Bollettino Unione Matematica Italiana

Nunzia Gavitone

Hardy-Sobolev Inequalities for Hessian Integrals

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 951–967.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_951_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 951-967

Hardy-Sobolev Inequalities for Hessian Integrals

NUNZIA GAVITONE

Sunto. – *Usando appropriate tecniche di simmetrizzazione, si provano disuguaglianze di tipo Hardy-Sobolev per integrali Hessiani che estendono quelle classiche, ben note per le funzioni di Sobolev. Per tali disuguaglianze viene dato il valore della costante ottimale. Infine si stabilisce un miglioramento delle suddette disuguaglianze con l'aggiunta di un secondo termine che presenta un peso singolare dato da un'opportuna potenza negativa della funzione* $\log(|x|)$.

Summary. – *Using appropriate symmetrization arguments, we prove the Hardy-Sobolev type inequalities for Hessian Integrals which extend the classical results, well known for Sobolev functions. For such inequalities the value of the best constant is given. Finally we give an improvement of these inequalities by adding a second term that, involves another singular weight which is a suitable negative power of* $\log(|x|)$.

1. – Introduction.

Let $n \geq 2$ and let Ω be an open and bounded set of $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$ the k-Hessian operator, for $k = 1, \ldots, n$, is defined by

$$F_k\big(D^2u\big) = \big[D^2u\big]_k = S_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq i_n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.1}$$

where D^2u denotes the Hessian matrix of u and λ_i is the i^{th} eigenvalue of D^2u. Well known examples of k-Hessian operators are $F_1 = \Delta u$ and $F_n = \det D^2u$ (Monge-Ampère operator). If for any $n \times n$ matrix $A = [a_{ij}]$ we put $F_k^{ij}(A) = \dfrac{\partial}{\partial a_{ij}} F_k[A]$, it is possible to define the so-called (p, k)-Hessian Integrals for $u \in C^2(\Omega)$, $p \geq 1$,

$$I_{p,k}[u, \Omega] = \int_\Omega F_k^{ij}(D^2u) D_i u D_j u |Du|^{p-k-1}\, dx \tag{1.2}$$

Such integrals are well known generalizations of energy integrals. For example, if $u = 0$ on $\partial\Omega$, we have for $p \geq 1$ and $k = 1$,

$$I_{p,1}[u,\Omega] = \int_\Omega |Du|^p \, dx \tag{1.3}$$

In analogy with the classical results obtained for Sobolev functions, in [15] and [20], under additional convexity assumptions on Ω and u, suitable Sobolev inequalities in the form

$$I_{p,k}[u;\Omega] \geq C\|u\|^p_{L^q(\Omega)} \tag{1.4}$$

have been proved. For such inequalities the value of the best constant C is available. In the present paper we investigate the question of finding an inequality in the form (1.4) when the norm of u is substituted by a norm with a weight which is a suitable negative power of $|x|$. For such inequalities, sometimes known as Hardy-Sobolev inequalities, we also give the best value of the constant. The inequalities are established for a particular class of functions $A_k(\Omega)$, with $n \geq k > 0$, called k-convex functions. This class is defined as the set of functions $u(x) \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, with Ω k-convex (see Section 2), such that $u = 0$ on $\partial\Omega$ and that verify the following properties:

$$\left[D\frac{Du}{|Du|}\right]_l \geq 0, \ \forall l = 1,\dots,k, \text{ and for } Du \neq 0 \tag{1.5}$$

$$[D^2u]_k \geq g|Du|^k \text{ for some } g \leq 0, \ g \in L^1(\Omega). \tag{1.6}$$

A consequence of (1.5) is that the functions of this class are non positive functions, (see [15]).

A typical inequality for $u \in A_{k-1}(\Omega)$, with Ω $(k-1)$-convex, (see Theorem 1) reads as:

$$I_{p,k}[u,\Omega] \geq C\left[\int_\Omega \frac{|u|^q}{|x|^s}\, dx\right]^{\frac{p}{q}} \tag{1.7}$$

where $1 < p < n-k+1, \quad 0 \leq s < p+k-1, \quad q \leq \dfrac{p(n-s)}{n-k-p+1}$.

Various improvements and generalizations are given. Here we only quote the fact that also the limit case, $s = p+k-1$, which for Sobolev functions corresponds to Hardy inequality, will be considered and the question of obtaining an improved inequality with an additional term on the right-hand side, is addressed (see Theorem 3). Similar results concerning improvements of Hardy inequality can be found for instance in [4] and [6] for Sobolev functions. Our results will be established using appropriate symmetrization arguments, in particular the symmetrization for quermassintegral, defined in [15]. In fact, we will see that using the properties of this symmetrization process, we can obtain our results proving them only for the symmetrand of the functions of $A_k(\Omega)$.

This paper is organized as follows. In Section 2, we introduce the notations that we use in the paper, we define the symmetrization for quermassintegral and we address the principal properties of this symmetrization which we need to prove our results. In Section 3 we prove the Hardy-Sobolev type inequalities for (p,k)-Hessian Integrals. In particular, we prove that these inequalities are sharp defining a suitable minimizing sequence of functions in the class $A_k(\Omega)$. In the last section, we give an improvement of Hardy inequality, established in Section 3, by adding a second term in the right-hand side that involves another singular weight which is a suitable negative power of $\log(|x|)$.

2. – Notations and Preliminaries.

We begin with an appropriate definition of quermassintegral for non-convex domains (see [15]). Let Ω be an open, bounded set of $\mathbb{R}^n$ with boundary $\partial\Omega \in C^2$, having principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ (oriented so that convex domains have non-negative curvatures).

For $k = 1, \dots, n-1$ we define the k^{th} mean curvature of $\partial\Omega$ by

$$H_k[\partial\Omega] = S_k(\kappa_1, \dots, \kappa_{n-1}), \tag{2.1}$$

while for $k = 0$ we assume

$$H_0 = 1. \tag{2.2}$$

For $k = 0, \dots, n-1$ the quermassintegral $V_k(\Omega)$ is defined by

$$V_k(\Omega) = \frac{1}{n\binom{n-1}{k}} \int_{\partial\Omega} H_{n-k-1}(\partial\Omega) d\mathcal{H}^{n-1}, \tag{2.3}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, and for $k = n$ we assume

$$V_n(\Omega) = |\Omega|. \tag{2.4}$$

When $k \leq n-2$ we restrict our attention to quermassintegral on domains Ω that are $(n-k-1)$-convex, i.e.

$$H_j(\partial\Omega) \geq 0, \; j = 1, \dots, n-k-1. \tag{2.5}$$

When $\partial\Omega$ is connected, (2.5) is equivalent to $H_{n-k-1} \geq 0$, while for $k = 0$ we have to assume that Ω is $(n-1)$-convex, that is the components of Ω are convex in the usual sense. Whenever Ω_1, Ω_2 are convex domains and $\Omega_1 \subset \Omega_2$, clearly $V_n(\Omega_1) \leq V_n(\Omega_2)$.

Moreover $V_k(\Omega_1) \leq V_k(\Omega_2)$, for every k (see [5]). But this property is not true in general when the domains are only $(n-k-1)$-convex. We get around this difficulty by only considering functions whose sub-level sets have this monotonicity property, that are functions belonging to the class $A_k(\Omega)$ defined in the introduction. Indeed in [15] it is proved that the sub-level sets of the functions in $A_k(\Omega)$, that, as already said, are non positive functions, verify the monotonicity property $V_{n-k+1}(\Omega_s) \leq V_{n-k+1}(\Omega_t)$, where $\Omega_\tau = \{x \in \Omega : u(x) < \tau\}$ and $\min_\Omega u < s \leq t < 0$.

Let Ω be $(n-k-1)$-convex; we introduce the k^{th}-mean radius of Ω defined by

$$\begin{aligned} \xi_k(\Omega) &= \left(\frac{V_k(\Omega)}{\omega_n}\right)^{\frac{1}{k}} \quad k = 1, \ldots, n \\ \xi_k(\emptyset) &= 0, \end{aligned}$$

when ω_n denotes the measure of the n-dimensional unit ball. The following isoperimetric inequality holds true (see [16])

$$\xi_l(\Omega) \leq \xi_k(\Omega) \quad \text{for} \quad 1 \leq k \leq l \leq n. \tag{2.6}$$

We observe that (2.6) includes the classical isoperimetric inequality when $l = n$ and $k = n-1$, because $\xi_{n-1}(\Omega) = \left(\frac{\mathcal{H}^{n-1}(\Omega)}{\omega_n}\right)^{\frac{1}{n-1}}$.

Now we can define the $(k-1)$-symmetrand of a function $u \in A_{k-1}(\Omega)$, $k = 1, \ldots, n$ as follows:

$$u^*_{k-1}(x) = \sup\{t \leq 0 : \xi_{n-k+1}(\Omega_t) \leq |x|, \quad Du \neq 0 \text{ on } \Sigma_t\} \tag{2.7}$$

for $|x| \leq R = \xi_{n-k+1}(\Omega)$, where Σ_t is the t level set of u, $\Sigma_t = \{x \in \Omega : u(x) = t\}$.

The following statements hold true (see [18] and [15]):

- writing $u^*_{k-1}(|x|) = \rho(r)$ for $r = |x|$, we have $\rho(0) = \min_\Omega u$ and $\rho(R) = 0$;
- $\rho(r)$ is a negative and non-decreasing function on $[0, R]$;
- $\rho(r) \in C^{0,1}([0, R])$ and moreover $0 \leq \rho'(r) \leq \sup_\Omega |Du|$ a.e.

REMARK 1. – For $k = 1$ we have that the 0-symmetrand of u, i. e. $u^*_0(x)$, coincides with the Schwarz symmetrand of u.
Moreover for $k = n-1$ the $(n-1)$-symmetrand of u, i.e. $u^*_{n-1}(x)$, is such that $u^*_{n-1}\left(\left(\frac{x}{n\omega_n}\right)^{n-1}\right)$ coincides with the rearrangement of u with respect to the perimeter of the level set.

By definition, through (2.6), it follows $\xi_k(\Omega_t) \leq \xi_k(\{u^*_m < t\})$, for $k \geq n-m$ and equality holds when $n-m = k$. This implies

$$\|u\|_{L^p(\Omega)} \leq \left\|u^*_k\right\|_{L^p(B_R)}, \quad p \geq 1. \tag{2.8}$$

Let $u \in A_{k-1}(\Omega)$ and let $m = \min_{\Omega} u < t < 0$. We recall the following Reilly equality (see [13])

$$\int_{\Omega_t} S_k(D^2 u)\, dx = \frac{1}{k} \int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t)\, d\mathcal{H}^{n-1}. \tag{2.9}$$

Now we can define the following functional, known as (p,k)-Hessian Integrals

$$I_{p,k}[u,\Omega] = \int_m^0 dt \int_{\Sigma_t} H_{k-1}(\Sigma_t) |Du|^{p-1}\, d\mathcal{H}^{n-1}, \tag{2.10}$$

and by the following statements (see [15])

$$H_{k-1}(\Sigma_t) = \left[D \frac{Du}{|Du|} \right]_{k-1}, \quad \text{if } Du \neq 0;$$

$$F_k^{ij}(D^2 u) D_{x_i} u D_{x_j} u = |Du|^{k+1} \left[D \frac{Du}{|Du|} \right]_{k-1}, \quad \text{if } Du \neq 0,$$

we see that definition (2.10) coincides with (1.2).

In the case $p = k+1$ and $u = 0$ on $\partial\Omega$, we have

$$I_{k+1,k}[u,\Omega] = k \int_{\Omega} F_k(D^2 u) |u|\, dx, \tag{2.11}$$

and $I_{k+1,k}$ is called k- Hessian Integral.

In the radial case the (p,k)-Hessian Integrals can be written as follows

$$I_{p,k}[u^*_{k-1}, B_R] = n \binom{n-1}{k-1} \omega_n \int_0^R f^p \left(\omega_n r^{n-k+1} \right) r^{n-k}\, dr \tag{2.12}$$

where $f\left(\omega_n |x|^{n-k+1} \right) = \left| \nabla u^*_{k-1}(x) \right|$.

Finally for (p,k)-Hessian Integrals the following extension of Polya-Szegö principle holds, (see [15]),

$$I_{p,k}[u,\Omega] \geq I_{p,k}\left[u^*_{k-1}, B_R \right], \quad p \geq 1. \tag{2.13}$$

3. – The Hardy-Sobolev inequality.

In this section we prove some inequalities that extend the result in [15] and which can be seen as analogous to Hardy-Sobolev ones which are known for Sobolev functions, (see [11], [12], [8],[2]).

THEOREM 1. – *Let Ω be an open, bounded and $(k-1)$-convex set of $\mathbb{R}^n$ and let be $u \in A_{k-1}(\Omega)$, (with $k = 1, \dots, n$). If $1 < p < n-k+1$ then exists a constant C depending only from n, k, p, s, Ω, such that*

$$I_{p,k}[u,\Omega] \geq C(n,p,k,q,s,\Omega)\left[\int_{\Omega} \frac{|u|^q}{|x|^s}\,dx\right]^{\frac{p}{q}} \tag{3.1}$$

for $q \leq \dfrac{p(n-s)}{n-k-p+1}$ *and* $0 \leq s < p+k-1$. *The constant C is given by*

$$C = \binom{n-1}{k-1}\left(\frac{p+k-s-1}{p}\right)^{\frac{p}{q}-1}\frac{\pi^{-\frac{p}{2}}}{n-s}\quad\left(\frac{p-1}{n-p-k+1}\right)^{p-1}$$

$$\left[\frac{\Gamma\left(1+\dfrac{p(n-s)}{2(p+k-s-1)}\right)\Gamma\left(\dfrac{p(n-s)}{p+k-s-1}\right)}{\Gamma\left(\dfrac{n-s}{p+k-s-1}\right)\Gamma\left(1+\dfrac{(p-1)(n-s)}{p+k-s-1}\right)}\right]^{\frac{p+k-s-1}{n-s}} \tag{3.2}$$

PROOF. – Let $u \in A_{k-1}(\Omega)$, and consider the $(k-1)$-symmetrand of u, u^*_{k-1}. If we set $\overline{R} = \xi_n(\Omega)$ and $R = \xi_{n-k+1}(\Omega)$ we have:

$$\int_{\Omega}\frac{|u|^q}{|x|^s}\,dx \leq \int_{B_{\overline{R}}}\frac{|u^*_0|^q}{|x|^s}\,dx \leq \int_{B_R}\frac{|u^*_{k-1}|^q}{|x|^s}\,dx \tag{3.3}$$

where the first inequality is a consequence of Hardy-Littlewood inequality for rearrangements (see [1], [2]) and the second inequality is a consequence of (2.6). Then Polya-Szegö principle for Hessian Integrals (2.13) and (3.3) allow us to prove inequality (3.1) only for the $(k-1)$-symmetrand of u, i.e. u^*_{k-1}. Now writing

$$u^*_{k-1}(x) = u^*_{k-1}(|x|) = \rho(r), \qquad r = |x|,$$

we have, by (2.12):

$$I_{p,k}\left[u^*_{k-1}(|x|), B_R\right] = n\omega_n\binom{n-1}{k-1}\int_0^R (\rho'(r))^p r^{n-k}\,dr \tag{3.4}$$

In order to prove (3.1) we have to show that the following one-dimensional inequality holds true

$$n\omega_n\binom{n-1}{k-1}\int_0^R (\rho'(r))^p r^{n-k}\,dr \geq n\omega_n C\left[\int_0^R |\rho|^q r^{n-1-s}\,dr\right]^{\frac{p}{q}} \tag{3.5}$$

Let $s < p + k - 1$ and let us make the change of variable $t = r^{p/(p+k-s-1)}$. Then the inequality (3.5) becames

$$\int_0^{R^{p/(p+k-s-1)}} (\rho'(t))^p t^{d-1}\,dt \geq \overline{C}\left[\int_0^{R^{p/(p+k-s-1)}} |\rho(t)|^q t^{d-1}\,dt\right]^{\frac{p}{q}} \tag{3.6}$$

where

$$d = \frac{p(n-s)}{p+k-s-1} \qquad \text{and} \tag{3.7}$$

$$\overline{C} = C\binom{n-1}{k-1}^{-1}\left(\frac{p}{p+k-s-1}\right)^{\frac{p}{q}+p-1}. \tag{3.8}$$

Now, extending to zero the function ρ in $\mathbb{R}^+$ inequality (3.6), and then (3.5), follows immediately from Lemma 2 in [14]. The constant in such Lemma is sharp and a straightforward computation gives the constant in (3.2). We finally observe that after the change of variables, the inequality (3.6) can be viewed as a Sobolev embedding theorem in the fractional dimension d in (3.8) and we must require

$$p < d \qquad \text{i.e.} \qquad p < n - k + 1. \qquad \square$$

REMARK 2. – We observe that in particular cases inequality (3.1) reads as follows:

(i) If we take $s = 0$ we obtain the Sobolev inequality for (p,k)-Hessian Integrals (see [15]).

(ii) If we take $k = 1$ then we have

$$I_{p,1}[u,\Omega] = \int_\Omega |Du|^p\,dx$$

hence the inequality (3.1) becames the well known Hardy-Sobolev inequality ([11], [2], [12], [9]) and if we take also $s = 0$ we have the classical Sobolev inequality with the best constant (see [14]).

Moreover if we take only $s = 0$ and we obtain the Sobolev inequality for (p,k)-Hessian Integrals (see [15]).

For inequality (3.1), as the classical case for $s = p$, in the limit case, $s = p + k - 1$, we have the following Hardy type inequality for (p,k)-Hessian Integrals.

THEOREM 2. – *Let Ω be an open, bounded and $(k-1)$-convex set of $\mathbb{R}^n$ and let $u \in A_{k-1}(\Omega)$, $k = 1,\dots,n$. If $1 < p < n - k + 1$ the following inequality holds:*

$$I_{p,k}[u,\Omega] \geq \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx \tag{3.9}$$

Before proving the inequality (3.9), we need following Lemma about the Hardy type inequalities in one dimension that we find in [2].

LEMMA 1. – *Let ψ be a nonnegative measurable function on* $(0, \infty)$ *and suppose* $-\infty < \lambda < 1$ *and* $1 < d \leq \infty$. *Then the following inequality holds:*

$$\left\{ \int_0^\infty \left(t^{1-\lambda} \int_t^\infty \psi(s) \frac{1}{s}\, ds \right)^d \frac{1}{t}\, dt \right\}^{\frac{1}{d}} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty \left(t^{1-\lambda} \psi(t) \right)^d \frac{1}{t}\, dt \right\}^{\frac{1}{d}} \tag{3.10}$$

Now we proceed with the proof of (3.9).

PROOF OF THEOREM 2. – We will see that the inequality (3.9) is an immediate consequence of the previous Lemma. In fact we can rewrite (3.9) as one-dimensional inequality.

Let $u \in A_{k-1}(\Omega)$ and we consider $u^*_{k-1}(x)$. Using the same arguments of the proof of Theorem 1, we again can prove inequality (3.9) only for $u^*_{k-1}(x)$.

Writing

$$u^*_{k-1}(x) = u^*_{k-1}(|x|) = \rho(r),$$

we must prove the following one dimensional inequality:

$$\begin{aligned} I_{p,k}[u, B_R] &= n\omega_n \binom{n-1}{k-1} \int_0^R (\rho')^p r^{n-k}\, dr \\ &\geq \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p} \right)^p \int_0^R |\rho|^p r^{n-k-p}\, dr. \end{aligned} \tag{3.11}$$

Then taking

$$\begin{aligned} \psi(s) &= \rho'(s)s \\ \lambda &= \frac{2p+k-n-1}{p} < 1 \\ d &= p \end{aligned}$$

and extending the function ρ to zero for $r > R$, by inequality (3.10) we have (3.11). $\square$

REMARK 3. – We observe that inequality (3.1) and (3.9) are sharp.

Now we show the optimality of the constant in (3.9). A similar argument can be repeated for (3.1). For all $\varepsilon > 0$ we define the following functions:

$$u_\varepsilon(x) = u_\varepsilon(|x|) = \left(\frac{1}{R^q + \varepsilon^q} \right)^{\frac{n-k+1}{p}} - \left(\frac{1}{r^q + \varepsilon^q} \right)^{\frac{n-k+1}{p}}$$

where $q = \dfrac{n-k-p+1}{n-k+1}$, $\quad R = \xi_{n-k+1}(\Omega)$ and $r = |x|$.

We prove that

$$\lim_{\varepsilon\to 0} \frac{I_{p,k}[u_\varepsilon, \Omega]}{\displaystyle\int_\Omega \frac{|u_\varepsilon|^p}{|x|^{p+k-1}}\,dx} = \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p. \tag{3.12}$$

First we prove that $u_\varepsilon \in A_{k-1}(\Omega)$. We observe that

- $u_\varepsilon \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega}), \qquad \forall \varepsilon > 0$
- $u_\varepsilon \leq 0 \quad$ in Ω and $u_\varepsilon = 0 \quad$ on $\partial\Omega$

Computing the derivative of u_ε we have:

$$S_l\left[D\left(\frac{Du_\varepsilon}{|Du_\varepsilon|}\right)\right] = S_l\left[D\left(\frac{x_i}{r}\right)\right] = \binom{n-1}{l-1}\left(\frac{1}{r}\right)^l \geq 0 \qquad \forall l = 1, \ldots, k-1$$

hence the property (1.5) in the definition of A_{k-1} is checked. Moreover

$$\begin{aligned}
S_{k-1}[D^2 u_\varepsilon] =& \left(q \;\frac{n-k+1}{p}\; \frac{r^{q-2}}{(r^q+\varepsilon^q)^{\frac{n-k+p+1}{p}}}\right)^{k-1} \\
&\left[\binom{n-1}{k-1} + \binom{n-1}{k-2}\left(q-1-q^2\;\frac{n-k+1}{p}\,\frac{r^q}{r^q+\varepsilon^q}\right)\right] \\
=& |Du_\varepsilon|^{k-1}\frac{1}{r^{k-1}}\left[\binom{n-1}{k-1} + \binom{n-1}{k-2}\left(q-1-q^2\;\frac{n-k+1}{p}\,\frac{r^q}{r^q+\varepsilon^q}\right)\right] \\
\geq& |Du_\varepsilon|^{k-1} g(|x|)
\end{aligned}$$

where

$$g(|x|) = C(n,k,p)\frac{1}{r^{k-1}}$$

and $C(n,k,p) = -\dbinom{n-1}{k-2}\left(1 + q^2\;\dfrac{n-k+1}{p}\right)$.

We note that $g \in L^1(\Omega)$ since $k \leq n$, hence also the property (1.6) is checked and we obtain that $u_\varepsilon \in A_{k-1}(\Omega)$ and the claim is proven.

Now we consider the limit (3.12). We have:

$$\begin{aligned}
I_{p,k}[u_\varepsilon, B_R] =& n\omega_n\binom{n-1}{k-1}\int_0^R (u'_\varepsilon)^p r^{n-k}\,dr \\
=& n\omega_n\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^{R/\varepsilon}\left(\frac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1}\right)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\,dr
\end{aligned}$$

On the other hand we have:

$$\int_\Omega \frac{|u_\varepsilon|^p}{|x|^{p+k-1}}\,dx = n\omega_n \int_0^{R/\varepsilon} \left[\left(\frac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1} \right)^{\frac{n-k+1}{p}} - \left(\frac{1}{\left(\frac{R}{\varepsilon}\right)^{\frac{n-k-p+1}{n-k+1}}+1} \right)^{\frac{n-k+1}{p}} \right]^p r^{n-k-p}\,dr. \tag{3.13}$$

Setting $\psi(r) = \dfrac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1}$, the limit (3.12) can be written as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{I_{p,k}[u_\varepsilon,\Omega]}{\displaystyle\int_\Omega \frac{|u_\varepsilon|^p}{|x|^{p+k-1}}\,dx} \\
&= \lim_{s\to\infty} \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\displaystyle\int_0^s \left[\psi(r)^{\frac{n-k+1}{p}} - \psi(s)^{\frac{n-k+1}{p}}\right]^p r^{n-k-p}\,dr} \\
&= \lim_{s\to\infty} \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\psi(s)^{n-k+1}\displaystyle\int_0^s \left[\left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{n-k+1}{p}} -1\right]^p r^{n-k-p}dr}
\end{aligned} \tag{3.14}$$

Now we set

$$\frac{1}{\sigma} = \frac{\psi(r)}{\psi(s)}$$

$$r = \left[\frac{\sigma}{\psi(s)} - 1\right]^{\frac{n-k+1}{n-k-p+1}}$$

Hence the integral in the denominator of (3.14) becomes:

$$\begin{aligned}
&\psi(s)^{n-k+1} \int_0^s \left[\left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{n-k+1}{p}} -1\right]^p r^{n-k-p}\,dr \\
&= \left(\frac{n-k+1}{n-k-p+1}\right) \psi(s)^{n-k} \int_{\psi(s)}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}} -1\right]^p \left(\frac{\sigma}{\psi(s)} - 1\right)^{n-k} d\sigma \\
&= \left(\frac{n-k+1}{n-k-p+1}\right) \sum_{j=0}^{n-k} (-1)^{n-k-j} \psi(s)^{n-k-j} \binom{n-k}{j} \int_{\psi(s)}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}} -1\right]^p \sigma^j\,d\sigma.
\end{aligned} \tag{3.15}$$

Now we observe that:

$$\lim_{s\to\infty}\psi(s)^{n-k-j}\int_{\psi(s)}^{1}\left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^{p}\sigma^{j}\,d\sigma=\frac{1}{n-k-j}\qquad\text{if } j\neq n-k, \tag{3.16}$$

and

$$\lim_{s\to\infty}\frac{\displaystyle\int_{1/t_s}^{1}\left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^{p}\sigma^{n-k}\,d\sigma}{-\log\psi(s)}=1 \tag{3.17}$$

Using (3.14), (3.15), (3.16), (3.17), we have,

$$\lim_{s\to\infty}\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^{p}\frac{\displaystyle\int_{0}^{s}\psi(r)^{n-k+p+1}r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\psi(s)^{n-k+1}\displaystyle\int_{0}^{s}\left[\left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{n-k+1}{p}}-1\right]^{p}r^{n-k-p}\,dr}$$

$$=\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^{p}\lim_{s\to\infty}\frac{\displaystyle\int_{0}^{s}\psi(r)^{n-k+p+1}r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\left(\dfrac{n-k+1}{n-k-p+1}\right)(-\log\psi(s))}$$

$$=\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^{p}.$$

Hence (3.12) holds and $\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^{p}$ is the best constant in the inequality (3.9).

4. – An improved Hardy inequality.

In the previous section we have proved the Hardy inequality for (p,k)-Hessian Integrals and we have given the best value of the constant.

In this section we show that an improved Hardy inequality holds for (p,k)-Hessian Integrals. Such inequality is obtained by adding on the right-hand side of (3.9) a second term involving the singular weight $\left(\frac{1}{\log\left(\frac{1}{|x|}\right)}\right)^{\gamma}$. Moreover we will prove that the optimal value of γ is $\gamma=2$.

The main result is the following

THEOREM 3. – *Let Ω be an open, $(k-1)$-convex and bounded set of $\mathbb{R}^n$ with $0<k<n$. Let C be a positive constant such that $C \geq \sup_{\Omega}\left(|x|e^{\frac{2}{p}}\right)$ and $1<p<n-k+1$. Then the following statements hold:*

i) *there exists a constant $C_1>0$ depending on n,p,k,C such that*

$$
\begin{aligned}
I_{p,k}[u,\Omega] \geq & \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}}\,dx \\
& + C_1 \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^{\gamma} dx
\end{aligned}
\tag{4.1}
$$

for every functions $u \in A_{k-1}(\Omega)$ if and only if $\gamma \geq 2$.

ii) *For $2 \leq p < n-k+1$ there exists a constant $C_2>0$ depending on n,p,q,C,Ω such that for any $u \in A_{k-1}(\Omega)$ we have*

$$
\begin{aligned}
I_{p,k}[u,\Omega] \geq & \binom{n-1}{k-1} \quad \left(\frac{n-k-p+1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}}\,dx \\
& C_1 \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^{2} dx + C_2 \left[\int_{\Omega} \frac{|u|^q}{|x|^{\beta}}\,dx\right]^{\frac{p}{q}}
\end{aligned}
\tag{4.2}
$$

where $1<q<\dfrac{p(n-\beta)}{n-k-p+1}$ and $0 \leq \beta < p+k-1$.

This theorem extends the results in [6] and [4].

PROOF. – We organize the proof in the following manner: first we prove the validity of inequalities (4.1) and (4.2) and finally we show the optimality of $\gamma=2$.

Let $u \in A_{k-1}(\Omega)$, since the singular weight $\left(\log\left(\frac{C}{|x|}\right)\right)^{\gamma}$ is a decreasing function with respect to $r=|x|$ under our assumption on C, we can prove both inequalities, (4.1) and (4.2), only for the $(k-1)$-symmetrand of u, i.e. $u^*_{k-1}(x)$ by Hardy-Littlewood inequality for rearrangements (see [1], [2]) and Polya-Szegö principle for Hessian Integral (2.13).

Writing

$$
u^*_{k-1}(|x|)=\rho(r),
$$

and setting $d=n-k+1$, the inequalities (4.1) and (4.2) that we have to prove become respectively the following one-dimensional inequalities

$$
\begin{aligned}
I_{p,k}\left[u_{k-1}^*(|x|),B_R\right] &= \binom{n-1}{k-1}\int_0^R (\rho'(r))^p r^{d-1}\,dr \\
&\geq \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^R \frac{|\rho|^p}{r^p} r^{d-1}\,dr \\
&\quad + C_1\int_0^R \frac{|\rho|^p}{r^p\left(\log\left(\frac{C}{r}\right)\right)^\gamma} r^{d-1}\,dr
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
I_{p,k}\left[u_{k-1}^*(|x|),B_R\right] &= \binom{n-1}{k-1}\int_0^R (\rho'(r))^p r^{d-1}\,dr \\
&\geq \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^R \frac{|\rho|^p}{r^p} r^{d-1}\,dr \\
&\quad + C_1\int_0^R \frac{|\rho|^p}{r^p \log\left(\frac{C}{r}\right)^2} r^{d-1}\,dr \\
&\quad + C_2\left[\int_0^R \frac{|\rho|^q}{r^{\beta-k+1}} r^{d-1}\,dr\right]^{\frac{p}{q}}.
\end{aligned} \tag{4.4}
$$

We observe that (4.3) and (4.4) follow from Theorem 1.1 in [6] where the dimension n is replaced by d.

Now we shall prove the optimality.

We suppose that $1 < p < n-k+1$ and $0 \leq \gamma < 2$. We observe that in the case $\gamma = 0$ the optimality holds true since $\beta_{n,p,k} = \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p$ is the best constant in the Hardy inequality (2). Hence we can suppose $0 < \gamma < 2$.

Now we define the following functional

$$
J_\gamma(u) = \frac{I_{p,k}[u,\Omega] - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx}{\int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx}. \tag{4.5}
$$

To prove optimality of $\gamma = 2$ we shall prove that

$$\inf_{\substack{u \in A_{k-1}(\Omega) \\ u \neq 0}} J_\gamma(u) = 0. \tag{4.6}$$

To prove (4.6), we will construct a sequence $u_\varepsilon \in A_{k-1}(\Omega) \quad \forall \varepsilon > 0$ such that

$$\lim_{\varepsilon \to 0} J_\gamma(u_\varepsilon) = 0. \tag{4.7}$$

Now, without loss of generality, we suppose that Ω is the unit ball $B_1(0)$, we set $a = \dfrac{n-k-p+1}{p}$, and for every $\varepsilon > 0$, we define the following functions

$$\tilde{u}_\varepsilon = \begin{cases} -\dfrac{2}{e a \varepsilon^{2a} \log\left(\dfrac{1}{\varepsilon}\right)} & 0 \le r \le \varepsilon^2 \\ -\dfrac{2}{e a \varepsilon^{2a} \log\left(\dfrac{1}{\varepsilon}\right)} + \dfrac{\log\left(\dfrac{r}{\varepsilon^2}\right)}{r^{a} \log\left(\frac{1}{\varepsilon}\right)} & \varepsilon^2 \le r \le \varepsilon^2 e^{\frac{1}{a}} \\ -\dfrac{\log\left(\frac{r}{\varepsilon^2}\right)}{r^{a} \log\left(\frac{1}{\varepsilon}\right)} & \varepsilon^2 e^{\frac{1}{a}} \le r \le \varepsilon \\ -\dfrac{\log(r)}{r^{a} \log\left(\frac{1}{\varepsilon}\right)} & \varepsilon \le r \le 1, \end{cases} \tag{4.8}$$

where $r = |x|$.

We observe that this function is negative, increasing, continuous and null on the boundary of $B_1(0)$ but it is not a smooth function, hence $\tilde{u}_\varepsilon \notin A_{k-1}(B_1(0))$.

Moreover it is easy to check that in every interval where $\tilde{u}_\varepsilon$ has the second derivatives, $\tilde{u}_\varepsilon$ fulfils the properties (1.5) and (1.6) of the class $A_{k-1}(B_1(0))$.

We observe that if we compute $J_\gamma(\tilde{u}_\varepsilon)$ we obtain that (4.7) holds true. In fact using the same arguments of [6] we have:

$$\int_{B_1} \frac{|\tilde{u}_\varepsilon|^p}{|x|^{p+k-1}}\, dx \ge \frac{2n\omega_n}{p+1} \log\left(\frac{1}{\varepsilon}\right) \tag{4.9}$$

Moreover we have

$$\begin{aligned} I_{p,k} &= n\omega_n \binom{n-1}{k-1} \int_0^1 \left(\frac{\partial \tilde{u}_\varepsilon}{\partial r}\right)^p r^{n-k}\, dr \\ &= \frac{2n\omega_n \binom{n-1}{k-1} a^p}{p+1} \log\left(\frac{1}{\varepsilon}\right) + O\left(\frac{1}{\log\left(\frac{1}{\varepsilon}\right)}\right). \end{aligned} \tag{4.10}$$

Hence by (4.9) and (4.10) we obtain that the numerator of $J_\gamma(\tilde{u}_\varepsilon)$ is:

$$I_{p,k}[u,\Omega] - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \leq O\left(\frac{1}{\log\left(\frac{1}{\varepsilon}\right)}\right). \tag{4.11}$$

Now we consider the denominator of $J_\gamma(\tilde{u}_\varepsilon)$ and we have

$$\int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx \geq \tilde{C}\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1-\gamma}, \tag{4.12}$$

where $\tilde{C}$ is a positive constant depending only by n, p.

Finally, by (4.12) and (4.11) and since $0 < \gamma < 2$ we have

$$J_\gamma(\tilde{u}_\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0, \tag{4.13}$$

hence the claim is proved.

Now in order, to have a sequence $u_\varepsilon \in A_{k-1}(B_1(0))$ that fulfils (4.7), we regularize $\tilde{u}_\varepsilon$. But before, to have that the regularization of $\tilde{u}_\varepsilon$ converges to $\tilde{u}_\varepsilon$ uniformly in $B_1(0)$, we extend $\tilde{u}_\varepsilon(r)$ up to $r = 1+\varepsilon$ as follows

$$\overline{u}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon(r) & 0 \leq r \leq 1 \\ \dfrac{1}{\log\left(\frac{1}{\varepsilon}\right)} r - \log\left(\dfrac{1}{\varepsilon}\right) & 1 \leq r \leq 1+\varepsilon. \end{cases} \tag{4.14}$$

Now we define the regularization of $\overline{u}_\varepsilon$ as follows

$$\overline{u}_{\varepsilon,h}(x) = h^{-n} \int_{B_{1+\varepsilon}(0)} \rho\left(\frac{x-y}{h}\right) \overline{u}_\varepsilon(y)\, dy, \tag{4.15}$$

where $h > 0$ and such that $h < \operatorname{dist}(x, \partial B_{1+\varepsilon}(0))$ and ρ is the usual mollifier.

Now we prove that $\overline{u}_{\varepsilon,h} \in A_{k-1}(B_1(0))$. First of all we need to have $\overline{u}_{\varepsilon,h}(x) = 0$ on $\partial B_1(0)$. Hence we define the following function

$$v_{\varepsilon,h}(x) = \overline{u}_{\varepsilon,h}(x) - \overline{C}, \tag{4.16}$$

where $\overline{C} = \overline{u}_{\varepsilon,h}(x)\,|_{\partial B_1(0)}$.

Using the property of $\tilde{u}_\varepsilon$, it is not difficult to prove the following statements:

- $v_{\varepsilon,h}$ is a radially function on $B_1(0)$.
- $v_{\varepsilon,h}$ is increasing with respect to $r = |x|$ on $B_1(0)$, for h sufficiently small.
- $v_{\varepsilon,h} = 0$ on $\partial B_1(0)$
- $v_{\varepsilon,h}$ is a negative function on $B_1(0)$.
- $v_{\varepsilon,h}$ fulfils the properties (1.5) (1.6) of the class $A_{k-1}(B_1(0))$.

Hence $v_{\varepsilon,h} \in A_{k-1}(B_1(0))$.

Finally we shall prove that

$$J_\gamma(v_{\varepsilon,h}) \to 0 \quad \text{for } \varepsilon \to 0. \tag{4.17}$$

Since $v_{\varepsilon,h}$ is a radially function and setting $v_{\varepsilon,h}(|x|) = \varphi(r)$, we have

$$J_\gamma(u) = \frac{I_{p,k}[u,\Omega] - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx}{\int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx}$$

$$= \frac{\binom{n-1}{k-1}\int_0^1 (\varphi'(r))^p r^{n-k}\,dr - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^1 |\varphi(r)|^p r^{n-p-k}\,dr}{\int_0^1 \frac{|\varphi(r)|^p}{\left(\log\left(\frac{C}{r}\right)\right)^\gamma} r^{n-p-k}\,dr}. \tag{4.18}$$

By the following

$$v_{\varepsilon,h} \to \tilde{u}_\varepsilon \quad \text{for } \varepsilon \to 0 \qquad \text{uniformly on } B_1(0);$$

$$Dv_{\varepsilon,h} = (Dv_\varepsilon)_h \to D\tilde{u}_\varepsilon \quad \text{for } \varepsilon \to 0 \qquad \text{uniformly on } B_1(0),$$

we have

$$J_\gamma(v_{\varepsilon,h}) \to J_\gamma(\tilde{u}_\varepsilon) \qquad \text{for } h \to 0. \tag{4.19}$$

Finally, by (4.13) and (4.19), we have

$$J_\gamma(v_{\varepsilon,h}) \to 0 \qquad \text{for } \varepsilon \to 0,$$

hence the optimality is proved.

This completes the proof of the theorem. □

REFERENCES

[1] C. BANDLE, *Isoperimetric Inequalities and Applications* Pitman, 1980.
[2] C. BENNETT - R. SHARPLEY, *Interpolation of Operators*, Academic Press, 1988.
[3] G.A. BLISS, *An integral inequality*, J. London Math. Soc., **5** (1930), 40-46.
[4] H. BREZIS - J. L. VÀZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Revista Mat. Univ. Complutense Madrid, **10** (1997), 443-469.
[5] Yu. D. BURAGO - V. A. ZALGALLER, *Geometric Inequalities*, Springer-Verlag, 1988.
[6] N. CHAUDHURI - ADIMURTHI - M. RAMASWAMY, *An improved Hardy-Sobolev inequality and its applications*, Proc. Am. Math. Soc., **130**, (2001), 489-505.
[7] K. S. CHOU - D. GENG, *Critical dimension of a Hessian Equation Involving Critical*

Exponent and a Related Asymptotic Result, J. Differential Equations, **129** (1996), 79-110.

[8] H. EGNELL, *Elliptic Boundary Value Problems with Singular Coefficients and Critical Nonlinearities*, Indiana Univ. Math. J., **38** (1989), 235-251.

[9] J. GARCIA AZORERO - I. PERAL ALONSO, *Hardy inequalities and some critical elliptic parabolic problems*, J. Diff. Eq., **144** (1998), 441-476.

[10] D. GILBARG - N. S. TRUDINGER, *Elliptic partial differential equation of second order*, (Second Edition), Springer-Verlag, 1983.

[11] V. G. MAZ′JA - *Sobolev Spaces*, Springer-Verlag, 1985.

[12] B. OPIC - A. KUFNER, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Group UK Limited, London, 1990.

[13] R. C. REILLY, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J., **20** (1974), 373-383.

[14] G. TALENTI, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., **110** (1976), 353-372.

[15] N. S. TRUDINGER, *On new isoperimetric inequalities and symmetrization*, J. Reine Angew. Math., **488** (1997), 203-220.

[16] N. S. TRUDINGER, *Isoperimetric inequalities for quermassintegrals*, Ann. Inst. Henri Poincaré Anal. Non Linéaire, **11** (1994), 411-425.

[17] K. TSO, *Remarks on critical exponents for Hessian operators*, Ann. Inst. H. Poincaré 7 (1990), 113-122.

[18] K. TSO, *On symmetrization and Hessian Equations*, J. Anal. Math., **52** (1989), 94-106.

[19] J. L. VÀZQUEZ - E. ZUAZUA, *The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential*, J. Func. Anal., **173** (2000), 103-153.

[20] X. J. WANG, *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ. Math. J., **43** (1994), 25-54.

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”,
Università degli studi di Napoli “Federico II”, via Cintia 80126 Napoli, Italy
E-mail: nunzia.gavitone@dma.unina.it

Pervenuta in Redazione
il 6 dicembre 2006