M. Chipot, I. Shafrir, G. Vergara Caffarelli

A Nonlocal Problem Arising in the Study of Magneto-Elastic Interactions


Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_1_197_0>

L’utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l’utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/
A Nonlocal Problem Arising in the Study of Magneto-Elastic Interactions

M. Chipot - I. Shafrir - V. Valente - G. Vergara Caffarelli

Dedicated to the memory of Guido Stampacchia

**Sunto.** – Si studia il funzionale non convesso che descrive l’energia di un materiale magneto-elastico. Sono considerati tre termini energetici: l’energia di scambio, l’energia elastica e l’energia magneto-elastica generalmente adottata per cristalli cubici. Si introduce un problema penalizzato monodimensionale e si studia il flusso di gradiente dell’associato funzionale del tipo Ginzburg-Landau. Si prova l’esistenza e unicità di una soluzione classica che tende asintoticamente, per sottosuccessione, a un punto stazionario del funzionale dell’energia.

**Abstract.** – The energy of magneto-elastic materials is described by a nonconvex functional. Three terms of the total free energy are taken into account: the exchange energy, the elastic energy and the magneto-elastic energy usually adopted for cubic crystals. We focus our attention to a one dimensional penalty problem and study the gradient flow of the associated type Ginzburg-Landau functional. We prove the existence and uniqueness of a classical solution which tends asymptotically for subsequences to a stationary point of the energy functional.

1. – Introduction.

The paper deals with the analysis of the equation

\[
\frac{du}{dt} = -\nabla F(u)
\]

where \(F(u)\) is a type Ginzburg-Landau functional, associated to the energy of a magneto-elastic material, which contains a nonlinear nonlocal term. The derivation of the energy functional \(F(u)\) is detailed in the next section starting from a general 3D-model depending on the displacements and the magnetization and assuming some simplifications. In particular in one-dimensional case the energy functional can be expressed in terms of the magnetization variable alone, and the equation (1.1) reduces to the fol-
lowing one

\begin{equation}
(1.2) \quad u_t = u_{xx} - \varepsilon^{-1}|u|^2 - 1)u + \mu A(u)[A(u) \cdot u - \int_0^1 A(u) \cdot u \, dx],
\end{equation}

where \( u = (u_1, u_2) \) and \( A(u) = (u_2, u_1) \).

The parameter \( \mu \) couples the elastic and magnetic processes and \( \varepsilon \) is a small positive parameter introduced to relax the constraint \(|u| = 1\).

We assume that the equation (1.2) is associated with the boundary and initial conditions

\begin{equation}
(1.3) \quad u_x(0, t) = u_x(1, t) = 0, \quad u(x, 0) = u_0(x).
\end{equation}

The paper is organized as follows. In Section 2 we introduce the general 3D model, and present the reduction to the simplified one dimensional model. In Section 3 we study the minimization problem involving the energy functional \( F_{\mu, \varepsilon}(u) \) associated with (1.2), namely

\[
F_{\mu, \varepsilon}(u) = \frac{1}{2} \int_0^1 |u_x|^2 \, dx + \frac{\varepsilon^{-1}}{4} \int_0^1 (|u|^2 - 1)^2 \, dx
\]

\[
- \mu \left[ \int_0^1 (A(u) \cdot u)^2 \, dx - \left( \int_0^1 A(u) \cdot u \, dx \right)^2 \right].
\]

We show that there exists a critical value of \( \mu \), explicitly given by \( \mu^* = \pi/2 \), such that:

(i) for \( \mu < \mu^* \) and \( \varepsilon \) small enough the only minimizers for \( F_{\mu, \varepsilon} \) are constant functions \( u \equiv \alpha \in S^1 \).

(ii) for \( \mu > \mu^* \) the minimizer for \( F_{\mu, \varepsilon} \) is nontrivial.

A similar bifurcation phenomenon was observed by Bethuel, Brezis, Coleman and Hélein in [2] in their study of nematics between cylinders. Finally, Section 4 is devoted to the study of the gradient flow. We prove existence and uniqueness of the solution \( u \) to (1.2), (1.3). Then we show that \( \lim_{t \to \infty} u(t) = u_\infty \) exists and that the function \( u_\infty \) is a stationary point of the energy functional.

2. – The model.

The behaviour of a magnetoelastic material is described by a system of differential equations in the two unknowns: the displacement vector and the magnetization vector. Let \( \Omega \subset \mathbb{R}^3 \) be the volume of the magnetoelastic material and \( \partial \Omega \) its boundary, the unknown magnetization vector \( m \) is a map from \( \Omega \) to \( S^2 \).
(the unit sphere of $\mathbb{R}^3$). The magnetization distribution is well described by a free energy functional which we assume composed of three terms namely the exchange energy $E_{\text{ex}}$, the elastic energy $E_{\text{el}}$ and the elastic-magnetic energy $E_{\text{em}}$. Let $\mathbf{v}$ be the displacement vector, then the total free energy $E$ for a deformable magnetoelastic material is given by

$$E(\mathbf{m}, \mathbf{v}) = E_{\text{ex}}(\mathbf{m}) + E_{\text{em}}(\mathbf{m}, \mathbf{v}) + E_{\text{el}}(\mathbf{v}).$$

We neglect here other contributions to the free energy due, for example, to anisotropy and demagnetization energy terms.

We refer to the books [3], [4]; moreover among the papers on this subject we quote [5], [6], [7], [8]. In the sequel we detail the three energetic terms and derive the governing differential equations. Some drastic hypotheses allows us to reach a reduced one dimensional problem and to carry out the variational analysis for the associated energy functional.

### 2.1 – The general 3D model.

Let $x_i$, $i = 1, 2, 3$ be the position of a point $\mathbf{x}$ of $\Omega$ and denote by

$$v_i = v_i(\mathbf{x}), \quad i = 1, 2, 3$$

the components of the displacement vector $\mathbf{v}$ and by

$$\varepsilon_{kl}(\mathbf{v}) = \frac{1}{2}(v_{k,l} + v_{l,k}), \quad k, l = 1, 2, 3$$

the deformation tensor where, as a common praxis, $v_{k,l}$ stands for $\frac{\partial v_k}{\partial x_l}$.

Moreover we denote by

$$m_j = m_j(\mathbf{x}), \quad j = 1, 2, 3$$

the component of the unit magnetization vector $\mathbf{m}$. In the sequel, where not specified, the Latin indices vary in the set $\{1, 2, 3\}$ and the summation of the repeated indices is assumed. We define

$$E_{\text{ex}}(\mathbf{m}) = \frac{1}{2} \int_{\Omega} a_{ij} m_{k,i} m_{k,j} \, d\Omega,$$

where $(a_{ij})$ is a symmetric positive definite matrix which is supposed diagonal for most materials with all diagonal elements equal to a positive number $a$. The magneto-elastic energy for cubic crystals is assumed. This implies

$$E_{\text{em}}(\mathbf{m}, \mathbf{v}) = \frac{1}{2} \int_{\Omega} \lambda_{ijkl} m_{i,j} m_{j,k} \varepsilon_{kl}(\mathbf{v}) \, d\Omega,$$
where \( \lambda_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 \delta_{ik} \delta_{jl} + \lambda_3 (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) \) with \( \delta_{ijkl} = 1 \) if \( i = j = k = l \) and \( \delta_{ijkl} = 0 \) otherwise. Finally we introduce the elastic energy

\[
E_{el}(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \sigma_{klmn} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{mn}(\mathbf{v}) d\Omega
\]

where \( \sigma_{klmn} \) is the elasticity tensor satisfying the following symmetry property

\[
\sigma_{klmn} = \sigma_{mnlk} = \sigma_{jkmn}
\]

and moreover the inequality

\[
\sigma_{klmn} \varepsilon_{kl} \varepsilon_{mn} \geq \beta \varepsilon_{kl} \varepsilon_{kl}
\]

holds for some \( \beta > 0 \).

We consider the energy functional \( E \) given by

\[
E(\mathbf{m}, \mathbf{v}) = E_{ex}(\mathbf{m}) + E_{em}(\mathbf{m}, \mathbf{v}) + E_{el}(\mathbf{v})
\]

We introduce two tensors \( S = (\sigma_{ijkl} \varepsilon_{ij}) \) and \( \mathcal{L} = (\lambda_{ijkl} m_i m_j) \), moreover we denote by \( \mathbf{p} \) the vector \( \mathbf{p} = (\lambda_{ijkl} m_i m_j \varepsilon_{kl}) \).

The system of differential equations associated to the functional (2.4) reads

\[
\begin{cases}
\text{div} \left( S + \frac{1}{2} \mathcal{L} \right) = 0 & \text{in } \Omega \\
\alpha A \mathbf{m} - \mathbf{p} + (\alpha |\nabla \mathbf{m}|^2 + \mathbf{p} \cdot \mathbf{m}) \mathbf{m} = 0 & \text{in } \Omega
\end{cases}
\]

with boundary conditions

\[
v = 0, \quad \frac{\partial \mathbf{m}}{\partial v} = 0 \quad \text{on } \partial \Omega
\]

where \( v \) is the outer unit normal at the boundary \( \partial \Omega \).

An alternative form for describing the magnetoelastic interactions (2.5) is

\[
\begin{cases}
\text{div} \left( S + \frac{1}{2} \mathcal{L} \right) = 0 & \text{in } \Omega \\
\mathbf{m} \times (\alpha A \mathbf{m} - \mathbf{p}) = 0, \quad |\mathbf{m}| = 1 & \text{in } \Omega
\end{cases}
\]

The dynamical systems associated to the problems (2.5), (2.7) are respectively

\[
\begin{cases}
\rho \mathbf{v}_{tt} = \text{div} \left( S + \frac{1}{2} \mathcal{L} \right) & \text{in } \Omega \times (0, T) \\
\mathbf{m}_t + \gamma (\mathbf{m}_t \times \mathbf{m}) = \alpha A \mathbf{m} - \mathbf{p} + (\alpha |\nabla \mathbf{m}|^2 + \mathbf{p} \cdot \mathbf{m}) \mathbf{m} & \text{in } \Omega \times (0, T)
\end{cases}
\]

and

\[
\begin{cases}
\rho \mathbf{v}_{tt} - \text{div} \left( S + \frac{1}{2} \mathcal{L} \right) = 0 & \text{in } \Omega \times (0, T) \\
\gamma \mathbf{m}_t = \mathbf{m} \times (\alpha A \mathbf{m} - \mathbf{m}_t - \mathbf{p}) & \text{in } \Omega \times (0, T)
\end{cases}
\]
with $\gamma$ and $\rho$ two positive constants. For results concerning the existence of weak solutions to the dynamical problems related to (2.8), (2.9), we refer the reader to [1], [9].

2.2 – The proposed 1D problem.

A simplified model and a simplified energy functional can be obtained assuming that $\Omega$ is a subset of $\mathbb{R}$ and neglecting some components of the unknowns $v$ and $m$. More precisely we consider the single space variable $x$ and assume $\Omega = (0, 1)$, $v = (0, w, 0)$ and $m = (m_1, m_2, 0)$. Then one has

(2.10) \[ \varepsilon_{kl}(v) = \varepsilon_{12}(v) = \varepsilon_{21}(v) = \frac{1}{2} w_x, \]

(2.11) \[ \lambda_{ijkl} = \lambda_{i1j2} = \lambda_3 (\delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1}) = \lambda_{i1j2}, \]

and the different energies are now

(2.12) \[ E_{ex}(m) = \frac{1}{2} \int_0^1 |m_x|^2 \, dx, \quad ((a_{ij}) = \alpha Id = Id), \]

(2.13) \[ E_{em}(m, v) = \frac{\lambda}{2} \int_0^1 \left( m_1 m_2 + m_2 m_1 \right) w_x \, dx \quad (\lambda_3 = \lambda), \]

(2.14) \[ E_{el}(v) = \frac{1}{2} \int_0^1 w_x^2 \, dx \quad (\sigma_{1221} = 1). \]

To deal with the constraint $|m| = 1$, especially when having in mind numerical computations, we introduce the penalization

(2.15) \[ \frac{1}{4\varepsilon} \int_0^1 (|m|^2 - 1)^2 \, dx. \]

If for $m = (m_1, m_2)$ we define the linear operator $A$ by $A(m) = (m_2, m_1)$, then the problem of minimization of the energy reduces to minimize

(2.16) \[ E_e(m, w) \]

\[ = \frac{1}{2} \int_0^1 |m_x|^2 \, dx + \frac{1}{4\varepsilon} \int_0^1 (|m|^2 - 1)^2 \, dx + \frac{\lambda}{2} \int_0^1 (A(m) \cdot m) w_x \, dx + \frac{1}{2} \int_0^1 w_x^2 \, dx, \]

over functions satisfying the boundary conditions

(2.17) \[ m_x = 0, \quad w = 0, \quad \text{on} \quad \partial \Omega = \{0, 1\}. \]
The corresponding Euler equation reads, for $m = m^\varepsilon$,

\begin{align}
    \begin{cases}
        m_{xx}^\varepsilon - \lambda A(m^\varepsilon)w_x - \varepsilon^{-1}(|m^\varepsilon|^2 - 1)m^\varepsilon = 0 \\
        w_{xx}^\varepsilon + \frac{\lambda}{2} (A(m^\varepsilon) \cdot m^\varepsilon)_x = 0.
    \end{cases}
\end{align}

(2.18)

Integrating the second equation leads to

\begin{equation}
    w_x = -\frac{\lambda}{2} (A(m^\varepsilon) \cdot m^\varepsilon) + C.
\end{equation}

(2.19)

The constant $C$ is obtained by integrating the above equation on $(0, 1)$ and using the boundary condition, i.e.,

\begin{equation}
    C = \frac{\lambda}{2} \int_0^1 (A(m^\varepsilon) \cdot m^\varepsilon) \, dx.
\end{equation}

(2.20)

Then replacing $w_x$ by its value in the first equation of (2.18) and setting $\mu = \lambda^2 / 2$ we obtain the following penalty nonlocal equation

\begin{equation}
    m_{xx}^\varepsilon - \varepsilon^{-1}(|m^\varepsilon|^2 - 1)m^\varepsilon + \mu A(m^\varepsilon)[A(m^\varepsilon) \cdot m^\varepsilon - \int_0^1 A(m^\varepsilon) \cdot m^\varepsilon \, dx] = 0,
\end{equation}

(2.21)

with boundary conditions

\begin{equation}
    m_x^\varepsilon(0) = m_x^\varepsilon(1) = 0.
\end{equation}

(2.22)

This is the problem we would like to address, as well as its parabolic analogue, i.e.,

\begin{equation}
    \begin{cases}
        u_t = u_{xx} - \varepsilon^{-1}(|u|^2 - 1)u + \mu A(u)[A(u) \cdot u - \int_0^1 A(u) \cdot u \, dx] \quad \text{in } \Omega \times (0, \infty) \\
        u_x = 0 \quad \text{on } \partial \Omega \times (0, \infty), \quad u(x, 0) = u_0.
    \end{cases}
\end{equation}

3. – The minimization problem.

The equation (2.21) is the Euler-Lagrange equation of the energy functional

\begin{equation}
    F_{\mu,\varepsilon}(m) = \frac{1}{2} \int_0^1 |m_x|^2 \, dx + \frac{\varepsilon^{-1}}{4} \int_0^1 (|m|^2 - 1)^2 \, dx
    \begin{aligned}
        - \frac{\mu}{4} & \left[ \int_0^1 (A(m) \cdot m)^2 \, dx - \left( \int_0^1 A(m) \cdot m \, dx \right)^2 \right]
    \end{aligned}
\end{equation}

(3.1)
Let us consider the minimization problem
\[(3.2) \quad F_{\mu,\varepsilon} = \inf_{m \in H^1(0,1)} F_{\mu,\varepsilon}(m).\]

Above we used the notation $H^1(0,1)$ for $H^1((0,1), \mathbb{R}^2)$.

**Theorem 3.1.** For each $\mu$ and for each positive $\varepsilon$ small enough, i.e., such that $\varepsilon^{-1} - \mu > 0$, the minimum of the functional $F_{\mu,\varepsilon}(m)$ is achieved by a function $m^\varepsilon = m^{\mu,\varepsilon} \in H^1(0,1)$. Furthermore, $m^\varepsilon$ is a solution (2.21)–(2.22) and is therefore of class $C^\infty$.

**Proof.** First of all we observe that by the Cauchy-Young inequality it holds, for any $\delta > 0$,
\[(3.3) \quad \left( \int_0^1 A(m) \cdot m \, dx \right)^2 \leq \int_0^1 (A(m) \cdot m)^2 \, dx \leq \int_0^1 |m|^4 \, dx \]
\[= \int_0^1 (|m|^2 - 1 + 1)^2 \, dx \leq \left( 1 + \frac{1}{\delta} \right)^2 + (1 + \delta) \int_0^1 (|m|^2 - 1)^2 \, dx.\]

So we have:

(i) If $\varepsilon^{-1} - \mu > 0$ then for $\delta$ small enough $\varepsilon^{-1} - (1 + \delta)\mu \geq 0$ and the functional $F_{\mu,\varepsilon}(m)$ is bounded from below. Indeed,

\[F_{\mu,\varepsilon}(m) \geq \frac{1}{2} \int_0^1 |m_x|^2 \, dx + \frac{\varepsilon^{-1} - (1 + \delta)\mu}{4} \int_0^1 (|m|^2 - 1)^2 \, dx - \left( 1 + \frac{1}{\delta} \right)^2 \frac{\mu}{4} \geq - \left( 1 + \frac{1}{\delta} \right)^2 \frac{\mu}{4}
\]

(ii) The functional $F_{\mu,\varepsilon}(m)$ is coercive, i.e.,

\[F_{\mu,\varepsilon}(m) \to +\infty, \quad \text{as} \quad ||m||_{H^1(0,1)} \to \infty.
\]

This follows easily from the inequality $(|m|^2 - 1)^2 \geq |m|^2 - 5/4$.

(iii) The functional is weakly lower semicontinuous, that is: if $\{m_n\}$ is a sequence of functions in $H^1(0,1)$ such that $m_n \rightharpoonup m$ weakly in $H^1(0,1)$, then

\[
\liminf_{n \to \infty} F_{\mu,\varepsilon}(m_n) \geq F_{\mu,\varepsilon}(m).
\]

Indeed, for such a weakly convergent sequence we have

\[
\int_0^1 |m_x|^2 \, dx \leq \liminf_{n \to \infty} \int_0^1 |(m_n)_x|^2 \, dx,
\]

$|m_n|^2 \to |m|^2$ and $A(m_n) \cdot m_n \to A(m) \cdot m$ strongly in $L^2(0,1)$. 

A NONLOCAL PROBLEM ARISING IN THE STUDY ETC.
Since the functional (3.1) is $C^1$, it follows that the stationary points of $F_{\mu,\varepsilon}$ are solutions to the Euler-Lagrange equations (2.21)–(2.22), and it is easily verified that any solution to this one-dimensional problem is of class $C^\infty$. □

**Remark 3.1.** – The result is sharp since for $\varepsilon > \frac{1}{\mu}$, $F_{\mu,\varepsilon}$ is unbounded from below. Indeed, suppose that $1 - \frac{1}{\mu\varepsilon} > 0$. Consider the function $f = (\delta - x)^+$. One has
\[
\left( \int_0^1 f^2 \right)^2 \int_0^1 f^4 = \left( \int_0^\delta (\delta - x)^2 \right)^2 \int_0^\delta (\delta - x)^4 = \frac{\delta^6}{9} / \frac{\delta^5}{5} = \frac{5}{9} \delta < 1 - \frac{1}{\mu\varepsilon}
\]
for $\delta$ small enough. So we may choose $\delta$ small enough such that
\[
\left( \int_0^1 f^2 \right)^2 < \left( 1 - \frac{1}{\mu\varepsilon} \right) \int_0^1 f^4.
\]

Next, consider $m^{(\alpha)} = \alpha f(x) \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. We have
\[
F_{\mu,\varepsilon}(m^{(\alpha)}) = \frac{1}{2} \alpha^2 \int_0^1 f'(x)^2 + \frac{1}{4\varepsilon} \int_0^1 (x^2 f'(x)^2 - 1)^2 - \frac{\mu}{4} \int_0^1 f^4 + \frac{\mu}{4} \left( \int_0^1 x^2 f' \right)^2
= \frac{1}{2} \alpha^2 \int_0^1 f'(x)^2 + \frac{\alpha^4}{4\mu} \left\{ \frac{1}{\varepsilon\mu} \int_0^1 \left( f^2 - \frac{1}{x^2} \right)^2 - \int_0^1 f^4 + \left( \int_0^1 f^2 \right)^2 \right\}.
\]

For $\alpha$ large enough the quantity
\[
\frac{1}{\varepsilon\mu} \int_0^1 \left( f^2 - \frac{1}{x^2} \right)^2 - \int_0^1 f^4 + \left( \int_0^1 f^2 \right)^2
\]
is close to $\left( \frac{1}{\varepsilon\mu} - 1 \right) \int_0^1 f^4 + \left( \int_0^1 f^2 \right)^2 < 0$ and thus $F_{\mu,\varepsilon}(m^{(\alpha)}) \to -\infty$ when $\alpha \to +\infty$.

The functional $F_{\mu,\varepsilon}(m)$ has some obvious symmetry properties. We have clearly $F_{\mu,\varepsilon}(S_i(m)) = F_{\mu,\varepsilon}(m)$ for each $S_i$ in the group
\[
G = \{ S_0, \ldots, S_7 \}
\]
generated by the rotation by $\pi/2$ and the complex conjugation.
LEMMA 3.1. Let $m$ be a solution of the problem (2.21)--(2.22) satisfying
$$F_{\mu,\varepsilon}(m) \leq 0,$$
for some $\varepsilon < \frac{1}{\mu}$. Then, the following a-priori estimate holds,

$$(3.5) \quad |m|^2 \leq K := \frac{\varepsilon^{-1} + \mu}{\varepsilon^{-1} - \mu} \sqrt{\frac{\varepsilon^{-1}}{\varepsilon^{-1} + \mu}}.$$

PROOF. By the assumption on $m$ we have

$$\frac{\varepsilon^{-1}}{4} \int_0^1 (|m|^2 - 1)^2 \, dx + \frac{\mu}{4} \left( \int_0^1 A(m) \cdot m \, dx \right)^2 - \int_0^1 (A(m) \cdot m)^2 \, dx \leq 0.$$

Combining this with (3.3) yields

$$\frac{\varepsilon^{-1}}{4} \int_0^1 (|m|^2 - 1)^2 \, dx + \frac{\mu}{4} \left( \int_0^1 A(m) \cdot m \, dx \right)^2 - (1 + \delta) \frac{\mu}{4} \int_0^1 (|m|^2 - 1)^2 \, dx \leq \left( 1 + \frac{1}{\delta} \right) \frac{\mu}{4}.$$

Therefore, for $\varepsilon^{-1} > \mu$ and any $\delta$ such that $\varepsilon^{-1} - (1 + \delta)\mu \geq 0$, i.e., $\frac{1}{\delta} \geq \frac{\mu}{\varepsilon^{-1} - \mu}$, we have

$$(3.6) \quad \left( \int_0^1 A(m) \cdot m \, dx \right)^2 \leq 1 + \frac{1}{\delta}.$$

Now we multiply the Euler equation (2.21) by $m$ and write the equation for $|m|^2$:

$$-\frac{1}{2} \frac{d^2}{dx^2} |m|^2 + |m|^2 + \varepsilon^{-1} (|m|^2 - 1) |m|^2 - \mu (A(m) \cdot m)^2 + \mu A(m) \cdot m \int_0^1 A(m) \cdot m \, dx = 0.$$

Using (3.6) we obtain

$$-\frac{1}{2} \frac{d^2}{dx^2} |m|^2 + \varepsilon^{-1} (|m|^2 - 1) |m|^2 - \mu |m|^4 - \mu \sqrt{1 + \frac{1}{\delta}} |m|^2 \leq 0,$$

that is

$$-\frac{1}{2} \frac{d^2}{dx^2} |m|^2 + (\varepsilon^{-1} - \mu) |m|^2 \left( |m|^2 - \frac{\varepsilon^{-1} + \mu \sqrt{1 + \frac{1}{\delta}}}{\varepsilon^{-1} - \mu} \right) \leq 0.$$
Choosing \( \frac{1}{\delta} = \frac{\mu}{\varepsilon - \mu} \) and setting \( K = \left( \varepsilon^{-1} + \mu \sqrt{\frac{\varepsilon^{-1}}{\varepsilon - \mu}} \right) / (\varepsilon - \mu) \) gives
\[
\frac{-1}{2} \frac{d^2}{d\varepsilon^2} (|\mathbf{m}|^2 - K) + (\varepsilon^{-1} - \mu) |\mathbf{m}|^2 (|\mathbf{m}|^2 - K) \leq 0.
\]
By the maximum principle, applied to the function \( h = |\mathbf{m}|^2 - K \), we get that \( h \leq 0 \), i.e., \( |\mathbf{m}|^2 \leq K \).

Let us denote by \( \lambda_2 \) the first nontrivial eigenvalue for the Neumann problem:
\[
\begin{aligned}
-\frac{f''}{x} &= \lambda f \quad \text{in } (0, 1), \\
f_x(0) &= f_x(1) = 0.
\end{aligned}
\]
It is well known that \( \lambda_2 = \pi^2 \) and that it yields the optimal constant in the following Poincaré inequality,
\[
\int_0^1 |g_x|^2 \, dx \geq \lambda_2 \int_0^1 (g(x) - \frac{1}{0} \int_0^1 g(t) \, dt)^2 \, dx, \quad \forall g \in H^1(0, 1).
\]
Next, we analyze the minimization problem (3.2) restricted to \( S^1 \)-valued maps. When applied to maps \( \mathbf{m} \in H^1((0, 1); S^1) \), all the functionals \( \{ F_{\mu, x} \}_{x > 0} \) take the same value, that we shall now use to define a new functional on \( H^1((0, 1); S^1) \):
\[
E_{\mu}(\mathbf{m}) = \frac{1}{2} \int_0^1 |\mathbf{m}_x|^2 \, dx - \frac{\mu}{4} \left[ \int_0^1 (A(\mathbf{m}) \cdot \mathbf{m})^2 \, dx - \left( \int_0^1 A(\mathbf{m}) \cdot \mathbf{m} \, dx \right)^2 \right].
\]
In the next proposition we shall apply a bifurcation analysis similar to the one used in [2] in a study of minimizing harmonic maps on an annulus.

**Proposition 3.1.** Put
\[
I(\mu) = \inf_{\mathbf{m} \in H^1((0, 1); S^1)} E_{\mu}(\mathbf{m}).
\]
Then:

(i) For \( \mu \leq \lambda_2 / 2 \) we have \( I(\mu) = 0 \) and the minimum is attained only by constant functions, \( \mathbf{m} \equiv \mathbf{x} \in S^1 \).

(ii) For \( \mu > \lambda_2 / 2 \) we have \( I(\mu) < 0 \) and the minimum is attained by \( \mathbf{m}^0 = e^{i\varphi^0} \) where \( \varphi^0 \) is a nontrivial solution of the problem
\[
\begin{aligned}
-\varphi_{xx} &= \mu \left( \sin 2\varphi^0 - \int_0^1 \sin 2\varphi^0 \, dt \right) \cos 2\varphi^0 \quad \text{in } (0, 1), \\
\varphi_x(0) &= \varphi_x(1) = 0.
\end{aligned}
\]
PROOF. – Each \( m \in H^1((0, 1); S^1) \) can be written as \( m = e^{i\phi} \) for some \( \phi \) in \( H^1((0, 1); \mathbb{R}) \). For such \( m \) we may rewrite the energy in (3.1) as

\[
E_\mu(m) = \frac{1}{2} \int_0^1 |\phi_x|^2 \, dx - \frac{\mu}{4} \int_0^1 \left( \sin 2\phi - \int_0^1 \sin 2\phi \, dt \right)^2 \, dx.
\]

The function \( f = \sin 2\phi \) satisfies \( f_x = 2(\cos 2\phi)\phi_x \), so that

\[
|\phi_x| = \frac{|f_x|}{2|\cos 2\phi|} \geq \frac{|f_x|}{2}.
\]

Write the r.h.s. of (3.11) as a sum of two integrals to obtain

\[
E_\mu(m) = \int_0^1 \left( \frac{1}{2} \phi_x^2 - \frac{1}{8} f_x^2 \right) \, dx \\
+ \int_0^1 \left( \frac{1}{8} f_x^2 - \frac{\mu}{4} \left( \sin 2\phi - \int_0^1 \sin 2\phi \, dt \right)^2 \right) \, dx := I_1 + I_2.
\]

Clearly, for \( \mu < \lambda_2/2 \) and any \( f \neq \text{const} \) we have by (3.12) and (3.8) that \( I_1 > 0 \) and \( I_2 > 0 \). For \( \mu = \lambda_2/2 \) and \( f \neq \text{const} \) we have still \( I_1 > 0 \) while \( I_2 \) is non-negative. This yields assertion (i) of the proposition.

Assume next that \( \mu > \lambda_2/2 \). From the optimality of \( \lambda_2 \) in (3.8) follows the existence of \( \tilde{f} \in H^1((0, 1); \mathbb{R}) \) with

\[
\int_0^1 \left( \frac{1}{8} |\tilde{f}_x|^2 - \frac{\mu}{4} \tilde{f}_x^2 \right) \, dx = -c < 0 \quad \text{and} \quad \int_0^1 \tilde{f} \, dx = 0.
\]

For \( t > 0 \) small enough set \( \psi(t) = \frac{1}{2} \arcsin (t\tilde{f}) \) and then \( m(t) = e^{i\psi(t)} \). Using (3.13) we get

\[
E_\mu(m(t)) = -ct^2 + O(t^4) < 0, \quad \text{for } t \text{ small enough}.
\]

This yields \( I(\mu) < 0 \), and the existence of a minimizer, \( m^0 = e^{i\phi^0} \) with \( \phi^0 \) a non-trivial solution of (3.10) is obvious. \( \square \)

A more precise description of the minimizers in the case \( \mu > \lambda_2/2 = \pi^2/2 \) is given by the next proposition.

**Proposition 3.2.** – In the case \( \mu > \lambda_2/2 \) the minimizer \( m^0 = e^{i\phi^0} \) is unique modulo the operation of the symmetry group \( \mathcal{G} \) (see (3.4)), namely, up to performing the operations:

\[
\phi^0 \leftarrow \phi^0 + k\pi/2 \quad \text{or} \quad \phi^0 \leftarrow -\phi^0 + k\pi/2, \quad k \in \mathbb{Z}.
\]

Such a unique representative of the minimizers can be chosen which is a strictly
increasing function on $[0, 1]$ that satisfies

\begin{equation}
\phi_0(x) = -\phi_0(1 - x) \quad x \in [0, 1].
\end{equation}

**Proof.** Setting

\[ a = \int_0^1 \sin 2\phi_0 \, dx, \]

we can rewrite (3.10) as

\begin{equation}
\begin{cases}
-\phi_{xx} = \mu(\sin 2\phi_0 - a) \cos 2\phi_0 & \text{in } (0, 1), \\
\phi_x(0) = \phi_x(1) = 0.
\end{cases}
\end{equation}

The rest of the proof is divided into several steps.

**Step 1:** $\phi^0$ is strictly monotone.

Replacing $\phi^0$ by its increasing rearrangement $(\phi^0)^*$ will decrease the first term on the r.h.s. of (3.11) (strictly, if $\phi^0$ is not a monotone function), without changing the second term on the r.h.s. of (3.11). Since we may replace $\phi^0$ by $-\phi^0$, we can assume in the sequel that $\phi^0 \geq 0$ in $[0, 1]$. We next claim that actually we have:

\begin{equation}
\phi_x^0 > 0 \quad \text{on } (0, 1).
\end{equation}

Indeed, the function $\psi = \phi_x^0$ satisfies

\begin{equation}
\begin{cases}
-\psi_{xx} = 2\mu(\cos 4\phi_0 + a \sin 2\phi_0)\psi & \text{in } (0, 1), \\
\psi \geq 0 \quad \text{in } (0, 1), \quad \psi(0) = \psi(1) = 0.
\end{cases}
\end{equation}

Since $\psi \neq 0$ we deduce (3.17) from the maximum principle.

**Step 2:** $|\sin 2\phi^0| < 1$ in $(0, 1)$ and $\sin 2\phi^0$ is strictly monotone increasing on $[0, 1]$.

Looking for contradiction, assume for example that $\sin 2\phi^0(x_0) = 1$ for some $x_0$ in $(0, 1)$. By (3.14) we may assume that $\phi(x_0) = \pi/4$. Set \( \tilde{\phi}(x) = \pi/2 - \phi^0(2x_0 - x) \). It is easy to verify that $\tilde{\phi}$ satisfies the equation in (3.16), and also $\tilde{\phi}(x_0) = \phi^0(x_0)$, $\tilde{\phi}_x(x_0) = \phi_x^0(x_0)$. By the uniqueness theory for ODE we deduce that $\tilde{\phi} = \phi^0$, i.e., $\phi^0(x) = \pi/2 - \phi^0(2x_0 - x)$. For the boundary conditions in (3.16) to hold, the only possibility is that $x_0 = 1/2$. We thus conclude that

\begin{equation}
\phi^0(x) = \pi/2 - \phi^0(1 - x), \quad x \in (0, 1).
\end{equation}

The relation (3.19) implies that

\[ a = \int_0^1 \sin 2\phi^0 \, dx = 2 \int_0^{1/2} \sin 2\phi^0 \, dx = \int_0^{1/2} \sin 2\phi^0 \, dx. \]
Defining the following functional on $H^1((0, 1/2); S^1)$,

\[
E^{(1/2)}_\mu(e^{i\phi}) = \frac{1}{2} \int_0^{1/2} |\phi_x|^2 \, dx - \frac{\mu}{4} \int_0^{1/2} \left( \sin 2\phi - \int_0^1 \sin 2\phi \, dt \right)^2 \, dx ,
\]

we conclude that

\[(3.20) \quad E_\mu(e^{i\phi^0}) = 2E^{(1/2)}_\mu(e^{i\phi^0}) .\]

Set, analogously to (3.9),

\[(3.21) \quad I_{1/2}(\mu) = \inf_{m \in H^1((0, 1/2); S^1)} E^{(1/2)}_\mu(m) .\]

The minimum in (3.21) is achieved by some function $\phi^1 \in H^1(0, 1/2)$. Since $\phi^0_x(1/2) > 0$, the restriction of $\phi^0$ to $(0, 1/2)$ is not a minimizer and therefore,

\[(3.22) \quad E^{(1/2)}_\mu(e^{i\phi^1}) < E^{(1/2)}_\mu(e^{i\phi^0}) .\]

We can extend $\phi^1$ to a function $\tilde{\phi}^1 \in H^1(0, 1)$ by setting

\[
\tilde{\phi}^1(x) = \phi^1(1 - x) \quad \text{for} \quad x \in [1/2, 1) .
\]

Combining it with (3.22) and (3.20) we deduce that $E_\mu(e^{i\phi^1}) < E_\mu(e^{i\phi^0})$. This contradiction completes the proof of the assertion $|\sin 2\phi^0| < 1$ in $(0, 1)$.

In view of the above and Step 1 we conclude that the function $\sin 2\phi^0$ is strictly increasing on $[0, 1]$. By adding an integer multiple of $\pi/4$, see (3.14), we may assume that the image of the interval $(0, 1)$ by $\phi^0$ is contained in $(-\pi/4, \pi/4)$. The uniqueness for that representative of the phase of the minimizer will be established in the sequel.

**Step 3: $a = 0$.**

Multiplying the equation in (3.16) by $\phi^0_x$ and integrating yields

\[(3.23) \quad (\phi^0_x)^2 = c^2 - \frac{\mu}{2} (\sin 2\phi^0 - a)^2 \quad \text{on} \quad [0, 1] ,
\]

for some constant $c > 0$. Write the roots of the polynomial $p(t) = c^2 - (\mu/2)(t - a)^2$ as $a - b$ and $a + b$ for some $b > 0$, i.e., $p(t) = (\mu/2)(a + b - t)(t - a + b)$. By Steps 1 and 2, (3.23), and the boundary condition in (3.16) it follows that

\[(3.24) \quad \sin 2\phi^0(0) = a - b \quad \text{and} \quad \sin 2\phi^0(1) = a + b .
\]

Assume by negation that $a \neq 0$. Next, we exploit the following two iden-
tities. First,

\begin{equation}
1 = \int_{0}^{1} dx = \int_{\frac{1}{2}\sin^{-1}(a+b)}^{\frac{1}{2}\sin^{-1}(a-b)} \frac{d\phi}{p^2\sin 2\phi} = \int_{a-b}^{a+b} \frac{dt}{\sqrt{2\mu(a+b-t)(t-a+b)(1-t^2)}}
\end{equation}

\begin{equation}
= \int_{-b}^{b} \frac{ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}.
\end{equation}

Similarly,

\begin{equation}
a = \int_{0}^{1} \sin 2\phi^0(x) dx = \int_{\frac{1}{2}\sin^{-1}(a+b)}^{\frac{1}{2}\sin^{-1}(a-b)} \frac{\sin 2\phi d\phi}{p^2\sin 2\phi}
\end{equation}

\begin{equation}
= \int_{-b}^{b} \frac{(s+a)ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}.
\end{equation}

From (3.25) and (3.26) we deduce that

\begin{equation}
0 = \int_{-b}^{b} \frac{sds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}
\end{equation}

\begin{equation}
= \int_{0}^{b} \frac{s}{\sqrt{2\mu(b-s)(b+s)}} \left( \frac{1}{\sqrt{1-(a+s)^2}} - \frac{1}{\sqrt{1-(a-s)^2}} \right) ds.
\end{equation}

But it is clear that the r.h.s. of (3.27) is strictly positive for \( a > 0 \) and strictly negative for \( a < 0 \), so in either case we are led to a contradiction.

STEP 4: Conclusion.

Going back to (3.23) we can now write

\begin{equation}
(\varphi^0)^2 = c^2 - \frac{\mu}{2} \sin^2 2\phi^0 = \frac{\mu}{2} (b - \sin 2\phi^0)(b + \sin 2\phi^0) \quad \text{on } [0, 1],
\end{equation}

with \( b = c\sqrt{2/\mu} \). The equation (3.25) now reads

\begin{equation}
\sqrt{2\mu} = \int_{-b}^{b} \frac{ds}{\sqrt{(b-s)(b+s)(1-s^2)}} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1-b^2 \sin^2 \theta}}.
\end{equation}

Since we assume that \( \mu > \frac{\pi^2}{2} \), it follows that there is a unique \( b > 0 \) for which (3.28) holds.
Next, there is a unique point \( x_0 \in (0, 1) \) where \( 0 = \tilde{\varphi}^0(x_0) = \sin 2\varphi^0(x_0) \). At that point, \( \dot{x}_x^0(x_0) = b \sqrt{\mu/2} \). The function \( \dot{x}(x) = -\varphi^0(2x_0 - x) \) solves the equation
\[\dot{x}_x = \mu \sin 2\dot{x} \cos 2\dot{x} \quad \text{in} \ (0, 1), \]
with the initial conditions
\[\dot{x}(x_0) = \dot{x}^0(x_0) = 0 \quad \text{and} \quad \dot{x}_x(x_0) = \dot{x}_x^0(x_0) = \sqrt{\mu b} \cdot\]
Since there is a unique solution to (3.29)–(3.30), it follows that \( \varphi^0 = \dot{x} \). Since \( \dot{x}_x(2x_0) = 0 \) we must have \( x_0 = 1/2 \) and the symmetry property (3.15) holds. The uniqueness assertion of the proposition follows from the uniqueness for the initial problem (3.29)–(3.30) for \( x_0 = 1/2 \).

Next we present a convergence result that will be used in our main theorem.

**Proposition 3.3.** For each \( \mu > 0 \), any sequence of minimizers \( \{m_{\varepsilon_n}\} \), with \( \varepsilon_n \to 0 \), has a subsequence which converges in \( H^1(0, 1) \) and in \( C[0, 1] \) to \( m^0 \in C^\infty([0, 1]; S^1) \) which is a minimizer for \( I(\mu) \).

**Proof.** Note that \( F_{\mu, \varepsilon}(m^\varepsilon) \leq F_{\mu, \varepsilon}(x) = E_\mu(x) = 0 \) for any constant \( x \in S^1 \).

Using (3.5) we conclude that for \( \varepsilon < \frac{1}{2\mu} \), we have
\[\int_0^1 |m_x^\varepsilon|^2 \, dx \leq C \quad \text{and} \quad \frac{1}{\varepsilon} \int_0^1 (1 - |m^\varepsilon|^2)^2 \, dx \leq C,\]
for some constant \( C \) (which is independent of \( \varepsilon \)). Since \( H^1(0, 1) \) is compactly embedded in \( C[0, 1] \), we can extract a subsequence, still denoted by \( \{m_{\varepsilon_n}\} \), that converges weakly in \( H^1(0, 1) \) and strongly in \( C[0, 1] \) to a limit \( m^0 \in H^1(0, 1; S^1) \).

Since for each \( \varepsilon_n \) and each \( m \in H^1((0, 1); S^1) \), \( F_{\mu, \varepsilon_n}(m^\varepsilon) \leq E_\mu(m) \), we get that
\[\limsup_{\varepsilon_n \to 0} F_{\mu, \varepsilon_n}(m^\varepsilon) \leq E_\mu(m), \quad \forall m \in H^1((0, 1); S^1).\]

On the other hand, the weak lower-semicontinuity of the \( L^2 \)-norm of the gradient, combined with the uniform convergence of \( \{m_{\varepsilon_n}\} \) towards \( m^0 \), yields
\[E_\mu(m^0) \leq \liminf_{\varepsilon_n \to 0} F_{\mu, \varepsilon_n}(m^\varepsilon).\]

Combining (3.31) with (3.32) we deduce that \( E_\mu(m^0) \leq E_\mu(m), \forall m \in H^1((0, 1); S^1) \), i.e., \( m^0 \) is a minimizer for \( I(\mu) \). It also follows that the convergence \( m^\varepsilon \to m^0 \) is actually strong in \( H^1(0, 1) \).

We are now in position to state our main result for the minimization problem (3.2).
Theorem 3.2. –

(i) For each $\mu < \lambda_2/2$ there exists $\varepsilon_0(\mu) > 0$ such that for $\varepsilon \leq \varepsilon_0(\mu)$ we have $F_{\mu, \varepsilon} = 0$ and the only minimizers for (3.2) are constant functions $m^\varepsilon = \alpha \in S^1$.

(ii) For $\mu > \lambda_2/2$ we have $F_{\mu, \varepsilon} < 0$ for every $\varepsilon > 0$. For each $\varepsilon > 0$ we may choose a representative for the minimizer $m^\varepsilon$ (by replacing $m^\varepsilon$ with $S_1(m^\varepsilon)$, see (3.4)) such that $\lim_{\varepsilon \to 0} m^\varepsilon = m^0$ in $H^1(0, 1)$ and in $C[0, 1]$, where $m^0 \in C^\infty([0, 1]; S^1)$ is a non-trivial minimizer for $I(\mu)$.

(iii) In the limiting case $\mu = \lambda_2/2$, we have for a subsequence, $\lim_{\varepsilon \to 0} m^\varepsilon = \alpha$ in $H^1(0, 1)$ and in $C[0, 1]$, for some constant $\alpha \in S^1$.

Proof. – (i) By Proposition 3.3 we have, in particular, that $\lim_{\varepsilon \to 0} |m^\varepsilon| = 1$, uniformly on $[0, 1]$. Hence, for any $\delta > 0$ we have, for $\varepsilon \leq \varepsilon_1(\delta)$,

$$1 - \delta \leq |m^\varepsilon(x)| \leq 1 + \delta, \quad x \in [0, 1].$$

In particular, if $\delta \leq 1/2$, say, then we may write $m^\varepsilon = \rho e^{i\phi}$, with $\rho = |m^\varepsilon|$. A simple computation gives

$$F_{\mu, \varepsilon}(m^\varepsilon) = \frac{1}{2} \int_0^1 (\rho^2 |\phi_x|^2 + |\rho_x|^2) dx + \frac{1}{4\varepsilon} \int_0^1 (1 - \rho^2)^2 dx$$

$$- \frac{\mu}{4} \int_0^1 \left( \rho^2 \sin 2\phi - \int_0^1 \rho^2 \sin 2\phi dt \right)^2 dx.$$

By the Cauchy-Schwarz inequality we get,

$$\int_0^1 \left( \rho^2 \sin 2\phi - \int_0^1 \rho^2 \sin 2\phi dt \right)^2 dx$$

$$= \int_0^1 \left( \left( \sin 2\phi - \int_0^1 \sin 2\phi dt \right) + (\rho^2 - 1) \sin 2\phi - \int_0^1 (\rho^2 - 1) \sin 2\phi dt \right)^2 dx$$

$$\leq (1 + \delta) \int_0^1 \left( \sin 2\phi - \int_0^1 \sin 2\phi dt \right)^2 dx$$

$$+ \left( 1 + \frac{1}{\delta} \right) \left( \int_0^1 (\rho^2 - 1)^2 \sin^2 2\phi dx - \left( \int_0^1 (\rho^2 - 1) \sin 2\phi dx \right)^2 \right)$$

$$\leq (1 + \delta) \int_0^1 \left( \sin 2\phi - \int_0^1 \sin 2\phi dt \right)^2 dx + \left( 1 + \frac{1}{\delta} \right) \int_0^1 (1 - \rho^2)^2 dx.$$
Combining (3.35) with (3.34) and (3.33) yields

\[(3.36) \quad F_{\mu, \varepsilon}(m^\varepsilon) \geq \frac{(1 - \delta)^2}{2} \frac{1}{0} \int |\phi_x|^2 - \frac{\mu(1 + \delta)}{4} \frac{1}{0} \left( \int \sin 2\phi - \int \sin 2\phi dt \right) \left( \frac{1}{4\varepsilon} - \frac{\mu}{4} \left( 1 + \frac{1}{\delta} \right) \right) \frac{1}{0} (1 - \rho^2)^2 dx. \]

Since \( \mu < \lambda_2/2 \) we can fix \( \delta \) small enough so that

\[ \tilde{\mu} := \frac{1 + \delta}{(1 - \delta)^2} \mu < \frac{\lambda_2}{2}. \]

For \( \varepsilon \) small enough such that \( \frac{1}{8\varepsilon} \geq \frac{\mu}{4} (1 + 1/\delta) \) we obtain from (3.36)

\[(3.37) \quad 0 \geq F(m^\varepsilon) \geq (1 - \delta)^2 \left\{ \frac{1}{2} \frac{1}{0} \int |\phi_x|^2 dx - \frac{\tilde{\mu}}{4} \frac{1}{0} \left( \int \sin 2\phi - \int \sin 2\phi dt \right) \frac{1}{0} \left( \frac{1}{4\varepsilon} - \frac{\mu}{4} \left( 1 + \frac{1}{\delta} \right) \right) \frac{1}{0} (1 - \rho^2)^2 dx \geq 0. \]

By Proposition 3.1 strict inequality holds for the last inequality on the r.h.s. of (3.37), unless \( m^\varepsilon \) equals identically a constant of modulus one, hence the result.

(ii) By Proposition 3.1 we have in this case,

\[ F_{\mu, \varepsilon} \leq I(\mu) < 0. \]

The convergence assertion follows from Proposition 3.3 and the uniqueness follows from Proposition 3.2.

(iii) This part is a direct consequence of Proposition 3.3 and Proposition 3.1.

\[ \square \]

Remark 3.2. – We do not know whether in the the limiting case \( \mu = \lambda_2/2 \) (case (iii)) the minimizer \( m^\varepsilon \) is necessarily a constant for \( \varepsilon \) small enough, as in case (i).

4. – The analysis of the gradient flow equation.

Let \( T \) be a positive number, we define \( Q_T = \Omega \times (0, T) \) and \( (\cdot, \cdot) \) the scalar product in \( L^2(\Omega) \) and in \( L^2(\Omega) \). Consider the initial boundary value problem

\[(4.1) \quad u_t = u_{xx} - \varepsilon^{-1}(|u|^2 - 1)u + \mu A(u) \left[ A(u) \cdot u - \int_0^1 A(u) \cdot u \, dx \right], \]
with the boundary conditions
\begin{equation}
\begin{aligned}
\mathbf{u}_x(0, t) = \mathbf{u}_x(1, t) = 0, & \quad t \in (0, T), \\
\end{aligned}
\end{equation}
and the initial condition
\begin{equation}
\begin{aligned}
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \quad \mathbf{x} \in \Omega \equiv (0, 1).
\end{aligned}
\end{equation}
Provided the solution \(\mathbf{u}(t)\) of (4.1), (4.2), (4.3) exists for all \(t\), we show that \(\lim_{t \to \infty} \mathbf{u}(t) = \mathbf{u}_\infty\) exists and, for suitable choice of the initial datum \(\mathbf{u}_0\), the function \(\mathbf{u}_\infty\) is a negative energy solution to (2.21), (2.22).

The following existence and uniqueness theorem holds.

**Theorem 4.1.** Let \(\mathbf{u}_0(\mathbf{x}) \in \mathbf{H}^1(\Omega)\) and \(\varepsilon^{-1} > 2\mu\) and set
\begin{equation}
N(\mathbf{u}) = -\varepsilon^{-1}(|\mathbf{u}|^2 - 1)\mathbf{u} + \mu A(\mathbf{u})[A(\mathbf{u}) \cdot \mathbf{u} - \int_0^1 A(\mathbf{u}) \cdot \mathbf{u} \, dx].
\end{equation}

Then, there exists a unique solution \(\mathbf{u} \in \mathbf{L}^\infty(Q_T)\) such that
\begin{equation}
\left\{
\begin{aligned}
\mathbf{u} & \in \mathbf{L}^2(0, T; H^1(\Omega)), & \mathbf{u}_t & \in \mathbf{L}^2(0, T; H^1(\Omega)'), \\
\|\mathbf{u}\|_{L^\infty(Q_T)} & \leq B, & (B \text{ independent of } T), \\
\frac{d}{dt} \langle \mathbf{u}, \mathbf{v} \rangle + \int_{\Omega} \mathbf{u}_x \cdot \mathbf{v}_x \, dx = \langle N(\mathbf{u}), \mathbf{v} \rangle, & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), & \text{in } \mathcal{D}'(0, T), \\
\mathbf{u}(0) & = \mathbf{u}_0.
\end{aligned}
\right.
\end{equation}
Then \(\mathbf{u}\) is a weak solution of (4.1)-(4.3) and since \(N(\mathbf{u})\) is bounded this is also a strong solution.

**Proof.** We use the Galerkin method. We consider \(w_1, \ldots, w_n\) an orthogonal basis in \(L^2(\Omega)\) of eigenvectors for the Neumann problem
\begin{equation}
\left\{
\begin{aligned}
-\lambda w & = \lambda w \quad \text{in } \Omega, \\
w_x(0) & = w_x(1) = 0.
\end{aligned}
\right.
\end{equation}
We consider then
\begin{equation}
\mathbf{u}_n = (u_{n,1}, u_{n,2}), \quad u_{n,j} = \sum_{i=1}^n y_{i,j}(t)w_i, \quad j = 1, 2,
\end{equation}
solution to the Cauchy problem
\begin{equation}
\left\{
\begin{aligned}
\mathbf{u}'_n & = (\mathbf{u}_n)_{xx} + N(\mathbf{u}_n) & t & \in (0, T), \\
u_{n,j}(0) & = \sum_{i=1}^n \langle w_i, u_{0,j} \rangle w_i, & j & = 1, 2.
\end{aligned}
\right.
\end{equation}
It is clear that (4.8) is a nonlinear system of ode’s with $2n$ unknowns. It has a unique solution locally.

**Claim 1:** $\mathbf{u}_n(0)$ is bounded in $H^1(\Omega)$. Indeed for $j = 1, 2$ one has

\[
\int_0^1 \left| (\mathbf{u}_{n,j}(0))_{x_1} \right|^2 \leq \sum_{i=1}^{\infty} \left| \langle w_i, u_{0,j} \rangle \right|^2 \leq \sum_{i=1}^{\infty} \left( \langle w_i, u_{0,j} \rangle \right)^2 \lambda_i \\
\int_0^1 \left| \mathbf{u}_{n,j}(0) \right|^2 = \sum_{i=1}^{n} \left( \langle w_i, u_{0,j} \rangle \right)^2 \leq \int_0^1 \left| (u_{0,j})_{x_1} \right|^2.
\]

(4.9)

To simplify our notation we do not write the measures of integration.

**Claim 2:** $\mathbf{u}_n$ is bounded in $L^\infty(\Omega \times (0, t))$ by a constant independent of $n$ and $t$.

We multiply the first equation of (4.8) by $\mathbf{u}_n'$ and integrate on $Q_t = \Omega \times (0, t)$ to get

\[
\int_{Q_t} \left| \mathbf{u}_n' \right|^2 = \int_{Q_t} (\mathbf{u}_n)_{x_1} \cdot \mathbf{u}_n' - \epsilon^{-1} \int_{Q_t} \left( \left| \mathbf{u}_n \right|^2 - 1 \right) \mathbf{u}_n \cdot \mathbf{u}_n' \\
+ \mu \int_{Q_t} (\mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n') \mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n - \frac{1}{2} \mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n.
\]

We remark then that

\[
\mathbf{u}_n \cdot \mathbf{u}_n' = \left( \frac{1}{2} \left| \mathbf{u}_n \right|^2 \right)', \\
\mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n' = \left( u_{n,1} u_{n,2} \right)' = \frac{1}{2} \left( \mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n \right)'.
\]

Then we obtain

\[
\int_{Q_t} \left| \mathbf{u}_n' \right|^2 = -\frac{1}{2} \left( \int_{\Omega} \left| \mathbf{u}_{n,x_1} \right|^2 \right)' - \frac{\epsilon^{-1}}{4} \int_{Q_t} \left( \left| \mathbf{u}_n \right|^2 - 1 \right)^2' \\
+ \frac{\mu}{4} \int_{Q_t} \left( \int_{\Omega} \left( \mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n \right)^2 \right)' - \frac{\mu}{4} \int_{Q_t} \left( \int_{\Omega} \left( \mathbf{A}(\mathbf{u}_n) \cdot \mathbf{u}_n \right)^2 \right)'.
\]

By integration we obtain

\[
\int_{Q_t} \left| \mathbf{u}_n' \right|^2 = F(\mathbf{u}_n)(0) - F(\mathbf{u}_n)(t)
\]

(4.10)
where we have set
\[ F(u_n) = \frac{1}{2} \int_{\Omega} |(u_n)_x|^2 + \frac{\varepsilon^{-1}}{4} \int_{\Omega} (|u_n|^2 - 1)^2 - \frac{\mu}{4} \int_{\Omega} (A(u_n) \cdot u_n)^2 + \frac{\mu}{4} \left( \int_{\Omega} A(u_n) \cdot u_n \right)^2. \]

By the Claim 1, \( u_n(0) \) is bounded in \( H^1(\Omega) \), and then also in \( L^\infty(\Omega) \), by a constant independent of \( n \). It follows that \( F(u_n)(0) \) is bounded by a constant \( A \) independent of \( n \), so from (4.10) we derive
\[ F(u_n)(t) \leq A. \]

Now we have
\[ \int_{\Omega} (A(u_n) \cdot u_n)^2 \leq \int_{\Omega} |u_n|^4 = \int_{\Omega} (|u_n|^2 - 1)^2 \leq 2 \int_{\Omega} (|u_n|^2 - 1)^2 + 2. \]

Then, from (4.11) and the definition of \( F(u_n) \) we get
\[ \frac{1}{2} \int_{\Omega} |(u_n)_x|^2 + \frac{\varepsilon^{-1} - 2\mu}{4} \int_{\Omega} (|u_n|^2 - 1)^2 - \frac{\mu}{2} + \frac{\mu}{4} \left( \int_{\Omega} A(u_n) \cdot u_n \right)^2 \leq A. \]

Since \( \varepsilon^{-1} - 2\mu > 0 \) it follows that
\[ \int_{\Omega} |(u_n)_x|^2 \leq 2A + \mu, \quad \int_{\Omega} (|u_n|^2 - 1)^2 \leq \frac{4A + 2\mu}{\varepsilon^{-1} - 2\mu}. \]

Due to the inequality \( \int_{\Omega} (|u_n|^2 - 1) \leq \left( \int_{\Omega} (|u_n|^2 - 1)^2 \right)^{1/2} \) we have that \( u_n \) is bounded in \( H^1(\Omega) \) by a constant independent of \( n \) and \( t \) and the Claim 2 follows from the imbedding of \( H^1(\Omega) \) into \( L^\infty(\Omega) \).

As a consequence of the Claim 2 the solution to (4.8) is global on \((0, T)\). It is also unique due to the fact that for \( u \) bounded, \( N(u) \) is Lipschitz continuous. Moreover \( u_n \) is also smooth in \( x \) and \( t \).

Let us denote by \( B \) the constant which bounds, uniformly in \( n \) and \( t \), the function \( u_n \) and set
\[ K = \{ v \in L^2(Q_T) \mid |v| \leq B \text{ a.e. in } Q_T \}. \]

It is clear that \( K \) is a closed convex set of \( L^2(Q_T) \). Due to the preceding analysis and the equation (4.8) it follows that for some constant \( C \) independent of \( n \) and \( T \) we have
\[ \|u_n\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \quad \|(u_n)_t\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \quad \|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \]
Since the imbedding
\[ \{ v \mid v \in L^2(0, T; H^1(\Omega)), v_t \in L^2(0, T; H^1(\Omega)') \} \subset L^2(0, T; L^2(\Omega)) \]
is compact – up to a subsequence – there exists \( u \) in \( L^2(0, T; H^1(\Omega)) \) such that
\[
\begin{align*}
&u_n \to u \quad \text{in } L^2(0, T; H^1(\Omega)), \\
&u_n \to u \quad \text{in } L^2(0, T; L^2(\Omega)), \\
&(u_n)_t \to u_t \quad \text{in } L^2(0, T; H^1(\Omega)').
\end{align*}
\]
Of course \( u \in K \). Going back to (4.4) we have
\[
N(u) = N_1(u) + N_2(u),
\]
where we have set
\[
\begin{align*}
N_1(u) &= -\varepsilon^{-1}|u|^2 - 1)u + \mu(A(u) \cdot u)A(u) \\
N_2(u) &= -\mu(A(u) \int_{\Omega} (A(u) \cdot u) \, dx.
\end{align*}
\]
Since \( N_1(u) \) is a smooth function we have for some constant \( L_1 \)
\[ |N_1(u) - N_1(v)| \leq L_1 |u - v|, \quad \forall u, v \in \mathbb{R}^2, \text{ bounded.} \]
Moreover for \( u, v \in K \)
\[ |N_2(u) - N_2(v)| = \left| -\mu(A(u) \int_{\Omega} (A(u) \cdot u) + \mu(A(v) \int_{\Omega} (A(v) \cdot v) \right| \\
= \left| -\mu(A(u) - A(v)) \int_{\Omega} (A(u) \cdot u) + \mu A(v) \int_{\Omega} (A(v) \cdot v - A(u) \cdot u) \right| \\
\leq C_1 |u - v| + C_2 \left\{ \int_{\Omega} |u - v|^2 \, dx \right\}^{1/2}. \]
From these estimates it follows that
\[ N(u_n) \to N(u) \quad \text{in } L^2(0, T; L^2(\Omega)). \]
We take now \( v \in H^1(\Omega) \) to get from (4.8)
\[
\frac{d}{dt} \langle (u_n, v) \rangle = -\int_{\Omega} u_{nx} \cdot v_x + \int_{\Omega} N(u_n) \cdot v, \quad \forall t \in (0, T).
\]
Passing to the limit in \( n \) we get easily the third equation of (4.5).
Let now \( v \in H^1(\Omega) \) and let \( \varphi \) be a smooth function such that
\[ \varphi(0) = 1, \quad \varphi(T) = 0. \]
From (4.5) we have
\[
\int_0^T \frac{d}{dt} \langle u, v \rangle \varphi = - \int_{Q_T} u_x \cdot v_x \varphi + \int_{Q_T} N(u) \cdot v \varphi
\]
\[
= \lim_{n} \int_{Q_T} u_n x \cdot v_x \varphi + \int_{Q_T} N(u_n) \cdot v \varphi = \lim_{n} \int_0^T \frac{d}{dt} \langle u_n, v \rangle \varphi
\]
\[
= \lim_{n} \int_0^T \frac{d}{dt} \left[ \langle u_n, v \rangle \varphi \right] - \int_0^T \langle u_n, v \varphi \rangle' = - \lim_{n} \langle u_n(0), v \rangle - \int_0^T \langle u, v \rangle \varphi' =
\]
\[
= - \langle u_0, v \rangle - \int_0^T \langle u, v \rangle \varphi'.
\]

Integrating the left hand side of this equality we arrive to
\[
\langle u(0), v \rangle = \langle u_0, v \rangle, \quad \forall v \in H^1(\Omega),
\]
which completes the existence result.

For uniqueness, starting from two solutions \(u_1, u_2\) we have
\[
\frac{d}{dt} (u_1 - u_2) = (u_1 - u_2)_{xx} + N(u_1) - N(u_2).
\]

Multiplying by \((u_1 - u_2)\) and integrating in \(\Omega\) we get by (4.13), (4.14)
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| u_1 - u_2 \right|^2 \leq C \int_{\Omega} \left| u_1 - u_2 \right|^2,
\]
and the result follows. \(\Box\)

**Corollary 4.1.** Let \(u\) be the solution of the problem (4.1), (4.2), (4.3). Then,
\[
\int_{Q_t} \left| u_t \right|^2 = F(u)(0) - F(u)(t).
\]

Moreover, there exists a positive constant \(\bar{A}\) independent of \(t\), such that
\[
\int_{Q_t} \left| u_t \right|^2 + \int_{\Omega} \left| u_x \right|^2 + \int_{\Omega} \left( \left| u \right|^2 - 1 \right)^2 \leq \bar{A}.
\]

**Proof.** The equality (4.15) easily follows from (4.10). Moreover, we have
\[
F(u) \geq \frac{1}{2} \int_{\Omega} \left| u_x \right|^2 + \frac{\varepsilon^{-1} - 2\mu}{4} \int_{\Omega} \left( \left| u \right|^2 - 1 \right)^2 - \frac{\mu}{2} + \frac{\mu}{4} \left( \int_{\Omega} A(u) \cdot u \right)^2
\]
for \(\varepsilon^{-1} - 2\mu \geq \bar{a} > 0\). We get then the estimate (4.16). \(\Box\)
Lemma 4.1. – Let $u$ be the solution of the problem (4.1), (4.2), (4.3). Then, there exists a positive constant $K$ such that the following estimate holds

\begin{equation}
\int_0^T \left| \frac{d}{dt} \|u_t\|_{L^2(0,1)}^2 \right| dt \leq K.
\end{equation}

Proof. – We look at the equation (4.1) in the form

\[ u_t = u_{xx} + N(u). \]

Differentiating with respect to $t$ and multiplying by $u_t$ we obtain

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_0^1 |u_t|^2 dx + \int_0^1 |u_{xt}|^2 dx = \int_0^1 \frac{d}{dt} N(u) \cdot u_t dx.
\end{equation}

Recall that $N(u) = N_1(u) + N_2(u)$ where $N_1$ is a $C^\infty$-function and

\[ N_2(u) = -\mu A(u) \int_\Omega (A(u) \cdot u) dx. \]

From this we deduce

\[ \frac{d}{dt} N_2(u) = -\mu A(u_t) \int_\Omega (A(u) \cdot u) dx - 2\mu A(u) \int_\Omega (A(u) \cdot u_t) dx, \]

and thus

\[ \left\| \frac{d}{dt} N_2(u) \right\| \leq C \|u_t\|. \]

Hence from (4.18), Theorem 4.1 and Corollary 4.1

\[ \int_0^T \left| \frac{d}{dt} \|u_t\|_{L^2(0,1)}^2 \right| dt \leq C \left( \int_0^T \|u_t\|_{L^2(0,1)}^2 dx \right) \leq K, \]

and the proof of the lemma easily follows. \hfill \Box

Now we can prove the following theorem

Theorem 4.2. – Let $u$ be the solution of the problem (4.1), (4.2), (4.3) for $T = \infty$. Then, there exists a sequence $t_k \to \infty$ such that

\begin{equation}
 u(x, t_k) \to u_{\infty}(x) \quad \text{in} \quad H^1(0,1),
\end{equation}

where $u_{\infty}(x)$ is a stationary point of (4.1). Moreover, all the weakly convergent sequences converge to stationary points.
PROOF. – Let \( u^k = u(\cdot, t_k) \) be the given solution of (4.1), (4.2), (4.3) at time \( t_k \). From the estimate (4.16) it follows that, passing to a subsequence if necessary,

\[
\begin{align*}
(4.20) & \quad u^k \to u_\infty \quad \text{weakly in } H^1(0, 1), \\
(4.21) & \quad u^k \to u_\infty \quad \text{strongly in } L^2(0, 1), \\
(4.22) & \quad u^k \cdot A(u^k) \to u_\infty \cdot A(u_\infty) \quad \text{strongly in } L^2(0, 1), \\
(4.23) & \quad |u^k|^2 \to |u_\infty|^2 \quad \text{strongly in } L^2(0, 1).
\end{align*}
\]

Now we have to prove that \( u_\infty \) is a solution of the stationary problem. For this we multiply the equation (4.1) by \( v \in H^1(0, 1) \) and integrate to get

\[
\begin{align*}
(4.24) & \quad \int_0^1 u_t^k \cdot v \, dx = -\int_0^1 u_x^k \cdot v_x \, dx - \varepsilon^{-1} \int_0^1 |u^k|^2 - 1 \, u^k \cdot v \, dx \\
& \quad + \mu \int_0^1 A(u^k) \cdot v \left[ A(u^k) \cdot u^k - \int_0^1 A(u^k) \cdot u^k \, dx \right]
\end{align*}
\]

From Lemma 4.1 we have that \( ||u_t^k|| \) is a Cauchy sequence (see (4.17)) and the limit can only be 0 since \( \int_0^\infty ||u_t||^2 \) is bounded. From the convergence established above it follows that \( u_\infty \) is a weak solution of the stationary problem. \( \square \)

COROLLARY 4.2. – Let \( u_0 \) be a function verifying the hypotheses of Theorem 4.1. If \( F(u_0) < 0 \) then the limit function \( u_\infty(x) \) defined in Theorem 4.2 is a negative energy stationary point of (3.1).

PROOF. – The proof easily follows from the energy estimate (4.15). Indeed since the system is dissipative we have

\[
F(u_\infty) \leq F(u_0)
\]

\( \square \)

Acknowledgments. The work of V.V. has been partially supported by the European Community under the contract HPRN-CT-2002-00284 “Smart Systems” and by the CNR/MIUR project: “Materiali compositi per applicazioni strutturali di rilevante interesse industriale”. M.C. acknowledges the support of the Swiss National Science Foundation under the contracts #20-105155/1 and #20-113287/1. The research of L.S. was partially supported by the Research Training Network “Fronts-Singularities” (RTN contract: HPRN-CT-2002-00274).
REFERENCES


Michel Chipot: University of Zürich, Institute of Mathematics
Winterthurerstr. 190, CH-8057 Zürich, Switzerland
E-mail: m.m.chipot@math.unizh.ch

Itai Shafrir: Technion - I.I.T.
Department of Mathematics, 32000 Haifa, Israel
E-mail: shafrir@math.technion.ac.il

Vanda Valente: Istituto per le Applicazioni del Calcolo “M. Picone”
CNR, V.le del Policlinico 137, 00161 Roma, Italy
E-mail: valente@iac.rm.cnr.it

Giorgio Vergara Caffarelli: University of Roma “La Sapienza”
Department MeMoMat, V. A. Scarpa 16, 00161 Roma, Italy
E-mail: vergara@dmmm.uniroma1.it

Received November 6, 2007 and in revised form December 3, 2007.