq C_\theta := \sin\Big(\theta \vee \dfrac{\pi}{2}\Big)$.*

PROOF. – Along the ray $\arg \lambda = \phi$ one has

$$\left|\frac{\lambda + 1}{\lambda}\right|^2 = |\lambda|^{-2} + 2\,|\lambda|^{-1}\cos\phi + 1.$$

If $\cos\phi \geq 0$ this is obviously ≥ 1, while if $\cos\phi < 0$ this function of $|\lambda|$ has minimum at $|\lambda| = |\cos\phi|^{-1}$, and its minimum value is $\sin^2\phi$. Hence the assertion follows.

Then we obtain

LEMMA 3. – *If A is a sectorial operator with spectral angle θ, then $\forall\, t > 0$ $(t+A)^{-1}$, $t(t+A)^{-1}$ and $A(t+A)^{-1}$ are sectorial operators with spectral angles $\leq \theta$.*

PROOF. – If $\theta' \in\,]\theta, \pi[$, then $\sup\limits_{z \in \Sigma_{\pi-\theta'}} |z(z+A)^{-1}| =: M(\theta') < \infty$. If $\lambda \in \Sigma_{\pi-\theta'}$, then by the foregoing lemmas

$$\|\lambda(\lambda + (t+A)^{-1})^{-1}\| = \|(t+A)(\lambda^{-1} + t + A)^{-1}\|$$

$$\leq \frac{t}{\lambda^{-1}+t}\|(\lambda^{-1}+t)(\lambda^{-1}+t+A)^{-1}\| + \|A(\lambda^{-1}+t+A)^{-1}\|$$

where $\|(\lambda^{-1}+t)(\lambda^{-1}+t+A)^{-1}\| \leq M(\theta')$, $\|A(\lambda^{-1}+t+A)^{-1}\| \leq M(\theta')+1$, and $\left|\frac{t}{\lambda^{-1}+t}\right| = \left|\frac{\lambda t}{\lambda t+1}\right| \leq C^{-1}_{\pi-\theta'}$. Similarly

$$\|\lambda(\lambda + t(t+A)^{-1})^{-1}\| = \|(t+A)(t+t\lambda^{-1}+A)^{-1}\|$$

$$\leq \frac{1}{1+\lambda^{-1}}\|(t+t\lambda^{-1})(t+t\lambda^{-1}+A)^{-1}\| + \|A(t+t\lambda^{-1}+A)^{-1}\| \leq M(\theta')\,C^{-1}_{\pi-\theta'} + M(\theta')+1$$

and

$$\|\lambda(\lambda + A(t+A)^{-1})^{-1}\| = \left\|\frac{\lambda}{\lambda+1}(t+A)\left(\frac{\lambda t}{\lambda+1}+A\right)^{-1}\right\|$$

$$\leq \left\|\frac{\lambda t}{\lambda+1}\left(\frac{\lambda t}{\lambda+1}+A\right)^{-1}\right\| + \left\|\frac{\lambda}{\lambda+1}\right\|\left\|A\left(\frac{\lambda t}{\lambda+1}+A\right)^{-1}\right\| \leq M(\theta') + C^{-1}_{\pi-\theta'}(M(\theta')+1).$$

PROOF OF THEOREM 2. – First we prove that $\forall a \in \mathbf{R}^+$ $\|(t+A)^{-a}\| \leq C(M,a)\,t^{-a}$, where $M := \sup\limits_{t>0}\|t(t+A)^{-1}\|$. To this end, since $(t+A)^{-a-\beta} = (t+A)^{-a}(t+A)^{-\beta}$, and the inequality holds trivially when a is a positive integer, with $C(M,a) = M^a$, we can assume that $0 < a < 1$. In this case (see [5, § 5.1])

$$(t+A)^{-a} = \frac{\sin(\pi a)}{\pi}\int_0^\infty \lambda^{a-1}(t+A)^{-1}(\lambda + (t+A)^{-1})^{-1}\,d\lambda$$

$$= \frac{\sin(\pi a)}{\pi}\int_0^\infty \lambda^{a-2}(\lambda^{-1}+t+A)^{-1}\,d\lambda.$$

Therefore

$$\|(t+A)^{-a}\| \le M\,\frac{\sin(\pi a)}{\pi}\int_0^\infty \frac{\lambda^{a-2}}{\lambda^{-1}+t}\,d\lambda$$

$$= M\,t^{1-a}\,\frac{\sin(\pi a)}{\pi}\int_0^\infty \frac{\xi^{a-1}}{1+\xi}\,t^{-1}\,d\xi = M\,t^{-a}.$$

Next, by Theorem 5.1.7 of [5] the range of $(t+A)^{-a}$, that is the domain of $(t+A)^a$, equals $\mathcal{D}(A^a)$; and as A^a is a closed operator this proves that $A^a(t+A)^{-a}$ is bounded. The same theorem gives the equality $A^a(t+A)^{-a} = (A(t+A)^{-1})^a$. Then

$$\|A^a(t+A)^{-a}\| = \left\|\frac{\sin(\pi a)}{\pi}\int_0^\infty \lambda^{a-1}A(t+A)^{-1}(\lambda + A(t+A)^{-1})^{-1}\,d\lambda\right\|$$

$$= \left\|\frac{\sin(\pi a)}{\pi}\int_0^\infty \frac{\lambda^{a-1}}{\lambda+1}A\Big(\frac{\lambda t}{\lambda+1}+A\Big)^{-1}\,d\lambda\right\| \le (M+1)\,\frac{\sin(\pi a)}{\pi}\int_0^\infty \frac{\lambda^{a-1}}{\lambda+1}\,d\lambda = M+1.$$

Finally, if $x \in X$, then $(t+A)^{-a-\beta}x \in \mathcal{D}(A^{a+\beta})$. Then by what we have just seen and the moment inequality (see e.g. [5, Corollary 5.1.13])

$$\|A^\beta(t+A)^{-a-\beta}x\| \le C(a,\beta,M)\,\|(t+A)^{-a-\beta}x\|^{\frac{a}{a+\beta}}\,\|A^{a+\beta}(t+A)^{-a-\beta}x\|^{\frac{\beta}{a+\beta}}$$

$$\le C_1(a,\beta,M)\,t^{-a}\,\|x\|.$$

* * *

Let X and Y be Banach spaces, and let $\mathcal{T}$ be a subset of $\mathcal{L}(X,Y)$ (the Banach space of the bounded linear operators on X to Y). $\mathcal{T}$ is said to be $\mathcal{R}$-bounded if there exists a non-negative constant M such that for arbitrary families $(T_i)_{i\in I}$ in $\mathcal{T}$ and $(x_i)_{i\in I}$ in X, indexed in the same finite set I, the following inequality holds:

$$\sum_{\varepsilon\in\{-1,1\}^I}\Big\|\sum_{i\in I}\varepsilon_i\,T_i x_i\Big\|_Y \le M \sum_{\varepsilon\in\{-1,1\}^I}\Big\|\sum_{i\in I}\varepsilon_i\,x_i\Big\|_X.$$

The best constant M in the above inequality is called the R-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. One can agree that $\mathcal{R}(\mathcal{T}) = \infty$ means that $\mathcal{T}$ is not R-bounded; hence R-boundedness can be expressed by $\mathcal{R}(\mathcal{T}) < \infty$. Here follows a list of properties of R-bounded sets of operators. For details and further information, the reader is referred to the papers [1], [2], [6].

PROPOSITION 1. – *Let X and Y be Banach spaces on the field* $\mathbf{K}$.

(a) *If* $T \in \mathcal{L}(X, Y)$, *then* $\mathcal{R}(\{T\}) = \|T\|$.
(b) *For any* $\mathcal{T} \subseteq \mathcal{L}(X, Y)$, $\sup_{T \in \mathcal{T}} \|T\| \leq \mathcal{R}(\mathcal{T})$.
(c) *If* $X = Y$ *and* $M \in \mathbf{R}^+$, *then* $\mathcal{R}(\{\lambda I; |\lambda| \leq M\}) \leq \beta_{\mathbf{K}} M$, *where* $\beta_{\mathbf{R}} = 1$ *and* $\beta_{\mathbf{C}} = 2$.

PROPOSITION 2. – *If X and Y are Banach spaces and $\mathcal{T}_0$, $\mathcal{T}_1$ are R-bounded subsets of $\mathcal{L}(X, Y)$, then:*

(a) $\mathcal{R}(\{A + B; A \in \mathcal{T}_0, B \in \mathcal{T}_1\}) \leq \mathcal{R}(\mathcal{T}_0) + \mathcal{R}(\mathcal{T}_1)$;
(b) $\mathcal{R}(\{AB; A \in \mathcal{T}_0, B \in \mathcal{T}_1\}) \leq \mathcal{R}(\mathcal{T}_0)\mathcal{R}(\mathcal{T}_1)$.

PROPOSITION 3. – *Let X and Y be Banach spaces and let $(T_{n,\lambda})_{(n,\lambda) \in \mathbf{N} \times \Lambda}$ be a family in $\mathcal{L}(X, Y)$. Assume that $\sum_{n=0}^{\infty} \mathcal{R}(\{T_{n,\lambda}; \lambda \in \Lambda\}) < \infty$. Then $\forall\, \lambda \in \Lambda$ the series $\sum_{n=0}^{\infty} T_{n,\lambda}$ is absolutely convergent in the operator norm, and if we set $T_\lambda = \sum_{n=0}^{\infty} T_{n,\lambda}$, we have $\mathcal{R}(\{T_\lambda; \lambda \in \Lambda\}) \leq \sum_{n=0}^{\infty} \mathcal{R}(\{T_{n,\lambda}; \lambda \in \Lambda\})$.*

Concerning Proposition 3, note that the convergence of the series follows from Proposition 1(b), while the estimate of the R-bound can be found in [6, Lemma 2.4].

A (possibly unbounded) linear operator A acting in a complex Banach space X is said to be R-sectorial if:

(i) the resolvent set $\rho(A)$ of A contains $\mathbf{R}^- := \{t \in \mathbf{R}; t < 0\}$;
(ii)′ $\mathcal{R}(\{t(t + A)^{-1}; t > 0\}) < \infty$.

Via the expansion of the resolvent operators (and Proposition 3) one can prove that if A is R-sectorial then for some $\psi \in\ ]0, \pi[$ such that $\sigma(A) \subseteq \overline{\Sigma}_\psi$, the set $\{\lambda(\lambda - A)^{-1}; \lambda \in \mathbf{C} \setminus \overline{\Sigma}_\psi\}$ is R-bounded. Then one can define the *R-spectral angle* as the g.l.b. of such ψ's, and it is obvious (as a consequence of Proposition 1(b)) that any R-sectorial operator is sectorial, with spectral angle not greater than the R-spectral angle.

LEMMA 4. – *Let X be a complex Banach space and let $\mathcal{T}$ be a R-bounded subset of $\mathcal{L}(X)$. Assume that there exists $\theta \in\ ]0, \pi[$ such that the set $\{\lambda(\lambda - T)^{-1}; \lambda \in \mathbf{C} \setminus \overline{\Sigma}_\theta, T \in \mathcal{T}\}$ is R-bounded. Then $\forall a > 0$ the set $\{T^a; T \in \mathcal{T}\}$ is R-bounded.*

PROOF. – It follows from the above mentioned Proposition 2(b) that it is enough to take $0 < a < 1$. With $M > \mathcal{R}(\mathcal{T})$ and $\varphi \in\ ]\theta, \pi[$, let us define Γ as the closed curve in the complex plane, oriented counterclockwise, that consists of the segments $[0, Me^{\pm i\varphi}]$ and of the arc $\{Me^{i\tau}; -\varphi \le \tau \le \varphi\}$. Then Γ runs around the spectrum of each one of the operators $T \in \mathcal{T}$, and so

$$T^a = \frac{1}{2\pi i} \int_{\Gamma} \lambda^a (\lambda - T)^{-1} \, d\lambda \qquad \forall T \in \mathcal{T}.$$

Now we write $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the polygonal part and Γ_1 the circular part of Γ, and we show that $\big\{ \int_{\Gamma_k} \lambda^a (\lambda - T)^{-1} \, d\lambda; T \in \mathcal{T} \big\}$ $(k = 0, 1)$ is R-bounded. Let $(T_i)_{i \in I}$, $(x_i)_{i \in I}$ be families in $\mathcal{T}$ and in X respectively, where I is a finite set. Then

$$\sum_{\varepsilon \in \{-1,1\}^I} \Big\| \sum_{i \in I} \varepsilon_i \int_{\Gamma_0} \lambda^a (\lambda - T_i)^{-1} x_i \, d\lambda \Big\|$$

$$\le \int_{\Gamma_0} |\lambda|^{a-1} \sum_{\varepsilon \in \{-1,1\}^I} \Big\| \sum_{i \in I} \varepsilon_i \, \lambda (\lambda - T_i)^{-1} x_i \Big\| \, d|\lambda| \le K \int_{\Gamma_0} |\lambda|^{a-1} \, d|\lambda| \sum_{\varepsilon \in \{-1,1\}^I} \Big\| \sum_{i \in I} \varepsilon_i \, x_i \Big\|,$$

where $K = \mathcal{R}(\{\lambda(\lambda - T)^{-1}; \lambda \in \mathbf{C} \setminus \overline{\Sigma}_\theta, \, T \in \mathcal{T}\})$. This proves the assertion concerning Γ_0.

If $\lambda \in \Gamma_1$, then $|\lambda| = M$, and hence $\forall \, T \in \mathcal{T}$

$$(\lambda - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

(uniformly with respect to λ). Then

$$\int_{\Gamma_1} \lambda^a (\lambda - T)^{-1} \, d\lambda = \sum_{n=0}^{\infty} \int_{\Gamma_1} \lambda^{a-n-1} \, d\lambda T^n = \sum_{n=0}^{\infty} \beta_n \, T^n,$$

where β_n does not depend on T, and $|\beta_n| \le 2 \, \varphi \, M^{a-n}$. Then

$$\mathcal{R}(\{\beta_n \, T^n; T \in \mathcal{T}\}) \le 4 \, \varphi \, M^{a-n} \mathcal{R}(\{T^n; T \in \mathcal{T}\}) \le 4 \, \varphi \, M^{a-n} (\mathcal{R}(\mathcal{T}))^n.$$

This is the n-th term of a convergent series, therefore $\big\{ \sum_{n=0}^{\infty} \beta_n \, T^n; T \in \mathcal{T} \big\}$ is R-bounded by the above mentioned Proposition 3.

Proof of Theorem 1. – Let us call θ the R-spectral angle of A; if $\theta < \theta' < \pi$ we also set $M(\theta') = \mathcal{R}(\{\lambda(\lambda + A)^{-1}; \lambda \in \Sigma_{\pi - \theta'}\})$. As $A(t + A)^{-1} = I - t(t + A)^{-1}$, it follows from proposition 1 that $\mathcal{R}(\{A(\lambda + A)^{-1}; \lambda \in \Sigma_{\pi - \theta'}\}) \le M(\theta') + 1$. Now (see lemmas 1 and 2), if $t > 0$ and $\lambda \in \Sigma_{\pi - \theta'}$ we have

$$\lambda(\lambda + t(t + A)^{-1})^{-1} = (t + A)(t + t\lambda^{-1} + A)^{-1}$$

$$= \frac{\lambda}{\lambda + 1} \, (t + t\lambda^{-1})(t + t\lambda^{-1} + A)^{-1} + A(t + t\lambda^{-1} + A)^{-1},$$

where $t + t\lambda^{-1} \in \Sigma_{\pi-\theta'}$ and $\left|\dfrac{\lambda}{\lambda+1}\right| \le C^{-1}_{\pi-\theta'}$. Therefore $\mathcal{R}(\{\lambda(\lambda + t(t+A)^{-1})^{-1};\ t > 0,\ \lambda \in \Sigma_{\pi-\theta'}\}) \le 2\,C^{-1}_{\pi-\theta'}\,M(\theta') + M(\theta') + 1$. Hence lemma 4 implies that the set $\{t^{\alpha}(t+A)^{-\alpha}; t > 0\} = \{(t(t+A)^{-1})^{\alpha}; t > 0\}$ is R-bounded.

Similarly, from

$$\lambda(\lambda + A(t+A)^{-1})^{-1} = \frac{\lambda}{\lambda+1}\,(t+A)\left(\frac{\lambda t}{\lambda+1} + A\right)^{-1}$$

$$= \frac{\lambda t}{\lambda+1}\left(\frac{\lambda t}{\lambda+1} + A\right)^{-1} + \frac{\lambda}{\lambda+1}\,A\left(\frac{\lambda t}{\lambda+1} + A\right)^{-1}$$

we get $\mathcal{R}(\{\lambda(\lambda + A(t+A)^{-1})^{-1}; t > 0,\ \lambda \in \Sigma_{\pi-\theta'}\}) \le M(\theta') + 2\,C^{-1}_{\pi-\theta'}(M(\theta') + 1)$, and from lemma 4 we deduce the R-boundedness of the set $\{(A(t+A)^{-1})^{\beta}; t > 0\} = \{A^{\beta}(t+A)^{-\beta}; t > 0\}$. Now proposition 2(b) above allows to conclude that the set

$$\{t^{\alpha} A^{\beta}(t+A)^{-\alpha-\beta}; t > 0\} = \{(A(t+A)^{-1})^{\beta}\,(t(t+A)^{-1})^{\alpha}; t > 0\}$$

is R-bounded.

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Received March 15, 2007 and in revised form April 17, 2007