Bollettino Unione Matematica Italiana

D. Sivaraj, V. Renuka Devi

Some Separation Axioms Via Ideals

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 917–931.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_917_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 917-931

Some Separation Axioms Via Ideals

D. Sivaraj - V. Renuka Devi

Sunto. – *Introduciamo una nuova classe di spazi, detti spazi di Hausdorff modulo $\mathcal{I}$ o T_2 mod $\mathcal{I}$ rispetto ad un ideale $\mathcal{I}$ che contiene la classe di tutti gli spazi di Hausdorff. Diamo delle caratterizzazioni di questi spazi e studiamo le loro proprietà. Il concetto di compattezza modulo un ideale $\mathcal{I}$ fu introdotto da Newcomb nel 1967 e studiato da Hamlett e Jankovic nel 1990. Studiamo le proprietà dei sottoinsiemi $\mathcal{I}$-compatti in spazi di Hausdorff modulo $\mathcal{I}$ e generalizziamo alcuni risultati di Hamlett e Jankovic. Gli spazi $\mathcal{I}$-regolari furono introdotti da Hamlett e Jankovic nel 1994. Studiamo ulteriormente il concetto di $\mathcal{I}$-regolarità rispetto alla sua conservazione da parte di funzioni, sottospazi e prodotto.*

Summary. – *We introduce a new class of spaces, called Hausdorff modulo $\mathcal{I}$ or T_2 mod $\mathcal{I}$ spaces with respect to an ideal $\mathcal{I}$, which contains the class of all Hausdorff spaces. Characterizations of these spaces are given and their properties are investigated. The concept of compactness modulo an ideal $\mathcal{I}$ was introduced by Newcomb in 1967 and studied by Hamlett and Jankovic in 1990. We study the properties of $\mathcal{I}$-compact subsets in Hausdorff modulo $\mathcal{I}$ spaces and generalize some results of Hamlett and Jankovic. $\mathcal{I}$-regular space was introduced by Hamlett and Jankovic in 1994. We further investigate the concept of $\mathcal{I}$-regularity with regard to its preservation by functions, subspaces and product.*

1. – Introduction and Preliminaries.

The subject of ideals in topological spaces has been studied by Kuratowski [13] and Vaidyanathaswamy [19]. An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a *local function* [13] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X, A^\star(\mathcal{I}, \tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local functions [11, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star - topology$, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$[18] and $\beta = \{U - I \mid U \in \tau$ and $I \in \mathcal{I}\}$ is a base for $\tau^\star$[11]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$

and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal space. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, *cl(A)* and *int(A)* will, respectively, denote the closure and interior of A in (X,τ) and $cl^\star(A)$ will denote the closure of A in $(X,\tau^\star)$. An open subset A of a space (X,τ) is said to be *regular open* if $A = int(cl(A))$. The complement of a regular open set is *regular closed*. Given an ideal space $(X,\tau,\mathcal{I}), \mathcal{I}$ is said to be *codense* [4] if $\mathcal{I}\cap\tau=\{\emptyset\}$ and $\mathcal{I}$ is said to be *regular* [3], if $RO(X)\cap\mathcal{I}=\{\emptyset\}$ where $RO(X)$ is the family of all regular open sets. Every codense ideal is regular but the converse need not be true [3].

2. – Hausdorff modulo an Ideal.

DEFINITION 2.1. – *An ideal space $(X,\tau,\mathcal{I})$ is said to be Hausdorff mod $\mathcal{I}$ or T_2 mod $\mathcal{I}$ if for every pair of distinct points x and y in X, there exist open sets U and V such that $x \in U,\ y \in V$ and $U \cap V \in \mathcal{I}$.*

Every Hausdorff space is T_2 mod $\mathcal{I}$, since $\emptyset \in \mathcal{I}$. The converse need not be true as the following Example shows.

EXAMPLE 2.2. – Let $X = \{a,b,c\},\ \tau = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{b,c\}\}$ and $\mathcal{I} = \{\emptyset,\{b\},\{c\},\{b,c\}\}$. Then X is T_2 mod $\mathcal{I}$ but not Hausdorff (even it is not T_1).

The easy proof of the following Theorems are omitted.

THEOREM 2.3. – *A space (X,τ) is Hausdorff if and only if (X,τ) is T_2 mod $\{\emptyset\}$.*

THEOREM 2.4. – *If an ideal space $(X,\tau,\mathcal{I})$ is T_2 mod $\mathcal{I}$ and $\mathcal{I}\subset\mathcal{J}$, then $(X,\tau,\mathcal{J})$ is T_2 mod $\mathcal{J}$.*

THEOREM 2.5. – *If $(X,\tau,\mathcal{I})$ is a T_2 mod $\mathcal{I}$ space and $\mathcal{I}$ is codense, then $(X,\tau,\mathcal{I})$ is Hausdorff.*

The following Theorem gives a characterization of T_2 mod $\mathcal{I}$ spaces.

THEOREM 2.6. – *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) *$(X,\tau,\mathcal{I})$ is T_2 mod $\mathcal{I}$.*
(b) *If $x \in X$, then for each $y \neq x$, there exists an open set U containing x such that $y \notin U^\star$.*
(c) *For every x in X, $\cap\{U_x^\star \mid U_x \in \tau(x)\}$ is either $\emptyset$ or $\{x\}$.*

PROOF. – $(a) \Rightarrow (b)$. Let $x \in X$ and $y \in X$ such that $x \neq y$. Then there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. $U \cap V \in \mathcal{I}$ implies that $(U \cap V)^\star = \emptyset$ and so $U^\star \cap V = \emptyset$. Thus $y \notin U^\star$ which proves (b).

$(b) \Rightarrow (c)$. If $x \in X$ and $y \in X$ such that $y \neq x$, by hypothesis, there is an open set U_x containing x such that $y \notin U_x^\star$. This implies that $y \notin \cap\{U_x^\star \mid U_x \in \tau(x)\}$ and so $\cap\{U_x^\star \mid U_x \in \tau(x)\}$ is either $\emptyset$ or $\{x\}$.

$(c) \Rightarrow (a)$. Let $x \in X$ and $y \in X$ such that $x \neq y$. By hypothesis, $y \notin \cap\{U_x^\star \mid U_x \in \tau(x)\}$ which implies that $y \notin U_x^\star$ for some $U_x \in \tau(x)$. Therefore, there exists $V_y \in \tau(y)$ such that $U_x \cap V_y \in \mathcal{I}$ and so $(X, \tau, \mathcal{I})$ is T_2 mod $\mathcal{I}$.

The following Theorem 2.7 shows that T_2 mod $\mathcal{I}$ is a property shared by both (X, τ) and (X,$\tau^\star$). Corollary 2.8 below follows from Theorems 2.5 and 2.7.

THEOREM 2.7. – *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then (X, τ) is T_2 mod $\mathcal{I}$ if and only if $(X, \tau^\star)$ is T_2 mod $\mathcal{I}$.*

PROOF. – If (X, τ) is T_2 mod $\mathcal{I}$, then clearly, $(X, \tau^\star)$ is T_2 mod $\mathcal{I}$. Conversely, suppose $(X, \tau^\star)$ is T_2 mod $\mathcal{I}$. Let $x \in X$ and $y \in X$ such that $x \neq y$. Then there exist $\tau^\star-$open sets U and V such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. Moreover, there exist open sets G and H and $I_1, I_2 \in \mathcal{I}$ such that $x \in G - I_1 \subset U$ and $y \in H - I_2 \subset V$. Therefore, (G$\cap$H)$-(I_1 \cup I_2)$=$(G - I_1) \cap (H - I_2) \subset (U \cap V) \in \mathcal{I}$, which implies that $G \cap H \in \mathcal{I}$. Hence (X, τ) is T_2 mod $\mathcal{I}$.

COROLLARY 2.8. – *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be codense. Then the following are equivalent.*

(a) *$(X, \tau, \mathcal{I})$ is Hausdorff.*
(b) *$(X, \tau, \mathcal{I})$ is T_2 mod $\mathcal{I}$.*
(c) *$(X, \tau^\star, \mathcal{I})$ is T_2 mod $\mathcal{I}$.*
(d) *$(X, \tau^\star, \mathcal{I})$ is Hausdorff.*

If X is any set, $A \subset X$ and $\mathcal{I}$ is an ideal on X, then $\mathcal{I}_A = \{A \cap I \mid I \in \mathcal{I}\} = \{I \in \mathcal{I} \mid I \subset A\}$ is an ideal on A [11]. The rest of the section is devoted to subspaces, product spaces and images and preimages of T_2 mod $\mathcal{I}$ spaces.

THEOREM 2.9. – *Let $(X, \tau, \mathcal{I})$ be a T_2 mod $\mathcal{I}$ space and $A \subset X$. Then $(A, \tau_A, \mathcal{I}_A)$ is T_2 mod $\mathcal{I}_A$.*

PROOF. – Let x and y be two distinct points of A. Then there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. Now $x \in U \cap A \in \tau_A$, $y \in V \cap A \in \tau_A$ and $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A \in \mathcal{I}_A$. Therefore, (A, τ_A) is T_2 mod $\mathcal{I}_A$.

If $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is a mapping, then $f(\mathcal{I})=\{f(I)\mid I\in\mathcal{I}\}$ is an ideal on Y [7, Theorem 1.8]. If f is an injection and $\mathcal{J}$ is any ideal on Y, then $f^{-1}(\mathcal{J})=\{f^{-1}(J)\mid J\in\mathcal{J}\}$ is an ideal on X [7, Theorem 1.11].

THEOREM 2.10. – *Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ be an open bijection and $(X,\tau,\mathcal{I})$ be T_2 mod $\mathcal{I}$. Then (Y,σ) is T_2 mod $f(\mathcal{I})$.*

PROOF. – Let y and u be distinct points in Y. Since f is bijective, there exist distinct points x and v in X such that $f(x)=y$ and $f(v)=u$. Since $(X,\tau,\mathcal{I})$ is T_2 mod $\mathcal{I}$, there exist open sets U and V such that $x\in U, v\in V$ and $U\cap V\in\mathcal{I}$. If $U\cap V=I\in\mathcal{I}$, then $f(U)\cap f(V)=f(U\cap V)=f(I)\in f(\mathcal{I})$. Since f is open, $f(U)$ and $f(V)$ are open sets in Y such that $y=f(x)\in f(U)$ and $u=f(v)\in f(V)$. Therefore, (Y,σ) is T_2 mod $f(\mathcal{I})$.

THEOREM 2.11. – *Let $f:(X,\tau)\to(Y,\sigma,\mathcal{J})$ be a continuous injection map where $(Y,\sigma,\mathcal{J})$ be T_2 mod $\mathcal{J}$. Then (X,τ) is T_2 mod $f^{-1}(\mathcal{J})$.*

PROOF. – Let x and v be distinct points in X. Since $(Y,\sigma,\mathcal{J})$ is T_2 mod $\mathcal{J}$, there exist open sets U and V in Y such that $f(x)\in U, f(v)\in V$ and $U\cap V\in\mathcal{J}$. If $U\cap V=J\in\mathcal{J}$, then $f^{-1}(U)\cap f^{-1}(V)=f^{-1}(U\cap V)=f^{-1}(J)\in f^{-1}(\mathcal{J})$. Since $f^{-1}(U)$ and $f^{-1}(V)$ are open sets containing x and v respectively, (X,τ) is T_2 mod $f^{-1}(\mathcal{J})$.

Given a space (X,τ) and a collection $\mathcal{A}$ of subsets of X, the smallest ideal on X containing $\mathcal{A}$ will be denoted by $\mathcal{I}(\mathcal{A})$ and is called the ideal generated by $\mathcal{A}$ [7]. It is easily seen that $\mathcal{I}(\mathcal{A})=\{A\mid A\subset A_1\cup A_2\cup\ldots\cup A_n, A_i\in\mathcal{A}$ for each i, $n\in\mathbb{N}\}$, where $\mathbb{N}$ is the set of all natural numbers.

THEOREM 2.12. – *Let $(X_a,\tau_a,\mathcal{I}_a)$ be a collection of ideal spaces for each a in an index set Δ, where each $(X_a,\tau_a,\mathcal{I}_a)$ be T_2 mod $\mathcal{I}_a$. If P_a is the projection map for each a, $\mathcal{A}=\{P_a^{-1}(I_a)\mid I_a\in\mathcal{I}_a,\ a\in\Delta\}$, $\mathcal{I}(\mathcal{A})$ is the smallest ideal generated by $\mathcal{A}$ and $\mathcal{I}$ is an ideal on $\prod X_a$ such that $\mathcal{I}\supset\mathcal{I}(\mathcal{A})$, then $\prod X_a$ with the product topology is T_2 mod $\mathcal{I}$.*

PROOF. – Let x and y be points in $\prod X_a$ such that $x\neq y$. Then for some $a\in\Delta$, $x_a\neq y_a$. Since each (X_a,τ_a) is T_2 mod $\mathcal{I}_a$, there exist open sets U_a and V_a such that $x_a\in U_a$, $y_a\in V_a$ and $U_a\cap V_a\in\mathcal{I}_a$. If $U_a\cap V_a=I_a\in\mathcal{I}_a$, then $P_a^{-1}(U_a)\cap P_a^{-1}(V_a)=P_a^{-1}(U_a\cap V_a)=P_a^{-1}(I_a)\in\mathcal{I}(\mathcal{A})\subset\mathcal{I}$. Since P_a is continuous, $P_a^{-1}(U_a)$ and $P_a^{-1}(V_a)$ are open sets containing x and y respectively. This completes the proof.

THEOREM 2.13. – *Let (X_a,τ_a) be a collection of spaces for each a in an index set Δ and $\mathcal{I}$ be an ideal on $\prod X_a$. If $\prod X_a$ is T_2 mod $\mathcal{I}$, then (X_a,τ_a) is T_2 mod $P_a(\mathcal{I})$ for each a in Δ, where each P_a is the projection map.*

PROOF. – It is well known that each X_a is homeomorphic to a subspace Y of the product space $\prod X_a$ and so each $P_a \mid Y : Y \longrightarrow X_a$ is a homeomorphism. By Theorem 2.9, Y is T_2 mod $\mathcal{I}_Y$ and so X_a is T_2 mod $(P_a \mid Y)(\mathcal{I}_Y)$. Since $(P_a \mid Y)(\mathcal{I}_Y) \subset (P_a)(\mathcal{I}_Y) \subset P_a(\mathcal{I})$, each (X_a, τ_a) is T_2 mod $P_a(\mathcal{I})$.

3. – $\mathcal{I}$-compact subsets and T_2 mod $\mathcal{I}$ spaces.

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I} -$ *compact* [7] if for every open cover $\{V_a \mid a \in \Delta\}$ of X, there is a finite subset Λ of Δ such that $A - \cup\{V_a \mid a \in \Lambda\} \in \mathcal{I}$. X is said to be $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset of X. Clearly, (X, τ) is compact if and only if (X, τ) is $\{\emptyset\}-$compact. X is QHC if every open cover of X has a finite subfamily whose closures cover X. (X, τ) is QHC if and only if (X, τ) is $\mathcal{N}-$compact [7] where $\mathcal{N}$ is the ideal of all nowhere dense sets in (X, τ). In [7, Theorem 2.3], it is stated that every $\mathcal{I}$-compact subset of a Hausdorff space is $\tau^\star-$closed. $\tau^\star-$closed sets are also called $\star-$closed sets. Note that, A is $\star-$closed if and only if $A^\star \subset$ A [11]. The following Theorem shows that the condition Hausdorff on X in Theorem 2.3 of [7] can be replaced by T_2 mod $\mathcal{I}$.

THEOREM 3.1. – *Let $(X, \tau, \mathcal{I})$ be a T_2 mod $\mathcal{I}$ space and $A \subset X$. If A is $\mathcal{I}$-compact, then the following holds.*

(a) *For every $x \notin A$, there exist open sets U and V and $I \in \mathcal{I}$ such that $x \in U$, $U \cap V \in \mathcal{I}$ and $A - I \subset V$.*

(b) *A is $\star-$closed.*

PROOF. – (a). For each $y \in A$, since $y \neq x$, there exist open sets U_y containing x and V_y containing y such that $U_y \cap V_y \in \mathcal{I}$. Since $\{V_y \mid y \in A\}$ is an open cover of A, there is a finite subset B of A such that $A - \cup\{V_y \mid y \in B\} \in \mathcal{I}$. If $V = \cup\{V_y \mid y \in B\}$, then V is an open set such that $A - V = I \in \mathcal{I}$ and so $A - I \subset V$. If $U = \cap\{U_y \mid y \in B\}$, then U is an open set containing x. Since $U_y \cap V_y \in \mathcal{I}$ for all $y \in B$, $U \cap V_y \in \mathcal{I}$ for all $y \in B$ and so $U \cap V \in \mathcal{I}$, which proves (a).

(b). If $x \notin A$, by (a), there exist open sets U and V and $I \in \mathcal{I}$ such that $x \in U, U \cap V = J \in \mathcal{I}$ and $A - I \subset V$. Now

$$A \cap U \subset (V \cup I) \cap U = (V \cap U) \cup (I \cap U) = J \cup (I \cap U) = (J \cup I) \cap (J \cup U) \subset J \cup I \in \mathcal{I}.$$

Therefore, $A \cap U \in \mathcal{I}$ for some open set U containing x and so $x \notin A^\star$ which implies that $A^\star \subset A$. Hence A is $\star-$closed.

COROLLARY 3.2. – *Let $(X, \tau, \mathcal{I})$ be a Hausdorff ideal space and $A \subset X$ be $\mathcal{I}$-compact. Then A is $\star-$closed* [7, Theorem 2.3].

If $\mathcal{I}=\{\emptyset\}$, in the above Theorem 3.1(a), then we have the following well known result: In Hausdorff spaces, *a point and a compact subset not containing the point are separated by disjoint open sets.*

THEOREM 3.3. – *Let $(X,\tau,\mathcal{I})$ be a T_2 mod $\mathcal{I}$ space and A and B be disjoint subsets of X which are $\mathcal{I}$-compact. Then there exist open sets G and H and $I,J\in\mathcal{I}$ such that $A-J\subset G,\ B-I\subset H$ and $G\cap H\in\mathcal{I}$.*

PROOF. – Fix a in A. By Theorem 3.1(a), there exist open sets U_a and V_a and $I_1\in\mathcal{I}$ such that $a\in U_a, U_a\cap V_a\in\mathcal{I}$ and $B-I_1\subset V_a$, equivalently, $B-V_a\in\mathcal{I}$. The collection $\{U_a \mid a\in A\}$ is an open cover for A and so there is a finite subset Λ of A such that $A-\cup\{U_a \mid a\in\Lambda\}\in\mathcal{I}$. If $G=\cup\{U_a \mid a\in\Lambda\}$ and $H=\cap\{V_a \mid a\in\Lambda\}$, then G and H are open sets such that $G\cap H\in\mathcal{I}$. Also, $A-G\in\mathcal{I}$ implies that $A-J\subset G$ for some $J\in\mathcal{I}$. Since $B-V_a\in\mathcal{I}$ for each $a\in\Lambda$, $B-H=B-\cap\{V_a \mid a\in\Lambda\}=\cup\{B-V_a \mid a\in\Lambda\}\in\mathcal{I}$ and so $B-I\subset H$ for some $I\in\mathcal{I}$. This completes the proof.

If $\mathcal{I}=\{\emptyset\}$, in the above Theorem 3.3, then we have the following well known result : *In Hausdorff spaces, disjoint compact subsets are separated by disjoint open sets.* If $\mathcal{I}$ is codense, we have Corollary 3.4 and if $\mathcal{I}=\mathcal{N}$, the ideal of all nowhere dense sets, we have Corollary 3.5.

COROLLARY 3.4. – *Let $(X,\tau,\mathcal{I})$ be a Hausdorff ideal space where $\mathcal{I}$ be codense and A and B be disjoint subsets of X which are $\mathcal{I}$-compact. Then, there exist disjoint open sets G and H and $I,J\in\mathcal{I}$ such that $A-J\subset G$ and $B-I\subset H$.*

COROLLARY 3.5. – *Let (X,τ) be a Hausdorff space and A and B be disjoint subsets of X which are $\mathcal{N}-$compact. Then there exist disjoint open sets G and H and nowhere dense sets I and J such that $A-J\subset G$ and $B-I\subset H$.*

THEOREM 3.6. – *The intersection of any collection of $\mathcal{I}$-compact subsets of a T_2 mod $\mathcal{I}$ space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact.*

PROOF. – Let $\{A_a \mid a\in\Delta\}$ be a collection of $\mathcal{I}$-compact subsets. Let $\mathrm{A}=\cap\{A_a \mid a\in\Delta\}$. By Theorem 3.1(b), each A_a is $\star-$closed and so A is $\star-$closed. By Theorem 2.4 of [7], A is $\mathcal{I}$-compact.

The proof of the following Theorem 3.7 is omitted.

THEOREM 3.7. – *The union of a finite collection of $\mathcal{I}$-compact subsets of any ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact.*

A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is said to be *pointwise $\mathcal{I}$-continuous* [12] if $f:(X,\tau^\star,\mathcal{I})\to(Y,\sigma)$ is continuous. Clearly, every continuous function is pointwise $\mathcal{I}$-continuous but the converse need not be true. A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ is called a $\star-$homeomorphism with respect to $\tau,\mathcal{I},\sigma$ and $\mathcal{J}$ [5] if $f:(X,\tau^\star(\mathcal{I}))\to(Y,\sigma^\star(\mathcal{J}))$ is a homeomorphism. We say that f is $\star-$closed if $f:(X,\tau^\star(\mathcal{I}))\to(Y,\sigma^\star(\mathcal{J}))$ is closed. The following Theorem 3.8 shows that in Corollary 2.8 of [7], the condition Hausdorff on Y can be replaced by T_2 mod $\mathcal{I}$. Theorem 3.9 below shows that in Theorem 2.9 of [7], the condition Hausdorff on Y can be replaced by T_2 mod $\mathcal{I}$. Corollary 3.10 and Corollary 3.11 are generalizations of Theorem 2.12 of [7] and Theorem 2.11 of [7], respectively.

THEOREM 3.8. – *Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,f(\mathcal{I}))$ be pointwise $\mathcal{I}$-continuous, $(X,\tau,\mathcal{I})$ be $\mathcal{I}$-compact and $(Y,\sigma,f(\mathcal{I}))$ be T_2 mod $f(\mathcal{I})$. Then $f:(X,\tau)\to(Y,\sigma)$ is a $\star-$closed function.*

PROOF. – Let $A\subset X$ be $\tau^\star-$closed. By Theorem 2.4 of [7], A is $\mathcal{I}$-compact in $(X,\tau,\mathcal{I})$ and so by Lemma 4.1 of [6], A is $\mathcal{I}$-compact in $(X,\tau^\star,\mathcal{I})$. By Theorem 2.2 of [7], $f(A)$ is $f(\mathcal{I})$-compact in Y. By Theorem 3.1(b), $f(A)$ is $\sigma^\star-$closed. Hence f is a $\star-$closed function.

THEOREM 3.9. – *Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,f(\mathcal{I}))$ be a pointwise $\mathcal{I}$-continuous bijection, $(X,\tau,\mathcal{I})$ be $\mathcal{I}$-compact and $(Y,\sigma,f(\mathcal{I}))$ be T_2 mod $f(\mathcal{I})$. Then $f:(X,\tau^\star)\to(Y,\sigma^\star)$ is a homeomorphism.*

PROOF. – By Theorem 3.8, f is $\star-$closed. It is enough to prove that $f:(X,\tau^\star)\to(Y,\sigma^\star)$ is continuous. Let $V\in\sigma^\star(f(\mathcal{I}))$ and $x\in f^{-1}(V)$. Then $f(x)=y\in V$ and so there exist $G\in\sigma$ and $I\in\mathcal{I}$ such that $y\in G-f(I)\subset V$ which implies that $x\in f^{-1}(G)-I\subset f^{-1}(V)$. Since f is pointwise $\mathcal{I}$-continuous and G is $\sigma-$open, $f^{-1}(G)$ is $\tau^\star-$open and so $f^{-1}(V)$ is $\tau^\star-$open, since I is $\tau^\star-$closed. Therefore, $f:(X,\tau^\star)\to(Y,\sigma^\star)$ is continuous. Hence f is a $\star-$homeomorphism.

COROLLARY 3.10. – *Let $(X,\tau,\mathcal{I})$ be an $\mathcal{I}$-compact space and σ be a topology on X such that $\sigma\subset\tau^\star$. If (X,σ,$\mathcal{I}$) is T_2 mod $\mathcal{I}$, then $\tau^\star=\sigma^\star$.*

PROOF. – The identity mapping $i:(X,\tau,\mathcal{I})\to$(X,$\sigma$,$\mathcal{I}$) is pointwise $\mathcal{I}$-continuous and so by Theorem 3.9, i is a $\star-$homeomorphism. Hence $\tau^\star=\sigma^\star$.

COROLLARY 3.11. – *Let $(X,\tau,\mathcal{I})$ be an $\mathcal{I}$-compact, T_2 mod $\mathcal{I}$ space. Then $\tau^\star$ is the largest $\mathcal{I}$-compact topology containing τ.*

PROOF. – Suppose $\tau\subset\sigma$ and $(X,\sigma,\mathcal{I})$ is $\mathcal{I}$-compact. Since $\tau\subset\sigma^\star$ and $(X,\sigma^\star,\mathcal{I})$ is $\mathcal{I}$-compact by Theorem 1.7 of [7], by Corollary 3.10, $\tau^\star=\sigma^\star$ and so $\tau^\star$ is the largest $\mathcal{I}$-compact topology containing τ.

An ideal space $(X, \tau, \mathcal{I})$ is said to be countably $\mathcal{I}$-compact [15] if for every countable open cover $\{V_i \mid i \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of all natural numbers, there is a finite subset $\mathbb{M}$ of $\mathbb{N}$ such that X$- \cup \{V_i \mid i \in \mathbb{M}\} \in \mathcal{I}$. A space (X, τ) is *feebly compact* or *lightly compact* [17] if for every countable open cover $\{V_i \mid i \in \mathbb{N}\}$, there is a finite subset $\mathbb{M}$ of $\mathbb{N}$ such that $X = \cup\{cl(V_i) \mid i \in \mathbb{M}\}$. The following Theorem 3.12 is a generalization of Theorem 1.4(1) of [7].

THEOREM 3.12. – *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact and $\mathcal{I}$ is regular, then (X, τ) is QHC.*

PROOF. – Let $\{V_a \mid a \in \Delta\}$ be an open cover of X. Then there exists a finite subset Λ of Δ such that $X - \cup \{V_a \mid a \in \Lambda\} \in \mathcal{I}$ and so $X - \cup\{cl(V_a) \mid a \in \Lambda\} \in \mathcal{I}$. Since $cl(V_a)$ is regular closed for every $a \in \Lambda$, $X - \cup\{cl(V_a) \mid a \in \Lambda\}$ is regular open and so $X - \cup \{cl(V_a) \mid a \in \Lambda\} = \emptyset$ which implies that X $= \cup\{cl(V_a) \mid a \in \Lambda\}$. Hence X is QHC.

The proof of the following Theorem 3.13 is similar to the proof of Theorem 3.12 which is a generalization of Theorem 2.4(1) of [9].

THEOREM 3.13. – *If $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact and $\mathcal{I}$ is regular, then (X, τ) is lightly compact.*

4. – $\mathcal{I}$-regular spaces.

$\mathcal{I}$-regular space was introduced and studied in [10]. In this section, we further investigate the properties of $\mathcal{I}$-regular spaces. An ideal space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-regular* if for every closed set F and $x \notin F$, there exist disjoint open sets U and V such that $x \in$ U and $F - V \in \mathcal{I}$. Clearly, if $\mathcal{I} = \{\emptyset\}$, regularity and $\mathcal{I}$-regularity of topological spaces coincide. Every regular space is $\mathcal{I}$-regular and the ideal space $(X, \tau, \mathcal{I})$ in Example 2.2 is $\mathcal{I}$-regular but not regular. The following Theorem characterize $\mathcal{I}$-regular spaces.

THEOREM 4.1. – *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) *X is $\mathcal{I}$-regular.*

(b) *For each $x \in X$ and open set U containing x, there is an open set V containing x such that $cl(V) - U \in \mathcal{I}$.*

(c) *For each $x \in X$ and open set U containing x, there is a closed neighborhood F of x such that $F - U \in \mathcal{I}$.*

(d) *For each $x \in X$ and closed set A not containing x, there is an open set V containing x such that $cl(V) \cap A \in \mathcal{I}$.*

PROOF. – (a) $\Rightarrow$ (b). Let $x \in X$ and U be an open set containing x. Then, there exist disjoint open sets V and W such that $x \in V$ and $(X - U) - W \in \mathcal{I}$. If $(X - U) - W = I \in \mathcal{I}$, then $(X - U) \subset W \cup I$. Now $V \cap W = \emptyset \Rightarrow V \subset X - W \Rightarrow cl(V) \subset X - W$ and so $cl(V) - U \subset (X - W) \cap (W \cup I) = (X - W) \cap I \subset I \in \mathcal{I}$.

(b) $\Rightarrow$ (d). Let A be closed in X such that $x \notin A$. Then, there exists an open set V containing x such that $cl(V) - (X - A) \in \mathcal{I}$ which implies that $cl(V) \cap A \in \mathcal{I}$.

(d) $\Rightarrow$ (a). Let A be closed in X such that $x \notin A$. Then, there is an open set V containing x such that $cl(V) \cap A \in \mathcal{I}$. If $cl(V) \cap A = I \in \mathcal{I}$, then $A - (X - cl(V)) = I \in \mathcal{I}$. V and $(X - cl(V))$ are the required disjoint open sets such that $x \in V$ and $A - (X - cl(V)) \in \mathcal{I}$. Hence X is $\mathcal{I}$-regular.

The equivalence of (b) and (c) is clear.

COROLLARY 4.2. – *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be codense. Then the following are equivalent.*

(a) *X is $\mathcal{I}$-regular.*

(b) *For each $x \in X$ and open set U containing x, there is an open set V containing x such that $V^\star - U \in \mathcal{I}$.*

(c) *For each $x \in X$ and closed set A not containing x, there is an open set V containing x such that $V^\star \cap A \in \mathcal{I}$.*

PROOF. – Follows from the fact that $\mathcal{I}$ is codense if and only if $cl(V) = V^\star$ for every open set V [16, Remark 4].

Theorem 4.3 below shows that $\mathcal{I}$-regular is a hereditary property and Theorem 4.4 shows that it is a topological property.

THEOREM 4.3. – *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular and $Y \subset X$, then $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y-$regular.*

PROOF. – Let $F \subset Y$ be closed in Y and $x \in Y$ such that $x \notin F$. Then $F = Y \cap K$, where K is closed in X and also $x \notin K$. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, there exist disjoint open sets U and V in X such that $x \in U$ and $K - V \in \mathcal{I}$. Now $Y \cap U$ and $Y \cap V$ are open sets in Y such that $x \in Y \cap U$ and $(Y \cap U) \cap (Y \cap V) = (U \cap V) \cap Y = \emptyset$. If $K - V = I \in \mathcal{I}$, then $K \subset I \cup V$ and so $Y \cap K \subset Y \cap (I \cup V) = (Y \cap I) \cup (Y \cap V)$ which implies that $F - (Y \cap V) \subset Y \cap I \in \mathcal{I}_Y$. Hence $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y-$regular.

THEOREM 4.4. – *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, f(\mathcal{I}))$ is a homeomorphism, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})-$regular.*

PROOF. – Let F be closed in Y and $y \in Y$ such that $y \notin F$. Let $x = f^{-1}(y)$. Since f is continuous, $f^{-1}(F)$ is a closed set in X not containing x. Since $(X, \tau, \mathcal{I})$ is

$\mathcal{I}$-regular, there exist disjoint open sets U and V in X such that $x \in U$ and $f^{-1}(F) - V \in \mathcal{I}$. Let $f^{-1}(F) - V = I \in \mathcal{I}$. Then $f^{-1}(F) \subset I \cup V$.

$$f^{-1}(F) \subset I \cup V \Rightarrow f(f^{-1}(F)) \subset f(I \cup V) \Rightarrow F \subset f(I) \cup f(V)$$
$$\Rightarrow F - f(V) \subset f(I) \in f(\mathcal{I}) \Rightarrow F - f(V) \in f(\mathcal{I}).$$

Since $f(U)$ and $f(V)$ are disjoint open sets in Y such that $y \in f(U)$ and $F - f(V) \in f(\mathcal{I})$, it follows that $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})-$regular.

THEOREM 4.5. – *Let (X_a, τ_a) be a collection of spaces for each a in an index set Δ and $\mathcal{I}$ be an ideal on $\prod X_a$. If $\prod X_a$ is $\mathcal{I}$-regular, then (X_a, τ_a) is $P_a(\mathcal{I})-$regular for each a in Δ, where P_a is the projection map.*

PROOF. – Similar to Theorem 2.13 and hence omitted.

THEOREM 4.6. – *Let $(X_a, \tau_a, \mathcal{I}_a)$ be a collection of ideal spaces for each a in an index set Δ where each $(X_a, \tau_a, \mathcal{I}_a)$ is $\mathcal{I}_a-$regular. If P_a is the projection map for each a, $\mathcal{A}=\{P_a^{-1}(I_a) \mid I_a \in \mathcal{I}_a,\ a \in \Delta\}$, $\mathcal{I}(\mathcal{A})$ is the smallest ideal generated by $\mathcal{A}$ and $\mathcal{I}$ is an ideal on $\prod X_a$ such that $\mathcal{I} \supset \mathcal{I}(\mathcal{A})$, then $\prod X_a$ with the product topology is $\mathcal{I}$-regular.*

PROOF. – Let $x = (x_a)$ be a point in $\prod X_a$ and U be an open set containing x. Then there exists a basic open set V in $\prod X_a$ such that $x \in V \subset U$. Since V is basic, there is a finite subset Λ of Δ such that $V = \prod V_a$ where $V_a \in \tau_a$ if $a \in \Lambda$ and $V_a = X_a$ if $a \notin \Lambda$. For each $a \in \Lambda$, $x_a = P_a(x) \in P_a(V) = V_a$. Since $(X_a, \tau_a, \mathcal{I}_a)$ is $\mathcal{I}_a-$regular, there exists an τ_a open set G_a such that $x_a \in G_a$ and $cl_a(G_a) - V_a \in \mathcal{I}_a$. Let $G = \cap\{P_a^{-1}(G_a) \mid a \in \Lambda\}$. Then G is an open set containing x and $cl_\pi(G) = \cap\{P_a^{-1}(cl_a(G_a)) \mid a \in \Lambda\}$, where $cl_\pi(A)$ is the closure of A in $\prod X_a$. Now, for each $a \in \Lambda$, $cl_a(G_a) - V_a \in \mathcal{I}_a$ implies that $cl_a(G_a) - V_a = I_a \in \mathcal{I}_a$ and so $cl_a(G_a) - I_a \subset V_a$. Therefore,

$$\cap\{(P_a^{-1}(cl_a(G_a)) - P_a^{-1}(I_a)) \mid a \in \Lambda\} \subset \cap\{P_a^{-1}(V_a) \mid a \in \Lambda\}$$
$$\Rightarrow \cap\{P_a^{-1}(cl_a(G_a)) \mid a \in \Lambda\} - \cup\{P_a^{-1}(I_a) \mid a \in \Lambda\} \subset \cap\{P_a^{-1}(V_a) \mid a \in \Lambda\}$$
$$\Rightarrow \cap\{P_a^{-1}(cl_a(G_a)) \mid a \in \Lambda\} - \cap\{P_a^{-1}(V_a) \mid a \in \Lambda\} \subset \cup\{P_a^{-1}(I_a) \mid a \in \Lambda\} \in \mathcal{I}(\mathcal{A}) \subset \mathcal{I}$$
$$\Rightarrow cl_\pi(G) - V \in \mathcal{I}.$$

Thus, there exists an open set G in $\prod X_a$ such that $x \in G$ and $cl_\pi(G) - V \in \mathcal{I}$ and so $\prod X_a$ with the product topology is $\mathcal{I}$-regular.

THEOREM 4.7. – *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-regular space and x and y be two distinct points in X. Then either $cl(\{x\}) = cl(\{y\})$ or $cl(\{x\}) \cap cl(\{y\}) \in \mathcal{I}$.*

PROOF. – Let $x \in cl(\{y\})$ and $y \in cl(\{x\})$. Then $cl(\{x\}) \subset cl(cl(\{y\})) = cl(\{y\}) \subset cl(cl(\{x\})) = cl(\{x\})$ and so $cl(\{x\}) = cl(\{y\})$. Suppose $y \notin cl(\{x\})$. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, by Theorem 4.1(d), there exists an open set V containing y such that $cl(V) \cap cl(\{x\}) \in \mathcal{I}$. Since $y \in V$, $cl(\{x\}) \cap cl(\{y\}) \subset cl(\{x\}) \cap cl(V) \in \mathcal{I}$. This completes the proof.

THEOREM 4.8. – *Let $(X, \tau, \mathcal{I})$ be an ideal space. If each point of X has a closed neighborhood A which is $\mathcal{I}_A$–regular, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular.*

PROOF. – Let $x \in$X and G be an open set containing x. By hypothesis, there is a closed neighborhood A of x such that (A,$\tau_A, \mathcal{I}_A$) is $\mathcal{I}_A$ regular. Then $A \cap G$ is an open set in A containing x. By Theorem 4.1(c), there is a closed neighborhood F of x in A such that $F - (A \cap G) \in \mathcal{I}_A$. Since F is a closed neighborhood of x in A, F is a closed neighborhood of x in X and $F - (A \cap G) \in \mathcal{I}_A$ implies that $F - G = I \in \mathcal{I}_A \subset \mathcal{I}$. Thus, F is a closed neighborhood of x in X such that $F - G \in \mathcal{I}$. Therefore, $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular.

THEOREM 4.9. – *If an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, then for every none-mpty set A and a closed set F in X such that $F \cap A = \emptyset$, there exist disjoint open sets U and V such that $A \cap U \neq \emptyset$ and $F - V \in \mathcal{I}$.*

PROOF. – Suppose $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular. Let F be closed in X and A be any nonempty set such that $F \cap A = \emptyset$. Then for $x \in A$, there exist disjoint open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$. Clearly, $A \cap U \neq \emptyset$.

THEOREM 4.10. – *If an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, then for every pair of disjoint sets A and B where A is $\mathcal{I}$-compact and B is closed in X, there exist disjoint open sets U and V, and I and J in $\mathcal{I}$ such that $A - I \subset U$ and $B - J \subset V$.*

PROOF. – Suppose $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular and A and B are disjoint subsets of X where A is $\mathcal{I}$-compact and B is closed. For each $x \in A$, there exist disjoint open sets U_x and V_x such that $x \in U_x$ and $B - V_x \in \mathcal{I}$. If $I_x = B - V_x \in \mathcal{I}$, then $B - I_x \subset V_x$. Since $\{U_x \mid x \in A\}$ is an open cover of A, there is a finite subset D of A such that $A - \cup\{U_x \mid x \in D\} \in \mathcal{I}$. If $U = \cup\{U_x \mid x \in D\}$ and $I = A - U \in \mathcal{I}$, then U is an open set such that $A - I \subset U$. If $V = \cap\{V_x \mid x \in D\}$ and $J = \cup\{I_x \mid x \in D\}$, then U and V are disjoint open sets and $J \in \mathcal{I}$. Since $B - V_x = I_x$ for every x in D, $\cup\{B - V_x \mid x \in D\} = J$ and so $B - V = J$ which implies that $B - J \subset V$. This proves the theorem.

COROLLARY 4.11. – *If a space (X, τ) is regular, then for every pair of disjoint sets A and B where A is compact and B is closed in X, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*

THEOREM 4.12. – *Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-regular and A be an $\mathcal{I}$-compact subset of X. If G is an open set containing A, then there exists a closed set F and $I, J \in \mathcal{I}$ such that $A - I \subset F \subset G \cup J$.*

PROOF. – Let A be an $\mathcal{I}$-compact subset of X and G be an open set containing A. Since $X - G$ is a closed set such that $(X - G) \cap A = \emptyset$, by Theorem 4.10, there exist disjoint open sets U and V and I and J in $\mathcal{I}$ such that $A - I \subset U$ and $(X - G) - J \subset V$. $(X - G) - J \subset V \Rightarrow (X - V) \subset G \cup J$ and so $cl(U) \subset G \cup J$. $cl(U)$ is the required closed set such that $A - I \subset cl(U) \subset G \cup J$. Hence the proof.

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 4.12, we have the following well known result. If $\mathcal{I} = \mathcal{N}$ in the above Theorem 4.12, we have Corollary 4.14.

COROLLARY 4.13. – *Let (X, τ) be a regular space and A be a compact subset of X. If G is an open set containing A, then there exists a closed set F such that $A \subset F \subset G$.*

COROLLARY 4.14. – *Let $(X, \tau, \mathcal{N})$ be $\mathcal{N}-$regular and A be an $\mathcal{N}-$compact subset of X. If G is an open set containing A, then there exist a closed set F and nowhere dense sets I and J such that $A - I \subset F \subset G \cup J$.*

In general, a topology larger than a regular topology need not be regular. But for ideal spaces, the following Theorem 4.15 shows that if an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, then $(X, \tau^\star, \mathcal{I})$ is always $\mathcal{I}$-regular.

THEOREM 4.15. – *If an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, then $(X, \tau^\star, \mathcal{I})$ is $\mathcal{I}$-regular.*

PROOF. – Let F be $\tau^\star-$closed and $p \notin F$. Since $X - F$ is a $\tau^\star-$open set containing p, there exists $U \in \tau$ and $I \in \mathcal{I}$ such that $p \in U - I \subset X - F$. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular, there exists an open set V such that $p \in V$ and $cl(V) - U \in \mathcal{I}$, by Theorem 4.1(b).

$$cl(V) - U \in \mathcal{I} \Rightarrow cl(V) - U = J \in \mathcal{I} \Rightarrow cl(V) - J \subset U \Rightarrow cl(V) - J \subset (X - F) \cup I$$
$$\Rightarrow cl(V) - (X - F) \subset J \cup I \in \mathcal{I} \Rightarrow cl(V) \cap F \in \mathcal{I} \Rightarrow cl^\star(V) \cap F \in \mathcal{I}.$$

Hence $(X, \tau^\star, \mathcal{I})$ is $\mathcal{I}$-regular, by Theorem 4.1(d).

The following Example 4.16 shows that the converse of the above Theorem 4.15 is not true but is true, if the ideal is codense, as shown by Theorem 4.17 below.

EXAMPLE 4.16. – Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then $(X, \tau, \mathcal{I})$ is not $\mathcal{I}$-regular. Since $\tau^\star = \wp(X)$, $(X, \tau^\star, \mathcal{I})$ is $\mathcal{I}$-regular.

THEOREM 4.17. – *If* $(X,\tau,\mathcal{I})$ *is an ideal space such that* $(X,\tau^\star,\mathcal{I})$ *is* $\mathcal{I}$*-regular and* $\mathcal{I}$ *is codense, then* $(X,\tau,\mathcal{I})$ *is* $\mathcal{I}$*-regular.*

PROOF. – Let A be closed and $x \notin A$. Since A is $\tau^\star-$closed, by Corollary 4.2(c), there exists a $\tau^\star-$open set V of x such that $V^\star \cap A \in \mathcal{I}$. Since V is a $\tau^\star-$open set containing x, there exists $U \in \tau$ and $I \in \mathcal{I}$ such that $x \in U - I \subset V$. Now $U - I \subset V \Rightarrow U^\star \subset V^\star$ and so $U^\star \cap A \subset V^\star \cap A \in \mathcal{I}$. By Corollary 4.2(c), $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-regular.

The ideal space $(X,\tau,\mathcal{I})$ in Example 4.16 is a T_2 mod $\mathcal{I}$ space which is not $\mathcal{I}$-regular. The ideal space $(X,\tau,\mathcal{I})$ in the following Example 4.18 is $\mathcal{I}$-regular which is not T_2 mod $\mathcal{I}$. Hence $\mathcal{I}$-regular and T_2 mod $\mathcal{I}$ are independent concepts.

EXAMPLE 4.18. – Let $\mathrm{X} = \{\mathrm{a,b,c,d}\}$, $\tau = \{\mathrm{X}, \emptyset, \{\mathrm{c}\}, \{\mathrm{a,b}\}, \{\mathrm{a,b,c}\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$. Then $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-regular space which is not T_2 mod $\mathcal{I}$.

5. – $\mathcal{I}$-Hausdorff and quasi $\mathcal{I}$-Hausdorff Spaces.

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$*-open* [8] if $A \subset int(A^\star)$. The family of all $\mathcal{I}$-open sets is denoted by IO$(X,\tau,\mathcal{I})$ or IO(X,τ) or IO(X). A subset A of an ideal space $(X,\tau,\mathcal{I})$ is *quasi* $\mathcal{I}$*-open* [1] if $A \subset cl(int(A^\star))$. Every $\mathcal{I}$-open set is a quasi $\mathcal{I}$-open set but the converse implication need not be true. The family of all quasi $\mathcal{I}$-open sets is denoted by $Q\mathcal{I}O(X,\tau)$. A space $(X,\tau,\mathcal{I})$ is called $\mathcal{I}$-Hausdorff [2] if for each pair of distinct points, there exist disjoint $\mathcal{I}$-open sets containing the points. A space $(X,\tau,\mathcal{I})$ is called quasi $\mathcal{I}$-Hausdorff [14] if for each pair of distinct points, there exist disjoint quasi $\mathcal{I}$-open sets containing the points. Every $\mathcal{I}$-Hausdorff space is quasi $\mathcal{I}$-Hausdorff [14]. The following examples show that T_2 mod $\mathcal{I}$ and $\mathcal{I}$-Hausdorff (resp. quasi $\mathcal{I}$-Hausdorff) are independent concepts.

EXAMPLE 5.1. – Let $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b,c\}\}$. Then $Q\mathcal{I}O(X,\tau) = IO(X,\tau) = \{\emptyset, \{a\}\}$. $(X,\tau,\mathcal{I})$ is T_2 mod $\mathcal{I}$ but not $\mathcal{I}$-Hausdorff.

EXAMPLE 5.2. – Let $X = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a,b\}\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then

$$Q\mathcal{I}O(X,\tau) = IO(X,\tau)$$
$$= \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}.$$

Then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Hausdorff but for a and b, there exist no open sets U and V such that $U \cap V \in \mathcal{I}$ and so $(X, \tau, \mathcal{I})$ is not T_2 mod $\mathcal{I}$.

Theorem 3.1 of [14] gives a characterization of quasi $\mathcal{I}$-Hausdorff spaces in which it is assumed that (i) if U and V are disjoint quasi $\mathcal{I}$-open sets, then $cl(U) \cap V = \emptyset$ and (ii) every open set is a quasi $\mathcal{I}$-open set. In Example 5.1 above, $\{a, b\}$ is an open set but not a quasi $\mathcal{I}$-open set and so condition (ii) is false. The following Example 5.3 shows that the condition (i) is also false.

EXAMPLE 5.3. – Consider the ideal space $(X, \tau, \mathcal{I})$ of Example 5.2. $\{a\}$ and $\{b\}$ are disjoint quasi $\mathcal{I}$-open sets containing a and b, respectively, such that $cl(\{a\}) = cl(\{b\}) = X$ and so $cl(\{a\}) \cap \{b\} \neq \emptyset$.

The following Theorem 5.4 gives a sufficient condition for an ideal space $(X, \tau, \mathcal{I})$ to be a quasi $\mathcal{I}$-Hausdorff space, if $\mathcal{I}$ is codense.

THEOREM 5.4. – *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be codense. Then $(X, \tau, \mathcal{I})$ is quasi $\mathcal{I}$-Hausdorff, if for every pair of distinct points x and $y \in X$, there exists a quasi $\mathcal{I}$-open set U such that $x \in U \subset U^\star \subset X - \{y\}$.*

PROOF. – Suppose the condition holds. Let x and $y \in X$ such that $x \neq y$. Then there exists a quasi $\mathcal{I}$-open set U such that $x \in U \subset U^\star \subset X - \{y\}$ and so $y \in X - U^\star$. Let $V = X - U^\star$. Then V is open. Since $\mathcal{I}$ is codense, $V \subset V^\star$ and so $U \cap V = \emptyset$. Since $cl(int(V^\star)) \supset cl(int(V)) = cl(V) \supset V$, V is quasi $\mathcal{I}$-open. Hence $(X, \tau, \mathcal{I})$ is quasi $\mathcal{I}$-Hausdorff.

The following Example 5.5 shows that the converse of the above theorem need not be true.

EXAMPLE 5.5. – Consider the ideal space $(X, \tau, \mathcal{I})$ of Example 5.2. $(X, \tau, \mathcal{I})$ is quasi $\mathcal{I}$-Hausdorff. For the points a and $b \in X$, there exists no quasi $\mathcal{I}$-open set U such that $a \in U \subset U^\star \subset X - \{b\}$.

REFERENCES

[1] M.E. ABD EL-MONSEF - R.A. MAHMOUD - A. A. NASEF, *On quasi $\mathcal{I}$-openness and quasi $\mathcal{I}$-continuity,* Tamkang J. Maths., **31** (2000), 101-108.

[2] J. DONTCHEV, *On Hausdorff spaces via topological ideals and $\mathcal{I}$-irresolute functions*, Annals of the New York Academy of Sciences, **767** (1995), 28-38.

[3] J. DONTCHEV - M. GANSTER, *On compactness with respect to countable extensions of ideals and the generalized Banach category Theorem*, Acta. Math. Hung., **88** (2000), 53-58.

[4] J. DONTCHEV - M. GANSTER - D. ROSE, *Ideal resolvability*, Topology and its Applications, **93** (1999), 1-16.
[5] T. R. HAMLETT - D. ROSE, $\star-$*topological properties*, Inter. J. Math. and Math. Sci., **13** (1990), 507-512.
[6] T. R. HAMLETT - D. ROSE, *Local compactness with respect to an ideal*, Kyungpook Math. J., **32** (1992), 31-43.
[7] T.R. HAMLETT - D. JANKOVIC, *Compactness with respect to an Ideal*, Boll. U.M.I., (7) **4**-B (1990), 849-861.
[8] D. JANKOVIC - T.R. HAMLETT, *Compatible Extensions of Ideals*, Boll.U.M.I., (7) **6**-B (1992), 453-465.
[9] T. R. HAMLETT - D. JANKOVIC - D. ROSE, *Countable Compactness with respect to an Ideal*, Math. Chronicle, **20** (1991), 109-126.
[10] T. R. HAMLETT - D. JANKOVIC, *On weaker forms of paracompactness,countable compactness and Lindelofness*, Annals of the New York Academy of Sciences, Papers on General Topology and Applications, **728** (1994), 41-49.
[11] D. JANKOVIC - T.R. HAMLETT, *New Topologies from old via Ideals*, Amer. Math. Monthly, **97**, No. 4 (1990), 295-310.
[12] J. KANIEWSKI - Z. PIOTROWSKI, *Concerning continuity apart from a meager set*, Proc. Amer. Math. Soc., **98** (1986), 324-328.
[13] K. KURATOWSKI, *Topology, Vol I*, Academic Press, New York,1966.
[14] A. A. NASEF, *On Hausdorff spaces via ideals and quasi $\mathcal{I}$-irresolute functions*, Chaos, Solitons and Fractals, **14** (2002), 619-625.
[15] R. L. NEWCOMB, *Topologies which are compact modulo an ideal*, Ph.D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.
[16] D. A. ROSE - D. JANKOVIC, *On functions having the property of Baire*, Real Analysis Exchange, **19** (2) (1993/94), 589-597.
[17] C. T. SCARBOROUGH - A. H. STONE, *Product of Nearly compact Spaces*, Trans. Amer. Math. Soc., **124** (1966), 131-147.
[18] R. VAIDYANATHASWAMY, *The Localization Theory in Set Topology*, Proc. Indian Acad. Sci., **20** (1945), 51-61.
[19] R. VAIDYANATHASWAMY, *Set Topology*, Chelsea Publishing Company, 1960.

D. Sivaraj: Department of Computer Applications,
D.J. Academy for Managerial Excellence, Coimbatore-641 032, Tamil Nadu, India.
E-mail: ttn_sivaraj@sancharnet.in

V. Renuka Devi: Department of Mathematics, Govindammal Aditanar College for Women,
Tiruchendur-628 215, Tamil Nadu, India.
E-mail: renu_siva2003@yahoo.com

Pervenuta in Redazione
il 3 novembre 2006