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Stefano Montaldo, Irene I. Onnis

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A Note on Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Stefano Montaldo - Irene I. Onnis

Sunto. – *In questo lavoro si considerano le superfici nel prodotto* $\mathbb{H}^2 \times \mathbb{R}$ *del piano iperbolico con la retta reale. I risultati principali sono: la descrizione geometrica di alcune proprietà dei grafici minimi; la determinazione di nuovi esempi di grafici minimi completi; la classificazione locale delle superfici totalmente ombelicali.*

Sammary. – *In this article we consider surfaces in the product space* $\mathbb{H}^2 \times \mathbb{R}$ *of the hyperbolic plane* $\mathbb{H}^2$ *with the real line. The main results are: a description of some geometric properties of minimal graphs; new examples of complete minimal graphs; the local classification of totally umbilical surfaces.*

1. – Introduction.

In the last decade the study of the geometry of surfaces in the three-dimensional Thurston geometries has grown considerably. One reason is that these spaces can be endowed with a complete metric with a large isometry group; another, more recent, is the announced proof of the Thurston geometric conjecture, which ensures the dominant role of these spaces among the three-dimensional geometries.

Leaving aside the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, among the remaining five Thurston geometries the Heisenberg space is probably the most studied and the geometry of surfaces is well understood. In recent years the study of the geometry of surfaces in the two product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ is growing very rapidly, and the interest is mainly focused on minimal and constant mean curvature surfaces [1, 2, 5, 6, 7, 8, 9, 10, 12, 14, 15].

The purpose of this paper is first to investigate on some geometric properties of minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ (Theorem 2.1 and Proposition 2.5) and to produce some new examples including complete ones. In the last part (Theorem 3.4) we classify, locally, the totally umbilical surfaces in $\mathbb{H}^2 \times \mathbb{R}$, giving their explicit parametrizations.

We shall recall some basic notions on $\mathbb{H}^2 \times \mathbb{R}$. Let $\mathbb{H}^2$ be the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$ of the hyperbolic plane endowed with the metric $g_{\mathbb{H}} = (dx^2 + dy^2)/y^2$, of constant Gauss curvature -1. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore the product $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the left invariant product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

With respect to the metric g an orthonormal basis of left invariant vector fields is

$$E_1 = y\frac{\partial}{\partial x}, \qquad E_2 = y\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}, \tag{1.1}$$

and the non zero components of the Christoffel symbols are:

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}, \qquad \Gamma^2_{11} = \frac{1}{y}. \tag{1.2}$$

2. – Minimal graphs.

The natural parametrization of a graph $\mathcal{M}$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$\phi(x, y) = (x, y, f(x, y)), \qquad (x, y) \in \Omega,$$

where the domain $\Omega \subseteq \mathbb{H}^2$ is relatively compact, with a differentiable boundary and $f : \Omega \to \mathbb{R}$ is a C^2-function. The unit normal vector field ξ to $\mathcal{M}$ is given by

$$\xi(x, y) = -\frac{f_x}{w\,y}\,E_1 - \frac{f_y}{w\,y}\,E_2 + \frac{1}{w\,y^2}\,E_3, \tag{2.1}$$

where

$$w = \frac{1}{y^2}\sqrt{y^2(f_x^2 + f_y^2) + 1}.$$

The coefficients of the induced metric $h = \phi^* g$ are

$$E = g(\phi_x, \phi_x) = f_x^2 + \frac{1}{y^2}, \quad F = g(\phi_x, \phi_y) = f_x f_y, \quad G = g(\phi_y, \phi_y) = f_y^2 + \frac{1}{y^2},$$

while the coefficients of the second fundamental form are given by

$$\begin{aligned} L &= -g(\nabla_{\phi_x}\xi, \phi_x) = \frac{y f_{xx} - f_y}{w\,y^3}, \\ M &= -g(\nabla_{\phi_x}\xi, \phi_y) = \frac{y f_{xy} + f_x}{w\,y^3}, \\ N &= -g(\nabla_{\phi_y}\xi, \phi_y) = \frac{y f_{yy} + f_y}{w\,y^3}, \end{aligned} \tag{2.2}$$

where ∇ is the Levi-Civita connection associated to the metric g. The mean curvature function is then

$$(2.3)\qquad H = \frac{y^2}{2}\,\mathrm{div}\Big(\frac{\nabla f}{\sqrt{1+y^2\,|\nabla f|^2}}\Big) = \frac{1}{2}\,\mathrm{div}_{\mathbb{H}^2}\Big(\frac{\nabla f}{w}\Big),$$

where ∇ and div stand for the Euclidean gradient and the Euclidean divergence, while $\mathrm{div}_{\mathbb{H}^2}$ is the divergence in $(\mathbb{H}^2, g_{\mathbb{H}})$. The equation $H = 0$ is called the *minimal surfaces equation* in $\mathbb{H}^2 \times \mathbb{R}$, and can be also written as

$$(2.4)\qquad (1+y^2f_y^2)f_{xx} - y\,(f_x^2+f_y^2)f_y - 2y^2 f_x f_y f_{xy} + (1+y^2f_x^2)f_{yy} = 0.$$

This equation was first found by B. Nelli and H. Rosenberg, in [9], where they showed that in $\mathbb{H}^2 \times \mathbb{R}$ there exist minimal surfaces of Catenoid-type, Helicoid-type and Scherk-type. Moreover, they proved that Bernstein's theorem fails, that is there exist complete minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ of rank different from zero.

The first geometric property of minimal graphs is that, as in the Euclidean case, solutions of (2.4) define graphs of "minimal" area.

THEOREM 2.1. – *If f satisfies the minimal surfaces equation* (2.4) *in Ω and f extends continuously to $\overline{\Omega}$, then the area of the surface $\mathcal{M}$, defined by f, is less than or equal to the area of any other surface $\widetilde{\mathcal{M}}$ defined by a function $\tilde f$ in Ω having the same values as f on $\partial\Omega$. Moreover, equality holds if and only if f and $\tilde f$ coincide on Ω.*

PROOF. – The theorem follows by an argument similar to that used for minimal graphs in $\mathbb{R}^3$ (see, for example, [13]) and for completeness we give the proof. In the domain $\Omega \times \mathbb{R}$ of $\mathbb{H}^2 \times \mathbb{R}$, consider the unit vector field $V(x, y, z)$ given by

$$V(x,y,z) := -\frac{f_x}{w\,y}\,E_1 - \frac{f_y}{w\,y}\,E_2 + \frac{1}{w\,y^2}\,E_3.$$

Writing $V = V^i\,(\partial/\partial x_i)$ and denoting by $\mathrm{div}_{\mathbb{H}^2\times\mathbb{R}}$ the divergence of $\mathbb{H}^2 \times \mathbb{R}$, we have

$$\mathrm{div}_{\mathbb{H}^2\times\mathbb{R}} V = -y^2\,\mathrm{div}\Big(\frac{\nabla f}{\sqrt{1+y^2\,|\nabla f|^2}}\Big).$$

Since $f(x, y)$ satisfies (2.4), it follows that

$$\mathrm{div}_{\mathbb{H}^2\times\mathbb{R}} V \equiv 0 \qquad \text{on} \quad \Omega\times\mathbb{R}.$$

The surfaces $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have the same boundary, and therefore $\mathcal{M} \cup \widetilde{\mathcal{M}}$ is an oriented boundary of an open set Θ in $\Omega \times \mathbb{R}$. Denoting by η the unit normal corresponding to the positive orientation on $\mathcal{M} \cup \widetilde{\mathcal{M}}$ and using the Divergence Theorem, we have:

$$0 = \int_{\Theta} \mathrm{div}_{\mathbb{H}^2\times\mathbb{R}} V = \int_{\mathcal{M}\cup\widetilde{\mathcal{M}}} g(V,\eta)\, dA\,. \tag{2.5}$$

From the definition of the vector V and from (2.1), it follows that

$$V \equiv \eta \qquad \text{on} \quad \mathcal{M},$$

hence, from (2.5), and since V and η are both unit vector fields, it results that

$$A(\mathcal{M}) = \int_{\mathcal{M}} g(V,\eta)\, dA = \int_{\widetilde{\mathcal{M}}} g(V,-\eta)\, dA \leq \int_{\widetilde{\mathcal{M}}} dA = A(\widetilde{\mathcal{M}}).$$

Furthermore, equality holds if and only if $g(V,-\eta) = 1$, that is, if and only if on Ω $\widetilde{f}_x = f_x$ and $\widetilde{f}_y = f_y$. Finally, since $f_{|\partial\Omega} = \widetilde{f}_{|\partial\Omega}$, we must have that $\widetilde{f}(x,y) = f(x,y)$ for all $(x,y) \in \Omega$. $\square$

In the following we show some solutions of (2.4).

EXAMPLE 2.2. – If a solution of (2.4) has the form $f(x,y) = \varphi(x)$, we have that $\varphi''(x) = 0$ and, then, $f(x,y) = ax + b$, with $a, b \in \mathbb{R}$. These are the only minimal planes in $\mathbb{H}^2 \times \mathbb{R}$ that can be described as graphs.

If now we look for solutions of type $f(x,y) = \psi(y)$, then (2.4) assumes the form $\psi''(y) - y\,\psi'(y)^3 = 0$, and, by integration, we get

$$\psi(y) = \arcsin(a\,y) + b, \qquad 0 < y \leq 1/a\,, \quad a, b \in \mathbb{R}, \quad a > 0.$$

EXAMPLE 2.3. – We can find interesting examples of minimal graphs seeking for radial solutions of (2.4) of type $f(x,y) = h(x^2+y^2)$. In this case, it results that $z\,h''(z) + h'(z) = 0$, thus the desired function is

$$f(x,y) = a\, \ln(x^2+y^2) + b, \qquad a, b \in \mathbb{R}. \tag{2.6}$$

This surface, called the *funnel surface*, defines a complete minimal graph. We observe that the Gauss map of this surface is of rank 1. On the right hand side of Figure 1 there is a plot of the image, under the Gauss map, of the funnel surface, which is plotted on the left hand side.

EXAMPLE 2.4. – Let $f(x,y)$ be a solution of the minimal surfaces equation of type

$$f(x,y) = \frac{a(x)}{x^2+y^2}\,,$$

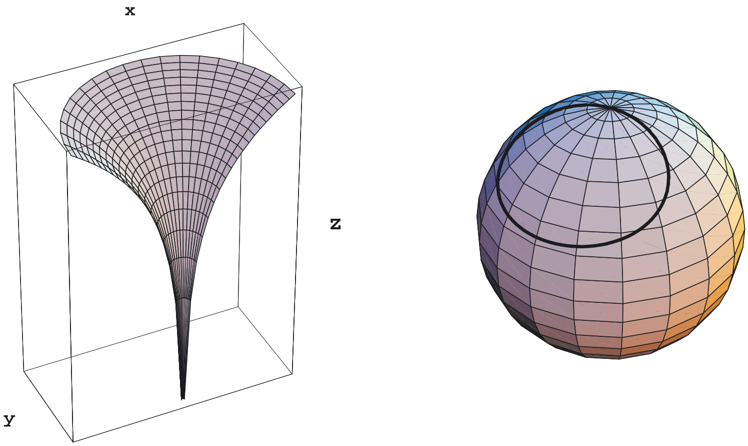

Fig. 1. – Complete minimal graph of rank 1 (left) and the image of its Gauss map (right).

where $a(x)$ is a real function. Then, (2.4) gives

$$[(x^2+y^2)^4 + 4\,y^4\,a(x)^2]\,a''(x) - 4(x^2+y^2)^3\,[x\,a'(x) - a(x)] = 0,$$

of which a solution is $a(x) = c\,x$, with $c \in \mathbb{R}$. The corresponding minimal function is

$$f(x,y) = \frac{c\,x}{x^2+y^2}, \qquad c \in \mathbb{R}, \tag{2.7}$$

which produces the minimal graph plotted in Figure 2 (left). We observe that the Gauss map of this complete graph is of rank 2.

This example can be generalized considering, for a given real function h, a solution of (2.4) of type $f(x,y) = h\big(\frac{x}{x^2+y^2}\big)$ or of type $f(x,y) = h\big(\frac{y}{x^2+y^2}\big)$. In the first case we essentially find (up to translations) the example given by (2.7). In the second case it results that $h''(z) - z\,h'(z)^3 = 0$ and, therefore,

$$h(z) = \arcsin(az) + b, \qquad a, b \in \mathbb{R}.$$

The corresponding minimal function

$$f(x,y) = \arcsin\left(\frac{a\,y}{x^2+y^2}\right) + b$$

does not define a complete graph. A plot of this surface is given in Figure 2 (right).

We now study the relations between the minimality of a surface in $(\mathbb{H}^2\times\mathbb{R}, g)$, defined as the graph of a differentiable function f, and the harmonicity of f.

Let $\mathcal{M}$ be a minimal graph of a C^2-function f, defined in a domain Ω of $\mathbb{H}^2$ and

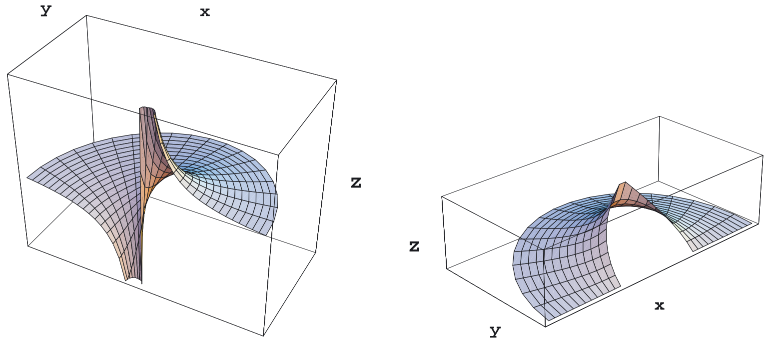

Fig. 2. – Minimal graphs of rank 2: a complete one (left) and a non complete one (right).

$\phi(x, y) = (x, y, f(x, y))$ its global parametrization. We know (see, for example, [4]) that $\phi : (\Omega, h) \to \mathbb{H}^2 \times \mathbb{R}$ is harmonic, that is ϕ satisfies the system

$$\Delta_h \phi^\alpha + h^{ij}\, \Gamma^\alpha_{\beta\gamma}\, \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} = 0, \quad \alpha = 1, 2, 3,$$

where Δ_h is the Beltrami-Laplace operator with respect to the induced metric $h = \phi^* g$. Consequently, from (1.2), it follows that $\Delta_h f = 0$.

As an immediate consequence of the this fact, we can prove that there exist no compact minimal surface without boundary in the product $\mathbb{H}^2 \times \mathbb{R}$. In fact, let

$$\begin{aligned} \psi : \mathcal{M} &\longrightarrow (\mathbb{H}^2 \times \mathbb{R}, g) \\ p &\longmapsto (\psi_1(p), \psi_2(p), \psi_3(p)) \end{aligned} \tag{2.8}$$

be a minimal immersion of a compact surfaces $\mathcal{M}$ in $\mathbb{H}^2 \times \mathbb{R}$. Then, $\triangle_{\psi^* g} \psi_3 = 0$ and, from the Hopf maximum principle, ψ_3 must be constant.

To state the next result we recall that a curve is called a *pregeodesic* if there exists a reparametrization which is a geodesic.

Proposition 2.5. – *Let $\mathcal{M} \subset \mathbb{H}^2 \times \mathbb{R}$ be a minimal surface defined as the graph of a non constant differentiable function $f : \Omega \subseteq \mathbb{H}^2 \to \mathbb{R}$. Then, the level curves of f are pregeodesics of $\mathbb{H}^2$ if and only if f is harmonic with respect to the flat Laplacian.*

Proof. – Let f be a solution of (2.4) and let $\gamma(t) = (x(t), y(t))$ be the parametrization of a level curve of f. If the function $x(t)$ is constant, it results that $f_y = 0$ and thus, using Example 2.2, the surface $\mathcal{M}$ is a piece of the plane $z = ax + b$, $a, b \in \mathbb{R}$.

Thus we can assume that there exists a point t_0 such that $x'(t) \neq 0$ in a neighborhood of t_0. Therefore, we can parametrize γ as $\gamma(x) = (x, y(x))$, with

$y(x) > 0$, in a neighborhood of $\gamma(t_0)$. It follows that

$$\gamma'(x) = \frac{E_1}{y(x)} + \frac{y'(x)}{y(x)}\, E_2,$$

and

$$\nabla_{\gamma'}\gamma' = -\frac{2y'(x)}{y(x)^2}\, E_1 + \Big(\frac{y(x)y''(x) - y'(x)^2 + 1}{y(x)^2}\Big)\, E_2.$$

The geodesic curvature k_g (in $\mathbb{H}^2$) of γ is then

$$k_g = \frac{g(\nabla_{\gamma'}\gamma', J\gamma')}{\|\gamma'\|^3} = \frac{y(x)y''(x) + y'(x)^2 + 1}{(1 + y'(x)^2)^{3/2}}.$$

Using

$$y'(x) = -\frac{f_x(x, y(x))}{f_y(x, y(x))}$$

and

$$y''(x) = -\frac{f_{xx} + 2f_{xy}y' + f_{yy}(y')^2}{f_y},$$

the minimal surfaces equation (2.4) can be written as

$$\Delta f = y(x)\, k_g\, |\nabla f|^3_{\mathbb{R}^2},$$

which completes the proof. □

REMARK 2.6. – Since the functions given in the equations (2.6) and (2.7) are harmonic with respect to the flat Laplacian, from the above result we conclude that the corresponding minimal graphs are ruled. Another example of harmonic minimal function is given by

$$f(x, y) = a \arctan\left(\frac{2x}{x^2 + y^2 - 1}\right) + b, \qquad a, b \in \mathbb{R},$$

which gives (locally) the helicoid.

3. – Totally umbilical surfaces of $\mathbb{H}^2 \times \mathbb{R}$.

We start this section by studying the totally geodesic surfaces of $\mathbb{H}^2 \times \mathbb{R}$. For this we need the following lemma.

LEMMA 3.1. ([11]). – *Let $\mathcal{M}$ be a regular surface in $\mathbb{H}^2 \times \mathbb{R}$. Then, there exists an open dense set in $\mathcal{M}$, of which the connected components admit one of the* *following parametrizations:*

$$\begin{aligned} X(u,v) &= (u, v, f(u,v)), && v > 0, \\ Y(u,v) &= (u, a(u), v), && a(u) > 0, \\ Z(u,v) &= (c, u, v), && u > 0, \end{aligned}$$

where $c \in \mathbb{R}$ *and* $a(u)$ *is a real function.*

The parametrizations $Y(u,v)$ and $Z(u,v)$ of Lemma 3.1 define surfaces that we shall call *vertical surfaces*.

THEOREM 3.2. – *The totally geodesic surfaces of* $\mathbb{H}^2 \times \mathbb{R}$ *are the horizontal planes* $z = c$, $c \in \mathbb{R}$, *and the vertical cylinders over the geodesics of* $\mathbb{H}^2$.

PROOF. – Since a totally geodesic surface in $\mathbb{H}^2 \times \mathbb{R}$ is not compact, using Lemma 3.1, we only have to classify the totally geodesic graphs and the totally geodesic vertical surfaces. We start by proving that the only totally geodesic graphs are the horizontal planes $z = c$, $c \in \mathbb{R}$. Let $\mathcal{M}$ be a totally geodesic surface defined as the graph of a differentiable function f. From (2.2), it follows that

$$\begin{cases} f_{xx} = \dfrac{f_y}{y}, \\ f_{yy} = -\dfrac{f_y}{y}, \\ f_{xy} = -\dfrac{f_x}{y}. \end{cases} \tag{3.1}$$

First, observe that $f(x,y) = c$, $c \in \mathbb{R}$, satisfies (3.1) and, therefore, defines a totally geodesic surface. Then, since $f_x = 0$ if and only if $f_y = 0$, there exist no totally geodesic graphs defined by a (non constant) function f that depends only on one variable. Thus, we can suppose that $f_x \neq 0$ and $f_y \neq 0$. From the second and third equations of (3.1), we find that there exist two functions $p(x)$ and $q(x)$ such that $f_y = p(x)/y$ and $f_x = q(x)/y$. Now, replacing in the first equation of (3.1), we have the contradiction

$$y\, q'(x) - p(x) = 0.$$

To complete the proof observe that the vertical cylinders over the geodesics of $\mathbb{H}^2$ are totally geodesic surfaces of $\mathbb{H}^2 \times \mathbb{R}$. In the following, we prove that these cylinders are the only vertical surfaces which are totally geodesic in $\mathbb{H}^2 \times \mathbb{R}$. Let $\mathcal{M}$ be a vertical surface. From Lemma 3.1, it follows that either $\mathcal{M}$ is the totally geodesic plane $x = c$, with $y > 0$, or it is parametrized by

$$X(u,v) = (u, a(u), v), \qquad a(u) > 0. \tag{3.2}$$

The unit normal vector field to the surface $\mathcal{M}$, defined by (3.2), is

$$\xi = \frac{a'}{\sqrt{1+(a')^2}}\,E_1 - \frac{1}{\sqrt{1+(a')^2}}\,E_2. \tag{3.3}$$

It is then straightforward to compute that

$$\nabla_{X_u}\xi = \Big[\Big(\frac{a'}{\sqrt{1+(a')^2}}\Big)_u + \frac{1}{a\,\sqrt{1+(a')^2}}\Big]E_1 + \Big[\frac{a'}{a\,\sqrt{1+(a')^2}} - \Big(\frac{1}{\sqrt{1+(a')^2}}\Big)_u\Big]E_2,$$

and

$$\nabla_{X_v}\xi = 0,$$

hence $M = N = 0$. Consequently, $\mathcal{M}$ is totally geodesic if and only if

$$L = -g(\nabla_{X_u}\xi, X_u) = 0,$$

that is, the function $a(u)$ satisfies the following ODE:

$$a\,a'' + (a')^2 + 1 = 0.$$

The latter equation implies that

$$a(u) = \sqrt{-u^2 + 2\,c_1\,u + c_2}, \qquad c_1, c_2 \in \mathbb{R} \quad \text{with} \quad c_1^2 + c_2 > 0,$$

and the curve $(u, a(u))$ is the geodesic of $\mathbb{H}^2$ given by the upper semi-circle with center at $(c_1, 0)$ and radius $\sqrt{c_1^2 + c_2}$. This completes the proof. □

REMARK 3.3. – From the proof of Theorem 3.2 it follows that, if $\mathcal{M}$ is a vertical surface, then $F = M = N = 0$ and so the mean curvature is given by $H = L/2E$. Thus a vertical surface is minimal if and only if it is totally geodesic.

We are now ready to state the main result of this section.

THEOREM 3.4. – *The totally umbilical surfaces of* $\mathbb{H}^2 \times \mathbb{R}$ *are locally:*

i) *the totally geodesic surfaces given in Theorem* 3.2;

ii) *the surface given as the graph of the function*

$$f(x,y) = \arctan\Big(\frac{\lambda(x,y)}{\sqrt{j - \lambda(x,y)^2}}\Big) + c, \qquad c \in \mathbb{R}$$

where

$$\lambda(x,y) = \frac{1}{y}\,\Big[\frac{c_1}{2}(x^2+y^2) + c_2\,x - c_3\Big], \qquad c_1, c_2, c_3 \in \mathbb{R},$$

and

$$j = 1 - c_2^2 - 2\,c_1\,c_3 > 0.$$

PROOF. – Using Lemma 3.1 we only have to classify the totally umbilical graphs and the totally umbilical vertical surfaces. Let $\mathcal{M}$ be a totally umbilical surface of $\mathbb{H}^2 \times \mathbb{R}$ parametrized by

$$\phi(x,y) = (x, y, f(x,y)), \quad (x,y) \in \Omega \subseteq \mathbb{H}^2.$$

Let $p \in \mathcal{M}$ and let ξ be an unit normal vector field defined in some neighborhood U of p. Since $\mathcal{M}$ is totally umbilical, by definition, there exists a function $\lambda : U \to \mathbb{R}$ such that the shape operator A_χ satisfies $A_\xi = \lambda I$ in U. The expression of A_ξ with respect to the basis $\{\phi_x, \phi_y\}$ is

$$A_\xi(p) = \begin{pmatrix} \left(\dfrac{f_x}{w}\right)_x - \dfrac{f_y}{y\,w} & y\left(\dfrac{f_x}{y\,w}\right)_y \\ \left(\dfrac{f_y}{w}\right)_x + \dfrac{f_x}{y\,w} & y\left(\dfrac{f_y}{y\,w}\right)_y \end{pmatrix}.$$

Thus $\mathcal{M}$ is totally umbilical if and only if

$$\tag{3.4} \begin{cases} \left(\dfrac{f_x}{w\,y}\right)_y = 0, \\ \left(\dfrac{f_y}{w}\right)_x = -\dfrac{f_x}{w\,y}, \\ \left(\dfrac{f_x}{w}\right)_x = \left(\dfrac{f_y}{w}\right)_y. \end{cases}$$

The first and second equations of (3.4) imply that there exist two functions $a(x)$ and $b(y)$ such that

$$\frac{f_x}{yw} = a(x), \qquad \frac{f_y}{w} = -\int a(x)dx + b(y).$$

Then, from the third equation of (3.4), we conclude that

$$a(x) = c_1 x + c_2, \qquad b(y) = \frac{c_1}{2} y^2 + c_3,$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Thus $f_y/w = \frac{c_1}{2}(y^2 - x^2) - c_2 x + c_3$, which implies that

$$\tag{3.5} \lambda(x,y) = y\left(\frac{f_y}{yw}\right)_y = \frac{1}{y}\left[\frac{c_1}{2}(x^2 + y^2) + c_2 x - c_3\right].$$

From the Codazzi's equation for totally umbilical surfaces (see, for example, [3])

$$(\mathrm{R}(\phi_x, \phi_y)\xi)^\top = \lambda_y \phi_x - \lambda_x \phi_y,$$

and using

$$(\mathrm{R}(\phi_x, \phi_y)\xi)^\top = \frac{f_y}{wy^3} E_1 - \frac{f_x}{wy^3} E_2,$$

it follows that

$$\left\{\begin{aligned} \lambda_x &= \frac{f_x}{w\,y^2}, \\ \lambda_y &= \frac{f_y}{w\,y^2}. \end{aligned}\right. \tag{3.6}$$

Now, using the identities

$$\left(\frac{1}{w\,y^2}\right)_y = -y\Big[f_x\left(\frac{f_x}{w\,y}\right)_y + f_y\left(\frac{f_y}{w\,y}\right)_y\Big]$$

$$\left(\frac{1}{w\,y^2}\right)_x = -\Big[f_x\left(\frac{f_x}{w}\right)_x + f_y\left(\frac{f_y}{w}\right)_x\Big],$$

and (3.5), a simple calculation gives

$$y^2\,w = \frac{1}{\sqrt{j-\lambda^2}}, \tag{3.7}$$

where $j(x) = 1 - c_2^2 - 2\,c_1\,c_3 > 0$. Substituting (3.7) in the first equation of (3.6) we have

$$f(x,y) = \int \frac{\lambda_x}{\sqrt{j-\lambda^2}}\,dx = \arctan\left(\frac{\lambda}{\sqrt{j-\lambda^2}}\right) + h(y),$$

for a certain function $h(y)$. From the second equation of (3.6), we conclude that $h(y)$ is constant.

To complete the proof, we must study the case when $\mathcal{M}$ is a totally umbilical vertical surface. From Lemma 3.1, it follows that either $\mathcal{M}$ is the totally geodesic plane $x = c$, or it is given by:

$$X(u,v) = (u, a(u), v), \qquad a(u) > 0.$$

In the last case, we have $A_\xi X_v = -\nabla_{X_v}\xi = 0$ and, therefore, $\lambda = 0$. This concludes the proof of the theorem. $\square$

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Stefano Montaldo: Università degli Studi di Cagliari
Dipartimento di Matematica e Informatica
Via Ospedale 72, 09124 Cagliari, Italia
e-mail: montaldo@unica.it

Irene I. Onnis; Departamento de Matematica, C.P. 668, ICMC,
USP, 13560-970, São Carlos, SP, Brasil
e-mail: onnis@icmc.usp.br

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