RALPH CHILL, SACHI SRIVASTAVA

$L^p$ Maximal Regularity for Second Order Cauchy Problems is Independent of $p$


Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_1_147_0>

L’utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l’utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

*Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI*  
http://www.bdim.eu/
\textbf{L^p Maximal Regularity for Second Order Cauchy Problems is Independent of } p \\

\textit{Ralph Chill - Sachi Srivastava (*)}

\textbf{Sunto.} – Si prova che se il problema del secondo ordine $\ddot{u} + Bu + Au = f$ ha regolarità massimale $L^p$ per qualche $p \in (1, \infty)$ allora ha regolarità massimale $L^p$ per ogni $p \in (1, \infty)$.

\textbf{Abstract.} – If the second order problem $\ddot{u} + Bu + Au = f$ has $L^p$ maximal regularity for some $p \in (1, \infty)$, then it has $L^p$-maximal regularity for every $p \in (1, \infty)$.

\section{Introduction.}

The notion of $L^p$ maximal regularity for the abstract linear second order problem

\begin{equation}
\ddot{u} + Bu + Au = f \text{ on } [0, T], \quad u(0) = \dot{u}(0) = 0,
\end{equation}

was first introduced and studied in [5]. Here $A$ and $B$ are two closed linear operators on a Banach space $\mathcal{X}$, with dense domains $\mathcal{D}_A$ and $\mathcal{D}_B$, respectively. We say that the problem (1) has $L^p$-maximal regularity, if for every $f \in L^p(0, T; \mathcal{X})$ there exists a unique (\textit{strong}) solution

$$u \in MR_{p,T} := \{v \in W^{2,p}(0, T; \mathcal{X}) \cap L^p(0, T; \mathcal{D}_A) : \dot{v} \in L^p(0, T; \mathcal{D}_B)\}$$

of the inhomogeneous problem (1). Strong solution means that $u(0) = \dot{u}(0) = 0$ and the differential equation (1) is satisfied almost everywhere. The space $MR_{p,T}$ is called \textit{maximal regularity space}.

The definition of $L^p$ maximal regularity is similar to that of $L^p$ maximal regularity for the first order problem $\dot{u} + Au = f$, and is closely related to the abstract notion of maximal regularity studied first by Da Prato and Grisvard [6],

(*) The second named author thanks the Paul Verlaine University of Metz for local hospitality during her visit to Metz when this work was started.
and then also by Acquistapace and Terreni [1], Dore and Venni [9], and Labbas and Terreni [11].

In this note, which may be considered as a sequel to [5], we show that $L^p$ maximal regularity for the second order problem is independent of the choice of $p$, $1 < p < \infty$. For the examples studied in [5], $L^p$ maximal regularity was known to hold for all $p \in (1, \infty)$, but at the end of this note we describe another example for which only $L^2$ maximal regularity was known. The fact that $L^p$ maximal regularity is independent of $p$ is well known for the first order problem; see for example, De Simon [8] in the case of Hilbert spaces, Sobolevskii [12], Cannarsa and Vespri [4], Hieber [10] in the general case.

2. – Initial value problem.

We first prove that $L^p$-maximal regularity of (1) implies existence and uniqueness of strong solutions of the initial value Cauchy problem

$$
\ddot{u} + B\dot{u} + Au = 0 \text{ on } [0, T], \quad u(0) = u_0, \ \dot{u}(0) = u_1,
$$

if the pair of initial values belongs to the trace space $\mathcal{Tr}_p$, associated with $\mathcal{X}, A$ and $B$, defined as

$$
\mathcal{Tr}_p := \{(u(0), \dot{u}(0)) : u \in MR_{p,1}\}.
$$

Subsequently, two consequences of this result are recalled.

The existence and uniqueness theorem was proven in [5] under the additional assumption $\mathcal{D}_A \hookrightarrow \mathcal{D}_B$. We present a proof which does not require this assumption.

**Theorem 2.1. – (Initial value problem).** Let $p \in (1, \infty)$. Suppose that (1) has $L^p$ maximal regularity for some $T > 0$. Then, for every $(u_0, u_1) \in \mathcal{Tr}_p$ and every $T > 0$, there exists a unique solution $u \in MR_{p,T}$ of the initial value problem (2).

**Proof.** – Suppose that (1) has $L^p$ maximal regularity for $T > 0$. Let $(u_0, u_1) \in \mathcal{Tr}_p$ be given. By [5, Lemma 6.3 (i), (ii)], for every $t \in [0, T]$,

$$
\{(u(t), \dot{u}(t)) : u \in MR_{p,T}\} = \mathcal{Tr}_p.
$$

Hence, there exists $v \in MR_{p,T}$ such that $u_0 = v(0)$ while $u_1 = \dot{v}(0)$. Let $f := \ddot{v} + B\dot{v} + Av \in L^p(0, T; \mathcal{X})$. By $L^p$ maximal regularity, there exists a solution $w \in MR_{p,T}$ of (1) with $f$ as chosen above. Then $w(t) = 0 = \dot{w}(0)$, Let $u := v - w \in MR_{p,T}$. Clearly $u$ is then a solution of (2) with initial values $u(0) = u_0$ and $\dot{u}(0) = u_1$. Uniqueness of this solution on $[0, T]$ follows from linearity and unique solvability of the problem (1).

This solution of (2) can be extended to a solution on $[0, 2T]$. Indeed, by the
same argument as above, for \((u(T), \dot{u}(T)) \in Tr_p\), there exists a unique solution
\(z \in MR_{p,T}\) of the initial value problem (2) satisfying \(z(0) = u(T), \dot{z}(0) = \dot{u}(T)\).
Then setting \(\ddot{u}(t) = u(t)\) if \(0 \leq t \leq T\) and equal to \(z(t - T)\) if \(T \leq t \leq 2T\), we obtain a solution of (2) in \(MR_{p,2T}\). Iterating this argument we see that the solution \(u\)
can be extended or restricted to a solution in \(MR_{p,T'}\) for any \(T' > 0\).

In order to show uniqueness on \([0, T')\) for any \(T' > 0\), let \(u\) and \(v\) be two solutions of (2) on \([0, T')\). Extending both solutions, if necessary, we can assume that \(T' = kT\) for some \(k \in \mathbb{N}\). Then it suffices to note that \(u = v\) on \([0, T]\) by uniqueness on the intervall \([0, T]\), and iterate this argument. □

The same arguments as in the proof of [5, Corollary 2.4] can now be used to show that if the problem (1) has \(L^p\) maximal regularity for some \(T > 0\) then it has \(L^p\) maximal regularity for all \(T > 0\). This strengthens the earlier result as the assumption that \(\mathcal{D}_A \hookrightarrow \mathcal{D}_B\) is no longer required. We record this statement formally as

**Corollary 2.2 [Independence of \(T > 0\)].** Let \(p \in (1, \infty)\). Suppose that the problem (1) has \(L^p\) maximal regularity on \([0, T]\) for some \(0 < T < \infty\). Then the problem (1) has \(L^p\) maximal regularity on \([0, T]\) for every \(T, 0 < T < \infty\).

Recall from [5, Lemma 6.1] that the trace space \(Tr_p\) is the product \(V_0 \times V_1\) of two Banach spaces which continuously embed into \(\mathcal{X}\). Hence, Theorem 2.1 implies that for every \(x \in V_1\) and every \(T > 0\) the initial value problem
\[
\ddot{u} + B\dot{u} + Au = 0 \text{ on } [0, T], \quad u(0) = 0, \quad \dot{u}(0) = x,
\]
admits a unique solution \(u \in MR_{p,T}\). Setting \(S(t)x := u(t)\), where \(u\) is this solution of the preceeding problem, we thus obtain a solution family \((S(t))_{t \geq 0}\) of operators in \(\mathcal{L}(V_1, V_0)\) which plays the role of a sine family. In a similar way one could define the cosine family associated with (2), but unlike in [2] (where \(B = 0\)), the sine family is in general not the primitive of the cosine family.

As in [5, Proposition 2.2], one can show that the \(S(t)\) extend to operators in \(\mathcal{L}(\mathcal{X})\), and that for every \(f \in L^p(0, T; \mathcal{X})\) the convolution \(S \ast f\) is the unique solution of the inhomogeneous problem (1). Infact, the following is true.

**Corollary 2.3 [Sine family and inhomogeneous problem].** Suppose that (1) has \(L^p\) maximal regularity and define the sine family \((S(t))\) as above. Then
\[
S \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X})) \cap C^\infty((0, \infty); \mathcal{L}(\mathcal{X}; \mathcal{D}_A \cap \mathcal{D}_B))
\]
and for every \(f \in L^p(0, T; \mathcal{X})\) the solution \(u\) of the inhomogeneous problem (1) is given by
\[
u(t) = (S \ast f)(t) := \int_{\mathbb{R}_+} S(t-s)f(s)ds.
\]
3. – Regularity of solutions.

The following result exhibits the regularity of the solutions of the initial value problem (2) by invoking the idea used in the proof of [5, Proposition 2.2].

**Theorem 3.1.** – Let \( p \in (1, \infty) \) and assume that (1) has \( L^p \) maximal regularity. Let \((u_0, u_1) \in Tr_p, T > 0, \) and let \( u \in MR_{p,T} \) be the unique solution of the initial value problem (2). Then \( u \in C^\infty((0, T); \mathcal{D}_A) \) and \( \dot{u} \in C^\infty((0, T); \mathcal{D}_B). \) Moreover, if for \( k \in \mathbb{N} \) one defines

\[
u_k(t) := t^k u^{(k)}(t), \quad t \in [0, T],
\]

then \( u_k \in MR_{p,T}. \)

**Remark 3.2.** – The proof below will actually show that the solution extends to an analytic function in a sector around the positive real axis.

**Proof.** – By Theorem 2.1 we know that the solution extends to a solution on \( \mathbb{R}_+. \) We consider the operator

\[
G : (-1, 1) \times MR_{p,T} \rightarrow L^p(0, T; \mathcal{X}) \times Tr_p,
\]

\[
(\lambda, \nu) \mapsto (\dot{\nu} + (1 + \lambda)B\dot{\nu} + (1 + \lambda)^2 A\nu, (v(0) - u_0, \dot{v}(0) - u_1)).
\]

The operator \( G \) is clearly analytic (see [13] for the definition of an analytic function between two Banach spaces). For \( \lambda \in (-1, 1) \) we put \( u^\lambda(t) := u(t + \lambda t). \) Then \( u^\lambda \in MR_{p,T}, u^0 = u, \) and \( G(\lambda, u^\lambda) = 0. \) Moreover, the partial derivative

\[
\frac{\partial G}{\partial \nu}(0, u) : MR_{p,T} \rightarrow L^p(0, T; \mathcal{X}) \times Tr_p,
\]

\[
v \mapsto (\dot{v} + B\dot{v} + A\nu, v(0), \dot{v}(0))
\]

is boundedly invertible by \( L^p \)-maximal regularity and Theorem 2.1. Hence, by the implicit function theorem, [13, Theorem 4.5, p. 150], there exists \( \varepsilon > 0, \) a neighbourhood \( U \) of \( u \) in \( MR_{p,T}, \) and an analytic function \( g : (-\varepsilon, \varepsilon) \rightarrow U \) such that

\[
\{ (\lambda, v) \in (-\varepsilon, \varepsilon) \times U : G(\lambda, v) = 0 \} = \{ (\lambda, g(\lambda)) : \lambda \in (-\varepsilon, \varepsilon) \}.
\]

From this we obtain \( g(\lambda) = u^\lambda, \) that is, the function \( \lambda \rightarrow u^\lambda \) is analytic in \( (-\varepsilon, \varepsilon). \)

In particular, the derivatives \( \frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0} \) exist in \( MR_{p,T} \) for every \( k \in \mathbb{N}. \) One easily checks that \( \frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0} \) coincides with the function \( u_k \) defined in the statement, so that one part of the claim is proved. The regularity of \( u \) and \( \dot{u} \) is an easy consequence of this part. \( \square \)
Our next Lemma is of a technical nature. We define, for $0 < T < \infty$, $k \in \mathbb{N}_0$, and $1 < p < \infty$, the spaces $D^{k,p}(0, T; \mathcal{X})$ as

$$D^{k,p}(0, T; \mathcal{X}) := \left\{ f \in W^{k,p}_{\text{loc}}((0, T); \mathcal{X}) : f_j \in L^p(0, T; \mathcal{X}) \text{ for every } 0 \leq j \leq k, \right.$$  

where $f_j(t) = t^j f^{(j)}(t)$.

Equipped with the norm $\| : \|_{D^{k,p}}$ given by

$$\| f \|_{D^{k,p}} := \sum_{j=0}^{k} \| f_j \|_{L^p}$$

these spaces are Banach spaces.

**Lemma 3.3.** Suppose the problem (1) has $L^p$ maximal regularity for some $p \in (1, \infty)$. Then for every $k \in \mathbb{N}_0$ the map $\psi_{k,T} : D^{k,p}(0, 2T; \mathcal{X}) \to MR_{p,T}$ given by $(\psi_{k,T} f)(t) = t^k u^{(k)}(t)$, where $u$ is the unique solution of (1) corresponding to $f$, is bounded. Moreover, there is a constant $C_k > 0$ such that $\| \psi_{k,T} \| \leq C_k$ for all $0 < T \leq 1$.

**Proof.** In the following, we will write, for convenience $\psi_{k,T} = \psi$, and only specify the indices when there is a chance of confusion. Let $f \in D^{k,p}(0, 2T; \mathcal{X})$ and let $u$ denote the unique solution of (1) for this $f$. Let $MR_{p,T}^0$ be the subspace of $MR_{p,T}$ given by

$$MR_{p,T}^0 = \{ v \in MR_{p,T} : v(0) = v'(0) = 0 \}.$$  

By considering the $C^k$ map

$$G : (-1, 1) \times MR_{p,T}^0 \to L^p(0, T; \mathcal{X})$$

$$G(\lambda, v)(t) = \dot{v}(t) + (1 + \lambda)B \dot{v}(t) + (1 + \lambda)^2 A v(t) - (1 + \lambda)^2 f((1 + \lambda)t)$$

and following the same strategy as in the proof of Theorem 3.1 we get an $\varepsilon > 0$, such that the function $\lambda \to u^{\lambda}$ is $C^k$ from $(-\varepsilon, \varepsilon)$ into $MR_{p,T}^0$; $u^{\lambda}(t) = u(t + \lambda t)$. In particular, the derivatives $\frac{d^j}{d\lambda^j} u^{\lambda} |_{\lambda=0}$ exist in $MR_{p,T}$ for every $j \in \{0, 1, \ldots, k\}$. But $\frac{d^j}{d\lambda^j} u^{\lambda} |_{\lambda=0}(t) = t^j \dot{u}^{(j)}(t)$. Therefore $\psi$ maps $D^{k,p}(0, 2T; \mathcal{X})$ into $MR_{p,T}$. Recall here that the operator that maps $f \in L^p(0, T; \mathcal{X})$ to the unique solution $u \in MR_{p,T}$ of problem (1) is bounded. It is straightforward to check then that $\psi$ is closed. By the closed graph theorem it follows that $\psi$ is bounded.

Let $E$ be a bounded operator that maps any $f \in D^{k,p}(0, 2; \mathcal{X})$ to an extension $Ef \in D^{k,p}(0, 4; \mathcal{X})$ in such a way that $(Ef)(t) = 0$ for $t \in (3, 4)$. 

For $a > 0$ and $\tau > 0$, let $D_a : D^{k,p}(0, \tau; \mathcal{X}) \to D^{k,p}(0, \frac{\tau}{a}; \mathcal{X})$ be the dilation given by $(D_a f)(t) = f(at)$. Then, for every $\tau > 0$,

$$\|D_a f\|_{D^{k,p}(0, \frac{\tau}{a}; \mathcal{X})} = \frac{1}{a} \|f\|_{D^{k,p}(0, \tau; \mathcal{X})}.$$

Let $0 < T \leq 1$. Then for any $f \in D^{k,p}(0, 2T; \mathcal{X})$ and $t \in [0, 2]$ set

$$(E_T f)(t) := \begin{cases} (D_{\frac{T}{2}} \circ E \circ D_T f)(t) & \text{for } t \in [0, 4T] \cap [0, 2], \\
0 & \text{otherwise.} \end{cases}$$

Then $E_T$ is bounded extension operator from $D^{k,p}(0, 2T; \mathcal{X})$ into $D^{k,p}(0, 2; \mathcal{X})$, and we have, on using (4),

$$\|E_T f\|_{D^{k,p}(0, 2; \mathcal{X})} \leq \|D_{\frac{T}{2}} \circ E \circ D_T f\|_{D^{k,p}(0, 4T; \mathcal{X})}$$

$$= T \|E \circ D_T f\|_{D^{k,p}(0, 4; \mathcal{X})}$$

$$\leq T \|E\| \|D_T f\|_{D^{k,p}(0, 2; \mathcal{X})}$$

$$= \|E\| \|f\|_{D^{k,p}(0, 2T; \mathcal{X})}.$$ 

Hence, for all $T \in (0, 1]$,

$$\|\psi_{k,T} f\|_{MR_{p,T}} \leq \|\psi_{k,1} (E_T f)\|_{MR_{p,1}}$$

$$\leq \|\psi_{k,1}\| \|E_T f\|_{D^{k,p}(0, 2; \mathcal{X})}$$

$$\leq \|\psi_{k,1}\| \|E\| \|f\|_{D^{k,p}(0, 2T; \mathcal{X})}.$$ 

Therefore $\|\psi_{k,T}\| \leq C_k := \|\psi_{k,1}\| \|E\|$ for all $T$, $0 < T \leq 1$. 

4. $p$ independence.

In this section we will establish that if the problem (1) has $L^p$ maximal regularity for some $p \in (1, \infty)$, then it has $L^p$ maximal regularity for all $p \in (1, \infty)$. The scheme followed will be similar to that in [10], where the corresponding result for the first order problem is proven. We will make use of the following theorem, which is a vector valued version of a result due to Benedek, Calderón and Panzone [3].

**Theorem 4.1 ([10], Theorem 4.3).** Suppose that $T$ is a bounded operator on $L^p(R; \mathcal{X})$ for some $p \in (1, \infty)$ and is represented by

$$(5) \quad Tf(t) := \int_{R} K(t - s)f(s)ds$$
for \( f \in L^\infty(\mathbb{R}; \mathcal{X}) \) with compact support and \( t \notin \text{supp} \, f \), and the kernel \( K \in L^1_{\text{loc}}(\mathbb{R} \setminus \{0\}, \mathcal{X}) \) satisfies

\[
\int_{|t| > c|s|} \|K(t - s) - K(t)\| \, dt \leq C \quad \text{for every } s \neq 0 \text{ and some } c > 1.
\]

Then \( T \) admits a bounded extension to \( L^r(\mathbb{R}, \mathcal{X}) \) for every \( r \in (1, \infty) \), that is, there exists a constant \( \tilde{C}_r \) such that

\[
\|Tf\|_{L^r(\mathbb{R}, \mathcal{X})} \leq \tilde{C}_r \|f\|_{L^r(\mathbb{R}, \mathcal{X})}.
\]

We now state our main result concerning \( L^p \) maximal regularity for problem (1) and the choice of \( p \in (1, \infty) \).

**Theorem 4.2.** – Suppose that for some \( p \in (1, \infty) \) the problem (1) has \( L^p \) maximal regularity. Then (1) has \( L^p \) maximal regularity for every \( p \in (1, \infty) \).

**Proof.** – Fix \( p \in (1, \infty) \). Suppose that problem (1) has \( L^p \) maximal regularity and let \( (S(t)) \) be the sine family from Corollary 2.3. By \( L^p \) maximal regularity and Corollary 2.3, the convolution operator \( f \mapsto S * f \) is a bounded linear operator from \( L^p(0, T; \mathcal{X}) \) into \( MR_{p,T} \).

It is enough for our purposes to show that the convolution operator \( f \mapsto S * f \) extends to a bounded linear operator from \( L^r \) to \( MR_r \) for every \( r \in (1, \infty) \).

Fix \( x \in X \) and \( 1 \geq T > 0 \). Set \( f(s) = x, \ s \geq 0 \). Then \( f \in D^{2,p}(0, 2T, \mathcal{X}) \) for all \( k \in \mathbb{N}_0 \). Therefore, applying Lemma 3.3 to \( f \) and the corresponding unique solution \( u := S * f \) of the problem (1), with \( k = 2 \) we have that \( t \mapsto t^2 u^{(2)}(t) \in MR_{p,T} \)

\[
\|\psi_{2,T}f\|_{MR_{p,T}} = \|t^2 u^{(2)}(t)\|_{MR_{p,T}} \\
\leq C_2 \|f\|_{D^{2,p}(0,2T,\mathcal{X})} \\
= C_2 (2T)^{\frac{3}{2}} \|x\|,
\]

where \( C_2 \) is the constant independent of \( T \) obtained in Lemma 3.3. Noting that 

\[
u(t) = \int_0^t S(t - s)f(s)ds = \int_0^t S(s)xds,
\]

we therefore obtain from the above inequality

\[
\|t^2 A^\ast \hat{S}(t)x\|_{L^p(0,T;\mathcal{X})} \leq C_2 (2T)^{\frac{3}{2}} \|x\|.
\]

Define the operator-valued kernel \( K \) as

\[
K(t) = \begin{cases} 
AS(t) & \text{if } t \in (0, T), \\
0 & \text{otherwise}.
\end{cases}
\]
Due to the $L^p$ maximal regularity of (1), the convolution operator $T_A$ given by 

$$T_A f(t) = \int K(t - s) f(s) \, ds$$

extends to a bounded linear operator on $L^p(0, T; \mathcal{X})$. Thus there is a constant $\tilde{C}_p$ such that for all $f \in L^p(\mathbb{R}, \mathcal{X})$,

$$\| T_A f \|_{L^p(0, T; \mathcal{X})} \leq \tilde{C}_p \| f \|_{L^p(0, T; \mathcal{X})}.$$

We claim that the kernel $K$ further satisfies condition (6) for some $c > 1$. Indeed, for $T \geq t > s > 0$, and $x \in \mathcal{X}$, we have, on using (7) and Hölder’s inequality

$$\| K(t - s)x - K(t)x \| = \| AS(t - s)x - AS(t)x \|$$

$$= \left\| \int_{t-s}^t A\tilde{S}(r)x \, dr \right\|$$

$$\leq \left\| \int_{t-s}^t r^{-2}r^2 A\tilde{S}(r)x \, dr \right\|$$

$$\leq \left( \int_{t-s}^t r^{-2q} \, dr \right)^{\frac{1}{q}} \left( \frac{1}{q} \int_0^t \| r^2 A\tilde{S}(r)x \|_p^p \, dr \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{t-s}^t r^{-2q} \, dr \right)^{\frac{1}{q}} C_2 (2t)^{\frac{1}{q}} \| x \|.$$

Therefore, for $c > 1$, and $s > 0$ we have

$$\int_{t > cs} \| K(t - s) - K(t) \| \, dt \leq C \int_{t > cs} (2t)^{\frac{1}{q}(1-2q)} \left| t^{(1-2q)} - (t - s)^{(1-2q)} \right|^{\frac{1}{p}} \, dt$$

$$\leq C \int_{t > cs} (2t)^{\frac{1}{q}(1-2)} \left| 1 - (1 - st^{-1})^{(1-2q)} \right|^{\frac{1}{p}} \, dt$$

$$\leq \tilde{C} \int_{t > cs} t^{\frac{1}{q} - 2} \, dt$$

$$\leq C' s^{\frac{1}{q}} \int_{T > t > cs} t^{(1-2)} \, dt$$

$$\leq M,$$

where $\tilde{C}, C'$, and $M$ are constants depending only on $\tilde{C}_p$, $q$, $c$ and $T$.

Therefore, from Theorem 4.1 it follows that $T_A$ is a bounded operator from $L^r(0, T; \mathcal{X})$ to $L^r(0, T; \mathcal{X})$, for every $r \in (1, \infty)$. Thus, for each $r \in (1, \infty)$ there exists a constant $C_r$ such that for all $f \in L^r(0, T; \mathcal{X})$,

$$\| AS * f \|_{L^r(0, T; \mathcal{X})} = \| T_A f \|_{L^r(0, T; \mathcal{X})} \leq C_r \| f \|_{L^r(0, T; \mathcal{X})}.$$

(8)
Now define another operator valued kernel $K_1$ as follows.

$$K_1(t) = \begin{cases} B\hat{S}(t) & \text{if } t \in (0, T), \\ 0 & \text{otherwise.} \end{cases}$$

Let $T_B$ be the operator given by $T_B f(t) := K_1 * f$. Due to $L^p$ maximal regularity, $T_B$ is a bounded operator on $L^p(0, T; \mathcal{X})$. Applying Lemma 3.3 to the constant function $f(t) = x$, $t \in (0, T)$, successively for $k = 1, 2$ we obtain for $u = S * f$,

$$\|t^2 B\hat{S}(t)x\|_{L^p(0,T;\mathcal{X})} \leq \|t^2 \hat{S}(t)x\|_{MR_{p,T}} + 2\|tS(t)x\|_{MR_{p,T}} + 2\|(S * f)(t)x\|_{MR_{p,T}}$$

$$\leq C_2\|f\|_{D^{2,p}} + 2C_1\|f\|_{D^{1,p}} + 2C_0\|f\|_{D^{0,p}}$$

$$= C(2T)^{\frac{1}{2}}\|x\|,$$

where the constant $C$ is independent of $T$ for $0 < T \leq 1$. Using this inequality and the same arguments as before, we can show that the kernel $K_1$ also satisfies (6) above. Thus, from Theorem 4.1 it follows that $f \mapsto K_1 * f$ is a bounded operator on $L^r(0, T; \mathcal{X})$ for every $r \in (1, \infty)$. Therefore, we have for every $f \in L^r(0, T; \mathcal{X})$,

$$\|B\hat{S} * f\|_{L^r(0, T; \mathcal{X})} = \|T_B f\|_{L^r(0, T; \mathcal{X})} \leq C^1_r\|f\|_{L^r(0, T; \mathcal{X})}.$$  

(9)

Since $u := S * f$ is a solution of the equation $\ddot{u}(t) + B\dot{u}(t) + A u(t) = f(t)$ a.e., it follows that $\ddot{u} \in L^r(0, T; \mathcal{X})$ and there exists a constant $C_r$ such that $\|u\|_{W^{2,r}} \leq C_r\|f\|$. This statement, together with (8) and (9) implies that the problem (1) has $L^r$ maximal regularity for every $r \in (1, \infty)$.

There are, as of now, only a few results that ensure that the problem (1) has $L^p$ maximal regularity for some $p \in (1, \infty)$. The main ingredient for showing $L^p$ maximal regularity in [5] is a characterization of $L^p$ maximal regularity in terms of Fourier multipliers. In a particular model problem $L^p$ maximal regularity was shown by using the Mikhlin-Weis Fourier multiplier theorem [5, Theorem 4.1]. This then implied $L^p$ maximal regularity for every $p \in (1, \infty)$, so that in the examples considered in [5], $L^p$ maximal regularity is independent of $p \in (1, \infty)$.

However, consider the following variational setting, not covered by the results in [5]. Let $V$ and $H$ be two separable Hilbert spaces such that $V$ embeds densely and continuously into $H$. We identify $H$ with its dual $H'$ so that $H$ is also densely and continuously embedded into $V'$.

**Corollary 4.3.** – Let $A$ and $B$ be two linear, maximal monotone, symmetric, not necessarily commuting operators from $V$ to $V'$. Then for every $p \in (1, \infty)$ and every $f \in L^p(0, T; V')$, every $u_0 \in V$ and every $u_1 \in (V', V)_{\frac{1}{p}, p}$ there exists a unique solution

$$u \in W^{2,p}(0, T; V') \cap W^{1,p}(0, T; V)$$

of the problem
\begin{equation}
\ddot{u} + B\dot{u} + Au = f \text{ on } [0, T], \quad u(0) = u_0, \ \dot{u}(0) = u_1.
\end{equation}

In other words, the above problem has $L^p$ maximal regularity for every $p \in (1, \infty)$.

PROOF. – By a result of J.-L. Lions [7, Théorème 1, p. 670] the problem (10) has $L^2$ maximal regularity in $V'$. By Theorem 4.2, the problem (10) has $L^p$ maximal regularity for every $p \in (1, \infty)$. Solvability of the initial value problem follows from Theorem 2.1. The fact that the associated trace space equals $V \times (V', V)_{rac{p}{p-1}}$ follows from [5, Lemma 6.2].

The fact that in the variational setting above the problem (1) has $L^2$ maximal regularity was proved in [7] by the Faedo-Galerkin method and a priori estimates. The proof thus heavily depends on the Hilbert space setting and it does not imply $L^p$ maximal regularity for $p$ different from 2. We point out that the conditions on $A$ and $B$ can be considerably relaxed; for the precise assumptions, see [7].

REFERENCES

[10] M. Hieber, Operator valued Fourier multipliers, Topics in nonlinear analysis. The Herbert Amann anniversary volume (J. Escher, G. Simonett, eds.), Progress in


Ralph Chill: Université Paul Verlaine - Metz, Laboratoire de Mathématiques et Applications de Metz - CNRS UMR 7122 Bât. A, Ile du Saulcy, 57045 Metz Cedex 1, France
E-mail: chill@univ-metz.fr

Sachi Srivastava: Department of Mathematics, Indian Institute of Science, Bangalore, India
E-mail: sachi_srivastava@math.iisc.ernet.in

Received July 2, 2007 and in revised form September 17, 2007