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Some Separation Axioms Via Ideals

D. SIVARAJ - V. RENUKA DEVI

Summary. – We introduce a new class of spaces, called Hausdorff modulo $I$ or $T_2$ mod $I$ spaces with respect to an ideal $I$, which contains the class of all Hausdorff spaces. Characterizations of these spaces are given and their properties are investigated. The concept of compactness modulo an ideal $I$ was introduced by Newcomb in 1967 and studied by Hamlett and Jankovic in 1990. We study the properties of $I$-compact subsets in Hausdorff modulo $I$ spaces and generalize some results of Hamlett and Jankovic. $I$-regular space was introduced by Hamlett and Jankovic in 1994. We further investigate the concept of $I$-regularity with regard to its preservation by functions, subspaces and product.

1. – Introduction and Preliminaries.

The subject of ideals in topological spaces has been studied by Kuratowski [13] and Vaidyanathaswamy [19]. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $\phi(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : \phi(X) \to \phi(X)$, called a local function [13] of $A$ with respect to $\tau$ and $I$ is defined as follows: for $A \subset X, A^*(I, \tau) = \{x \in X | U \cap A \not\in I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. We will make use of the basic facts concerning the local functions [11, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $\text{cl}^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than $\tau$ is defined by $\text{cl}^*(A) = A \cup A^*(I, \tau)[18]$ and $\beta = \{U - I | U \in \tau \text{ and } I \in I\}$ is a base for $\tau^*[11]$. When there is no chance for confusion, we will simply write $A^*$ for $A^*(I, \tau)$.
and \( \tau^* \) or \( \tau^*(\mathcal{I}) \) for \( \tau^*(\mathcal{I}, \tau) \). If \( \mathcal{I} \) is an ideal on \( X \), then \((X, \tau, \mathcal{I})\) is called an ideal space. By a space, we always mean a topological space \((X, \tau)\) with no separation properties assumed. If \( A \subset X \), \( cl(A) \) and \( int(A) \) will, respectively, denote the closure and interior of \( A \) in \((X, \tau)\) and \( cl^*(A) \) will denote the closure of \( A \) in \((X, \tau^*)\). An open subset \( A \) of a space \((X, \tau)\) is said to be \textit{regular open} if \( A = int(cl(A)) \). The complement of a regular open set is \textit{regular closed}. Given an ideal space \((X, \tau, \mathcal{I})\), \( \mathcal{I} \) is said to be \textit{codense} [4] if \( \mathcal{I} \cap \tau = \{\emptyset\} \) and \( \mathcal{I} \) is said to be \textit{regular} [3], if \( RO(X) \cap \mathcal{I} = \{\emptyset\} \) where \( RO(X) \) is the family of all regular open sets. Every codense ideal is regular but the converse need not be true [3].

2. – Hausdorff modulo an Ideal.

\textbf{Definition 2.1.} – An ideal space \((X, \tau, \mathcal{I})\) is said to be \textit{Hausdorff mod} \( \mathcal{I} \) or \( T_2 \mod \mathcal{I} \) if for every pair of distinct points \( x \) and \( y \) in \( X \), there exist open sets \( U \) and \( V \) such that \( x \in U, \ y \in V \) and \( U \cap V \in \mathcal{I} \).

Every Hausdorff space is \( T_2 \mod \mathcal{I} \), since \( \emptyset \in \mathcal{I} \). The converse need not be true as the following Example shows.

\textbf{Example 2.2.} – Let \( X = \{a, b, c\} \), \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \) and \( \mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \). Then \( X \) is \( T_2 \mod \mathcal{I} \) but not Hausdorff (even it is not \( T_1 \)).

The easy proof of the following Theorems are omitted.

\textbf{Theorem 2.3.} – A space \((X, \tau)\) is Hausdorff if and only if \((X, \tau)\) is \( T_2 \mod \{\emptyset\} \).

\textbf{Theorem 2.4.} – If an ideal space \((X, \tau, \mathcal{I})\) is \( T_2 \mod \mathcal{I} \) and \( \mathcal{I} \subset \mathcal{J} \), then \((X, \tau, \mathcal{J})\) is \( T_2 \mod \mathcal{J} \).

\textbf{Theorem 2.5.} – If \((X, \tau, \mathcal{I})\) is a \( T_2 \mod \mathcal{I} \) space and \( \mathcal{I} \) is codense, then \((X, \tau, \mathcal{I})\) is Hausdorff.

The following Theorem gives a characterization of \( T_2 \mod \mathcal{I} \) spaces.

\textbf{Theorem 2.6.} – Let \((X, \tau, \mathcal{I})\) be an ideal space. Then the following are equivalent.

(a) \((X, \tau, \mathcal{I})\) is \( T_2 \mod \mathcal{I} \).

(b) If \( x \in X \), then for each \( y \neq x \), there exists an open set \( U \) containing \( x \) such that \( y \notin U^* \).

(c) For every \( x \in X \), \( \cap\{U_x^* \mid U_x \in \tau(x)\} \) is either \( \emptyset \) or \( \{x\} \).
proof. – (a) ⇒ (b). Let \( x \in X \) and \( y \in X \) such that \( x \neq y \). Then there exist open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V \in \mathcal{I} \). \( U \cap V \in \mathcal{I} \) implies that \((U \cap V)^* = \emptyset\) and so \( U^* \cap V = \emptyset \). Thus \( y \not\in U^* \) which proves (b).

(b) ⇒ (c). If \( x \in X \) and \( y \in X \) such that \( y \neq x \), by hypothesis, there is an open set \( U_x \) containing \( x \) such that \( y \not\in U_x^* \). This implies that \( y \not\in \cap(U_x^* \mid U_x \in \tau(x)) \) and so \( \cap(U_x^* \mid U_x \in \tau(x)) \) is either \( \emptyset \) or \( \{x\} \).

(c) ⇒ (a). Let \( x \in X \) and \( y \in X \) such that \( x \neq y \). By hypothesis, \( y \not\in \cap(U_x^* \mid U_x \in \tau(x)) \) which implies that \( y \not\in U_x^* \) for some \( U_x \in \tau(x) \). Therefore, there exists \( V_y \in \tau(y) \) such that \( U_x \cap V_y \in \mathcal{I} \) and so \((X, \tau, \mathcal{I}) \) is \( T_2 \) mod \( \mathcal{I} \).

The following Theorem 2.7 shows that \( T_2 \) mod \( \mathcal{I} \) is a property shared by both \((X, \tau)\) and \((X, \tau^*)\). Corollary 2.8 below follows from Theorems 2.5 and 2.7.

**Theorem 2.7.** – Let \((X, \tau, \mathcal{I})\) be an ideal space. Then \((X, \tau)\) is \( T_2 \) mod \( \mathcal{I} \) if and only if \((X, \tau^*)\) is \( T_2 \) mod \( \mathcal{I} \).

**Proof.** – If \((X, \tau)\) is \( T_2 \) mod \( \mathcal{I} \), then clearly, \((X, \tau^*)\) is \( T_2 \) mod \( \mathcal{I} \). Conversely, suppose \((X, \tau^*)\) is \( T_2 \) mod \( \mathcal{I} \). Let \( x \in X \) and \( y \in X \) such that \( x \neq y \). Then there exist \( \tau^* \)-open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V \in \mathcal{I} \). Moreover, there exist open sets \( G \) and \( H \) and \( I_1, I_2 \in \mathcal{I} \) such that \( x \in G - I_1 \subseteq U \) and \( y \in H - I_2 \subseteq V \). Therefore, \((G \cap H) - (I_1 \cup I_2) = (G - I_1) \cap (H - I_2) \subseteq (U \cap V) \in \mathcal{I} \), which implies that \( G \cap H \in \mathcal{I} \). Hence \((X, \tau)\) is \( T_2 \) mod \( \mathcal{I} \).

**Corollary 2.8.** – Let \((X, \tau, \mathcal{I})\) be an ideal space where \( \mathcal{I} \) be codense. Then the following are equivalent.

(a) \((X, \tau, \mathcal{I})\) is Hausdorff.
(b) \((X, \tau, \mathcal{I})\) is \( T_2 \) mod \( \mathcal{I} \).
(c) \((X, \tau^*, \mathcal{I})\) is \( T_2 \) mod \( \mathcal{I} \).
(d) \((X, \tau^*, \mathcal{I})\) is Hausdorff.

If \( X \) is any set, \( A \subseteq X \) and \( \mathcal{I} \) is an ideal on \( X \), then \( \mathcal{I}_A = \{A \cap I \mid I \in \mathcal{I}\} = \{I \in \mathcal{I} \mid I \subseteq A\} \) is an ideal on \( A \) [11]. The rest of the section is devoted to subspaces, product spaces and images and preimages of \( T_2 \) mod \( \mathcal{I} \) spaces.

**Theorem 2.9.** – Let \((X, \tau, \mathcal{I})\) be a \( T_2 \) mod \( \mathcal{I} \) space and \( A \subseteq X \). Then \((A, \tau_A, \mathcal{I}_A)\) is \( T_2 \) mod \( \mathcal{I}_A \).

**Proof.** – Let \( x \) and \( y \) be two distinct points of \( A \). Then there exist open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V \in \mathcal{I} \). Now \( x \in U \cap A \subseteq \tau_A \), \( y \in V \cap A \subseteq \tau_A \) and \((U \cap A) \cap (V \cap A) = (U \cap V) \cap A \in \mathcal{I}_A \). Therefore, \((A, \tau_A)\) is \( T_2 \) mod \( \mathcal{I}_A \).
If \( f : (X, \tau, I) \to (Y, \sigma) \) is a mapping, then \( f(I) = \{ f(I) \mid I \in I \} \) is an ideal on \( Y \) [7, Theorem 1.8]. If \( f \) is an injection and \( J \) is any ideal on \( Y \), then \( f^{-1}(J) = \{ f^{-1}(J) \mid J \in J \} \) is an ideal on \( X \) [7, Theorem 1.11].

**Theorem 2.10.** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be an open bijection and \( (X, \tau, I) \) be \( T_2 \mod I \). Then \( (Y, \sigma) \) is \( T_2 \mod f(I) \).

**Proof.** Let \( y \) and \( u \) be distinct points in \( Y \). Since \( f \) is bijective, there exist distinct points \( x \) and \( v \) in \( X \) such that \( f(x) = y \) and \( f(v) = u \). Since \( (X, \tau, I) \) is \( T_2 \mod I \), there exist open sets \( U \) and \( V \) in \( X \) such that \( x \in U, v \in V \) and \( U \cap V \in I \). If \( U \cap V = I \in I \), then \( f(U) \cap f(V) = f(U \cap V) = f(I) \in f(I) \). Since \( f \) is open, \( f(U) \) and \( f(V) \) are open sets in \( Y \) such that \( y = f(x) \in f(U) \) and \( u = f(v) \in f(V) \). Therefore, \( (Y, \sigma) \) is \( T_2 \mod f(I) \).

**Theorem 2.11.** Let \( f : (X, \tau) \to (Y, \sigma, J) \) be a continuous injection map where \( (Y, \sigma, J) \) be \( T_2 \mod J \). Then \( (X, \tau) \) is \( T_2 \mod f^{-1}(J) \).

**Proof.** Let \( x \) and \( v \) be distinct points in \( X \). Since \( (Y, \sigma, J) \) is \( T_2 \mod J \), there exist open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U, f(v) \in V \) and \( U \cap V \in J \). If \( U \cap V = J \in J \), then \( f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(J) \in f^{-1}(J) \). Since \( f^{-1}(U) \) and \( f^{-1}(V) \) are open sets containing \( x \) and \( v \) respectively, \( (X, \tau) \) is \( T_2 \mod f^{-1}(J) \).

Given a space \( (X, \tau) \) and a collection \( A \) of subsets of \( X \), the smallest ideal on \( X \) containing \( A \) will be denoted by \( I(A) \) and is called the ideal generated by \( A \) [7]. It is easily seen that \( I(A) = \{ A \mid A \subset A_1 \cup A_2 \cup \ldots \cup A_n, A_i \in A \text{ for each } i, n \in \mathbb{N} \} \), where \( \mathbb{N} \) is the set of all natural numbers.

**Theorem 2.12.** Let \( (X_a, \tau_a, I_a) \) be a collection of ideal spaces for each \( a \) in an index set \( A \), where each \( (X_a, \tau_a, I_a) \) be \( T_2 \mod I_A \). If \( P_a \) is the projection map for each \( a \), \( A = \{ P_a^{-1}(I_a) \mid I_a \in I_A, a \in A \} \), \( I(A) \) is the smallest ideal generated by \( A \) and \( I \) is an ideal on \( \prod X_a \) such that \( I \supseteq I(A) \), then \( \prod X_a \) with the product topology is \( T_2 \mod I \).

**Proof.** Let \( x \) and \( y \) be points in \( \prod X_a \) such that \( x \neq y \). Then for some \( a \in A, x_a \neq y_a \). Since each \( (X_a, \tau_a) \) is \( T_2 \mod I_a \), there exist open sets \( U_a \) and \( V_a \) such that \( x_a \in U_a, y_a \in V_a \) and \( U_a \cap V_a \in I_a \). If \( U_a \cap V_a = I_a \in I_a \), then \( P_a^{-1}(U_a) \cap P_a^{-1}(V_a) = P_a^{-1}(U_a \cap V_a) = P_a^{-1}(I_a) \in I(A) \subset I \). Since \( P_a \) is continuous, \( P_a^{-1}(U_a) \) and \( P_a^{-1}(V_a) \) are open sets containing \( x \) and \( y \) respectively. This completes the proof.

**Theorem 2.13.** Let \( (X_a, \tau_a) \) be a collection of spaces for each \( a \) in an index set \( A \) and \( I \) be an ideal on \( \prod X_a \). If \( \prod X_a \) is \( T_2 \mod I \), then \( (X_a, \tau_a) \) is \( T_2 \mod P_a(I) \) for each \( a \in A \), where each \( P_a \) is the projection map.
3. – $\mathcal{I}$-compact subsets and $T_2 \mod \mathcal{I}$ spaces.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$–compact [7] if for every open cover $\{V_a \mid a \in A\}$ of $X$, there is a finite subset $A$ of $A$ such that $A = \bigcup \{V_a \mid a \in A\} \in \mathcal{I}$. $X$ is said to be $\mathcal{I}$-compact if $X$ is $\mathcal{I}$-compact as a subset of $X$. Clearly, $(X, \tau)$ is compact if and only if $(X, \tau)$ is $\emptyset$–compact. $X$ is QHC if every open cover of $X$ has a finite subfamily whose closures cover $X$. $(X, \tau)$ is QHC if and only if $(X, \tau)$ is $\mathcal{N}$–compact [7] where $\mathcal{N}$ is the ideal of all nowhere dense sets in $(X, \tau)$. In [7, Theorem 2.3], it is stated that every $\mathcal{I}$-compact subset of a Hausdorff space is $\tau^*\mathcal{I}$–closed. $\tau^\ast\mathcal{I}$-closed sets are also called $\ast\mathcal{I}$-closed sets. Note that, $A$ is $\ast\mathcal{I}$-closed if and only if $A^\ast \subset A$ [11]. The following Theorem shows that the condition Hausdorff on $X$ in Theorem 2.3 of [7] can be replaced by $T_2 \mod \mathcal{I}$.

**Theorem 3.1.** – Let $(X, \tau, \mathcal{I})$ be a $T_2 \mod \mathcal{I}$ space and $A \subset X$. If $A$ is $\mathcal{I}$-compact, then the following holds.

(a) For every $x \notin A$, there exist open sets $U$ and $V$ and $I \in \mathcal{I}$ such that $x \in U$, $U \cap V \in \mathcal{I}$ and $A \cap I \subset V$.

(b) $A$ is $\ast\mathcal{I}$-closed.

**Proof.** – (a). For each $y \in A$, since $y \neq x$, there exist open sets $U_y$ containing $x$ and $V_y$ containing $y$ such that $U_y \cap V_y \in \mathcal{I}$. Since $\{V_y \mid y \in A\}$ is an open cover of $A$, there is a finite subset $B$ of $A$ such that $A = \bigcup \{V_y \mid y \in B\} \in \mathcal{I}$. If $V = \bigcup \{V_y \mid y \in B\}$, then $V$ is an open set such that $A \cap V = I \in \mathcal{I}$ and so $A \cap I \subset V$. If $U = \bigcap \{U_y \mid y \in B\}$, then $U$ is an open set containing $x$. Since $U_y \cap V_y \in \mathcal{I}$ for all $y \in B$, $U \cap V \in \mathcal{I}$ for all $y \in B$ and so $U \cap V \in \mathcal{I}$, which proves (a).

(b). If $x \notin A$, by (a), there exist open sets $U$ and $V$ and $I \in \mathcal{I}$ such that $x \in U$, $U \cap V = I \in \mathcal{I}$ and $A \cap I \subset V$. Now $A \cap U \subset (V \cup I) \cap U = (V \cap U) \cup (I \cap U) = (J \cup I) \cap (J \cup U) \subset J \cup I \in \mathcal{I}$. Therefore, $A \cap U \in \mathcal{I}$ for some open set $U$ containing $x$ and so $x \notin A^\ast$ which implies that $A^\ast \subset A$. Hence $A$ is $\ast\mathcal{I}$-closed.

**Corollary 3.2.** – Let $(X, \tau, \mathcal{I})$ be a Hausdorff ideal space and $A \subset X$ be $\mathcal{I}$-compact. Then $A$ is $\ast\mathcal{I}$-closed [7, Theorem 2.3].
If $I = \{\emptyset\}$, in the above Theorem 3.1(a), then we have the following well known result: In Hausdorff spaces, a point and a compact subset not containing the point are separated by disjoint open sets.

**Theorem 3.3.** – Let $(X, \tau, I)$ be a $T_2$ mod $I$ space and $A$ and $B$ be disjoint subsets of $X$ which are $I$-compact. Then there exist open sets $G$ and $H$ and $I, J \in I$ such that $A - J \subset G$, $B - I \subset H$ and $G \cap H \in I$.

**Proof.** – Fix $a$ in $A$. By Theorem 3.1(a), there exist open sets $U_a$ and $V_a$ and $I_1 \in I$ such that $a \in U_a, U_a \cap V_a \in I$ and $B - I_1 \subset V_a$, equivalently, $B - V_a \in I$. The collection $\{U_a \mid a \in A\}$ is an open cover for $A$ and so there is a finite subset $A'$ of $A$ such that $A - \bigcup\{U_a \mid a \in A'\} \in I$. If $G = \bigcup\{U_a \mid a \in A'\}$ and $H = \bigcap\{V_a \mid a \in A\}$, then $G$ and $H$ are open sets such that $G \cap H \in I$. Also, $A - G \in I$ implies that $A - J \subset G$ for some $J \in I$. Since $B - V_a \in I$ for each $a \in A$, $B - H = B - \bigcap\{V_a \mid a \in A\} = \bigcup\{B - V_a \mid a \in A\} \in I$ and so $B - I \subset H$ for some $I \in I$. This completes the proof.

If $I = \{\emptyset\}$, in the above Theorem 3.3, then we have the following well known result: In Hausdorff spaces, disjoint compact subsets are separated by disjoint open sets. If $I$ is codense, we have Corollary 3.4 and if $I = N$, the ideal of all nowhere dense sets, we have Corollary 3.5.

**Corollary 3.4.** – Let $(X, \tau, I)$ be a Hausdorff ideal space where $I$ be codense and $A$ and $B$ be disjoint subsets of $X$ which are $I$-compact. Then, there exist disjoint open sets $G$ and $H$ and $I, J \in I$ such that $A - J \subset G$ and $B - I \subset H$.

**Corollary 3.5.** – Let $(X, \tau)$ be a Hausdorff space and $A$ and $B$ be disjoint subsets of $X$ which are $N$-compact. Then there exist disjoint open sets $G$ and $H$ and nowhere dense sets $I$ and $J$ such that $A - J \subset G$ and $B - I \subset H$.

**Theorem 3.6.** – The intersection of any collection of $I$-compact subsets of a $T_2$ mod $I$ space $(X, \tau, I)$ is $I$-compact.

**Proof.** – Let $\{A_a \mid a \in A\}$ be a collection of $I$-compact subsets. Let $A = \bigcap\{A_a \mid a \in A\}$. By Theorem 3.1(b), each $A_a$ is $\ast$-closed and so $A$ is $\ast$-closed. By Theorem 2.4 of [7], $A$ is $I$-compact.

The proof of the following Theorem 3.7 is omitted.

**Theorem 3.7.** – The union of a finite collection of $I$-compact subsets of any ideal space $(X, \tau, I)$ is $I$-compact.

A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be pointwise $I$-continuous [12] if $f : (X, \tau^*, I) \to (Y, \sigma)$ is continuous. Clearly, every continuous function is
pointwise $\mathcal{I}$-continuous but the converse need not be true. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called a $\ast$-homeomorphism with respect to $\tau, \mathcal{I}, \sigma$ and $\mathcal{J}$ [5] if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma^*(\mathcal{J}))$ is a homeomorphism. We say that $f$ is $\ast$-closed if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma^*(\mathcal{J}))$ is closed. The following Theorem 3.8 shows that in Corollary 2.8 of [7], the condition Hausdorff on $Y$ can be replaced by $T_2 \mod \mathcal{I}$. Theorem 3.9 below shows that in Theorem 2.9 of [7], the condition Hausdorff on $Y$ can be replaced by $T_2 \mod \mathcal{I}$. Corollary 3.10 and Corollary 3.11 are generalizations of Theorem 2.12 of [7] and Theorem 2.11 of [7], respectively.

**Theorem 3.8.** Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, f(\mathcal{I}))$ be pointwise $\mathcal{I}$-continuous, $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-compact and $(Y, \sigma, f(\mathcal{I}))$ be $T_2 \mod f(\mathcal{I})$. Then $f : (X, \tau) \to (Y, \sigma)$ is a $\ast$-closed function.

**Proof.** Let $A \subset X$ be $\tau^*$-closed. By Theorem 2.4 of [7], $A$ is $\mathcal{I}$-compact in $(X, \tau, \mathcal{I})$ and so by Lemma 4.1 of [6], $A$ is $\mathcal{I}$-compact in $(X, \tau^*, \mathcal{I})$. By Theorem 2.2 of [7], $f(A)$ is $f(\mathcal{I})$-compact in $Y$. By Theorem 3.1(b), $f(A)$ is $\sigma^*$-closed. Hence $f$ is a $\ast$-closed function.

**Theorem 3.9.** Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, f(\mathcal{I}))$ be a pointwise $\mathcal{I}$-continuous bijection, $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-compact and $(Y, \sigma, f(\mathcal{I}))$ be $T_2 \mod f(\mathcal{I})$. Then $f : (X, \tau^*) \to (Y, \sigma^*)$ is a homeomorphism.

**Proof.** By Theorem 3.8, $f$ is $\ast$-closed. It is enough to prove that $f : (X, \tau^*) \to (Y, \sigma^*)$ is continuous. Let $V \in \sigma^*(f(\mathcal{I}))$ and $x \in f^{-1}(V)$. Then $f(x) = y \in V$ and so there exist $G \in \sigma$ and $I \in \mathcal{I}$ such that $y \in G - f(I) \subset V$ which implies that $x \in f^{-1}(G) - I \subset f^{-1}(V)$. Since $f$ is pointwise $\mathcal{I}$-continuous and $G$ is $\sigma$-open, $f^{-1}(G)$ is $\tau^*$-open and so $f^{-1}(V)$ is $\tau^*$-open, since $I$ is $\tau^*$-closed. Therefore, $f : (X, \tau^*) \to (Y, \sigma^*)$ is continuous. Hence $f$ is a $\ast$-homeomorphism.

**Corollary 3.10.** Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-compact space and $\sigma$ be a topology on $X$ such that $\sigma \subset \tau^*$. If $(X, \tau, \mathcal{I})$ is $T_2 \mod \mathcal{I}$, then $\tau^* = \sigma^*$.

**Proof.** The identity mapping $i : (X, \tau, \mathcal{I}) \to (X, \tau, \mathcal{I})$ is pointwise $\mathcal{I}$-continuous and so by Theorem 3.9, $i$ is a $\ast$-homeomorphism. Hence $\tau^* = \sigma^*$.

**Corollary 3.11.** Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-compact, $T_2 \mod \mathcal{I}$ space. Then $\tau^*$ is the largest $\mathcal{I}$-compact topology containing $\tau$.

**Proof.** Suppose $\tau \subset \sigma$ and $(X, \sigma, \mathcal{I})$ is $\mathcal{I}$-compact. Since $\tau \subset \sigma^*$ and $(X, \sigma^*, \mathcal{I})$ is $\mathcal{I}$-compact by Theorem 1.7 of [7], by Corollary 3.10, $\tau^* = \sigma^*$ and so $\tau^*$ is the largest $\mathcal{I}$-compact topology containing $\tau$. 
An ideal space \((X, \tau, \mathcal{I})\) is said to be countably \(\mathcal{I}\)-compact [15] if for every countable open cover \(\{V_i \mid i \in \mathbb{N}\}\), where \(\mathbb{N}\) is the set of all natural numbers, there is a finite subset \(\mathcal{M}\) of \(\mathbb{N}\) such that \(X - \bigcup \{V_i \mid i \in \mathcal{M}\} \in \mathcal{I}\). A space \((X, \tau)\) is feebly compact or lightly compact [17] if for every countable open cover \(\{V_i \mid i \in \mathbb{N}\}\), there is a finite subset \(\mathcal{M}\) of \(\mathbb{N}\) such that \(X = \bigcup \{cl(V_i) \mid i \in \mathcal{M}\}\). The following Theorem 3.12 is a generalization of Theorem 1.4(1) of [7].

**Theorem 3.12.** – If \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-compact and \(\mathcal{I}\) is regular, then \((X, \tau)\) is QHC.

**Proof.** – Let \(\{V_a \mid a \in A\}\) be an open cover of \(X\). Then there exists a finite subset \(A\) of \(A\) such that \(X - \bigcup \{V_a \mid a \in A\} \in \mathcal{I}\) and so \(X - \bigcup \{cl(V_a) \mid a \in A\} \in \mathcal{I}\). Since \(cl(V_a)\) is regular for every \(a \in A\), \(X - \bigcup \{cl(V_a) \mid a \in A\}\) is regular open and so \(X - \bigcup \{cl(V_a) \mid a \in A\} = \emptyset\) which implies that \(X = \bigcup \{cl(V_a) \mid a \in A\}\). Hence \(X\) is QHC.

The proof of the following Theorem 3.13 is similar to the proof of Theorem 3.12 which is a generalization of Theorem 2.4(1) of [9].

**Theorem 3.13.** – If \((X, \tau, \mathcal{I})\) is countably \(\mathcal{I}\)-compact and \(\mathcal{I}\) is regular, then \((X, \tau)\) is lightly compact.

4. – \(\mathcal{I}\)-regular spaces.

\(\mathcal{I}\)-regular space was introduced and studied in [10]. In this section, we further investigate the properties of \(\mathcal{I}\)-regular spaces. An ideal space \((X, \tau, \mathcal{I})\) is said to be \(\mathcal{I}\)-regular if for every closed set \(F\) and \(x \not\in F\), there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(F - V \in \mathcal{I}\). Clearly, if \(\mathcal{I} = \{\emptyset\}\), regularity and \(\mathcal{I}\)-regularity of topological spaces coincide. Every regular space is \(\mathcal{I}\)-regular and the ideal space \((X, \tau, \mathcal{I})\) in Example 2.2 is \(\mathcal{I}\)-regular but not regular. The following Theorem characterize \(\mathcal{I}\)-regular spaces.

**Theorem 4.1.** – Let \((X, \tau, \mathcal{I})\) be an ideal space. Then the following are equivalent.

(a) \(X\) is \(\mathcal{I}\)-regular.

(b) For each \(x \in X\) and open set \(U\) containing \(x\), there is an open set \(V\) containing \(x\) such that \(cl(V) - U \in \mathcal{I}\).

(c) For each \(x \in X\) and open set \(U\) containing \(x\), there is a closed neighborhood \(F\) of \(x\) such that \(F - U \in \mathcal{I}\).

(d) For each \(x \in X\) and closed set \(A\) not containing \(x\), there is an open set \(V\) containing \(x\) such that \(cl(V) \cap A \in \mathcal{I}\).
Proof. – (a) ⇒ (b). Let \( x \in X \) and \( U \) be an open set containing \( x \). Then, there exist disjoint open sets \( V \) and \( W \) such that \( x \in V \) and \((X - U) - W \in I\). If \((X - U) - W = I \in I\), then \((X - U) \subset W \cup I\). Now \( V \cap W = \emptyset \Rightarrow V \subset X - W \Rightarrow cl(V) \subset X - W \) and so \( cl(V) - U \subset (X - W) \cap (W \cup I) = (X - W) \cap I \subset I \in I\).

(b) ⇒ (d). Let \( A \) be closed in \( X \) such that \( x \notin A \). Then, there exists an open set \( V \) containing \( x \) such that \( cl(V) - (X - A) \in I \) which implies that \( cl(V) \cap A \in I \).

(d) ⇒ (a). Let \( A \) be closed in \( X \) such that \( x \notin A \). Then, there is an open set \( V \) containing \( x \) such that \( cl(V) \cap A \in I \). If \( cl(V) \cap A = I \in I \), then \( A - (X - cl(V)) = I \in I \). \( V \) and \((X - cl(V))\) are the required disjoint open sets such that \( x \in V \) and \( A - (X - cl(V)) \in I \). Hence \( X \) is \( I \)-regular.

The equivalence of (b) and (c) is clear.

Corollary 4.2. – Let \((X, \tau, I)\) be an ideal space where \( I \) be codense. Then the following are equivalent.

(a) \( X \) is \( I \)-regular.

(b) For each \( x \in X \) and open set \( U \) containing \( x \), there is an open set \( V \) containing \( x \) such that \( V^* - U \in I \).

(c) For each \( x \in X \) and closed set \( A \) not containing \( x \), there is an open set \( V \) containing \( x \) such that \( V^* \cap A \in I \).

Proof. – Follows from the fact that \( I \) is codense if and only if \( cl(V) = V^* \) for every open set \( V \) [16, Remark 4].

Theorem 4.3 below shows that \( I \)-regular is a hereditary property and Theorem 4.4 shows that it is a topological property.

Theorem 4.3. – If \((X, \tau, I)\) is \( I \)-regular and \( Y \subset X \), then \((Y, \tau_Y, I_Y)\) is \( I_Y \)-regular.

Proof. – Let \( F \subset Y \) be closed in \( Y \) and \( x \in Y \) such that \( x \notin F \). Then \( F = Y \cap K \), where \( K \) is closed in \( X \) and also \( x \notin K \). Since \((X, \tau, I)\) is \( I \)-regular, there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( K \setminus V \in I \). Now \( Y \setminus U \) and \( Y \cap V \) are open sets in \( Y \) such that \( x \in Y \setminus U \) and \((Y \setminus U) \cap (Y \cap V) = (U \setminus V) \cap Y = \emptyset \). If \( K \setminus V = I \in I \), then \( K \subset I \cup V \) and so \( Y \cap K \subset Y \setminus (I \cup V) = (Y \setminus I) \cup (Y \cap V) \) which implies that \( F - (Y \cap V) \subset Y \setminus I \in I_Y \). Hence \((Y, \tau_Y, I_Y)\) is \( I_Y \)-regular.

Theorem 4.4. – If \((X, \tau, I)\) is \( I \)-regular and \( f : (X, \tau, I) \rightarrow (Y, \sigma, f(I)) \) is a homeomorphism, then \((Y, \sigma, f(I))\) is \( f(I) \)-regular.

Proof. – Let \( F \) be closed in \( Y \) and \( y \in Y \) such that \( y \notin F \). Let \( x = f^{-1}(y) \). Since \( f \) is continuous, \( f^{-1}(F) \) is a closed set in \( X \) not containing \( x \). Since \((X, \tau, I)\) is
\( I \)-regular, there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( f^{-1}(F) - V \in I \). Let \( f^{-1}(F) - V = I \in I \). Then \( f^{-1}(F) \subseteq I \cup V \).

\[
f^{-1}(F) \subseteq I \cup V \Rightarrow f(f^{-1}(F)) \subseteq f(I \cup V) \Rightarrow F \subseteq f(I) \cup f(V)
\]

\[
\Rightarrow F - f(V) \subseteq f(I) \subseteq f(I) \Rightarrow F - f(V) \in f(I).
\]

Since \( f(U) \) and \( f(V) \) are disjoint open sets in \( Y \) such that \( y \in f(U) \) and \( F - f(V) \in f(I) \), it follows that \( (Y, \sigma, f(I)) \) is \( f(I) \)-regular.

**Theorem 4.5.** – Let \((X_a, \tau_a)\) be a collection of spaces for each \( a \) in an index set \( A \) and \( I \) be an ideal on \( \prod X_a \). If \( \prod X_a \) is \( I \)-regular, then \((X_a, \tau_a)\) is \( P_a(I) \)-regular for each \( a \) in \( A \), where \( P_a \) is the projection map.

**Proof.** – Similar to Theorem 2.13 and hence omitted.

**Theorem 4.6.** – Let \((X_a, \tau_a, I_a)\) be a collection of ideal spaces for each \( a \) in an index set \( A \) where each \((X_a, \tau_a, I_a)\) is \( I_a \)-regular. If \( P_a \) is the projection map for each \( a \), \( A=\{P_a^{-1}(I_a) \mid I_a \in I_a, a \in A\}, I(A) \) is the smallest ideal generated by \( A \) and \( I \) is an ideal on \( \prod X_a \) such that \( I \supseteq I(A) \), then \( \prod X_a \) with the product topology is \( I \)-regular.

**Proof.** – Let \( x = (x_a) \) be a point in \( \prod X_a \) and \( U \) be an open set containing \( x \). Then there exists a basic open set \( V \) in \( \prod X_a \) such that \( x \in V \subseteq U \). Since \( V \) is basic, there exists a finite subset \( A \) of \( A \) such that \( V = \prod V_a \) where \( V_a \in \tau_a \) if \( a \in A \) and \( V_a=X_a \) if \( a \notin A \). For each \( a \in A \), \( x_a = P_a(x) \in P_a(V) = V_a \). Since \((X_a, \tau_a, I_a)\) is \( I_a \)-regular, there exists an \( \tau_a \)-open set \( G \) such that \( x_a \in G \) and \( cl(G) - V_a \subset I_a \). Let \( G = \cap\{P_a^{-1}(cl(G)) \mid a \in A\} \). Then \( G \) is an open set containing \( x \) and \( cl(G) = \cap\{P_a^{-1}(cl(G)) \mid a \in A\} \), where \( cl(G) \) is the closure of \( G \) in \( \prod X_a \).

Now, for each \( a \in A \), \( cl(G) - V_a = I_a \in I_a \) implies that \( cl(G) - V_a = I_a \in I_a \) and so \( cl(G) - I_a \subset V_a \). Therefore,

\[
\cap\{P_a^{-1}(cl(G)) \mid a \in A\} \subset \cap\{P_a^{-1}(V_a) \mid a \in A\}
\]

\[
\Rightarrow \cap\{P_a^{-1}(cl(G)) \mid a \in A\} \cup \{P_a^{-1}(cl(G)) \mid a \in A\} \subset \cap\{P_a^{-1}(V_a) \mid a \in A\}
\]

\[
\Rightarrow \cap\{P_a^{-1}(cl(G)) \mid a \in A\} - \cap\{P_a^{-1}(V_a) \mid a \in A\} \subseteq \cup\{P_a^{-1}(I_a) \mid a \in A\} \in I(A) \subset I
\]

\[
\Rightarrow cl(G) - V \in I.
\]

Thus, there exists an open set \( G \) in \( \prod X_a \) such that \( x \in G \) and \( cl(G) - V \in I \) and so \( \prod X_a \) with the product topology is \( I \)-regular.

**Theorem 4.7.** – Let \((X, \tau, I)\) be an \( I \)-regular space and \( x \) and \( y \) be two distinct points in \( X \). Then either \( cl\{x\} = cl\{y\} \) or \( cl\{x\} \cap cl\{y\} \in I \).
Proof. – Let $x \in cl(\{y\})$ and $y \in cl(\{x\})$. Then $cl(\{x\}) \subset cl(cl(\{y\})) = cl(\{y\}) \subset cl(cl(\{x\})) = cl(\{x\})$ and so $cl(\{x\}) = cl(\{y\})$. Suppose $y \notin cl(\{x\})$. Since $(X, \tau, I)$ is $I$-regular, by Theorem 4.1(d), there exists an open set $V$ containing $y$ such that $cl(V) \cap cl(\{x\}) \in I$. Since $y \in V$, $cl(\{x\}) \cap cl(\{y\}) \subset cl(\{x\}) \cap cl(V) \in I$. This completes the proof.

Theorem 4.8. – Let $(X, \tau, I)$ be an ideal space. If each point of $X$ has a closed neighborhood $A$ which is $I_A$-regular, then $(X, \tau, I)$ is $I$-regular.

Proof. – Let $x \in X$ and $G$ be an open set containing $x$. By hypothesis, there is a closed neighborhood $A$ of $x$ such that $(A, \tau_A, I_A)$ is $I_A$ regular. Then $A \cap G$ is an open set in $A$ containing $x$. By Theorem 4.1(c), there is a closed neighborhood $F$ of $x$ in $A$ such that $F - (A \cap G) \in I_A$. Since $F$ is a closed neighborhood of $x$ in $A$, $F$ is a closed neighborhood of $x$ in $X$ and $F - (A \cap G) \in I_A$ implies that $F - G = I \in I_A \subset I$. Thus, $F$ is a closed neighborhood of $x$ in $X$ such that $F - G \in I$. Therefore, $(X, \tau, I)$ is $I$-regular.

Theorem 4.9. – If an ideal space $(X, \tau, I)$ is $I$-regular, then for every nonempty set $A$ and a closed set $F$ in $X$ such that $F \cap A = \emptyset$, there exist disjoint open sets $U$ and $V$ such that $A \cap U \neq \emptyset$ and $F - V \in I$.

Proof. – Suppose $(X, \tau, I)$ is $I$-regular. Let $F$ be closed in $X$ and $A$ be any nonempty set such that $F \cap A = \emptyset$. Then for $x \in A$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F - V \in I$. Clearly, $A \cap U \neq \emptyset$.

Theorem 4.10. – If an ideal space $(X, \tau, I)$ is $I$-regular, then for every pair of disjoint sets $A$ and $B$ where $A$ is $I$-compact and $B$ is closed in $X$, there exist disjoint open sets $U$ and $V$, and $I$ and $J$ in $I$ such that $A - I \subset U$ and $B - J \subset V$.

Proof. – Suppose $(X, \tau, I)$ is $I$-regular and $A$ and $B$ are disjoint subsets of $X$ where $A$ is $I$-compact and $B$ is closed. For each $x \in A$, there exist disjoint open sets $U_x$ and $V_x$ such that $x \in U_x$ and $B - V_x \in I$. If $I_x = B - V_x \in I$, then $B - I_x \subset V_x$. Since $\{U_x \mid x \in A\}$ is an open cover of $A$, there is a finite subset $D$ of $A$ such that $A - \cup\{U_x \mid x \in D\} \in I$. If $U = \cup\{U_x \mid x \in D\}$ and $I = A - U \in I$, then $U$ is an open set such that $A - I \subset U$. If $V = \cap\{V_x \mid x \in D\}$ and $J = \cup\{I_x \mid x \in D\}$, then $U$ and $V$ are disjoint open sets and $I \in I$. Since $B - V_x = I_x$ for every $x$ in $D$, $\cup\{B - V_x \mid x \in D\} = J$ and so $B - V = J$ which implies that $B - J \subset V$. This proves the theorem.

Corollary 4.11. – If a space $(X, \tau)$ is regular, then for every pair of disjoint sets $A$ and $B$ where $A$ is compact and $B$ is closed in $X$, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$. 

...
Theorem 4.12. – Let \((X, \tau, \mathcal{I})\) be \(\mathcal{I}\)-regular and \(A\) be an \(\mathcal{I}\)-compact subset of \(X\). If \(G\) is an open set containing \(A\), then there exists a closed set \(F\) and \(I, J \in \mathcal{I}\) such that \(A - I \subset F \subset G \cup J\).

Proof. – Let \(A\) be an \(\mathcal{I}\)-compact subset of \(X\) and \(G\) be an open set containing \(A\). Since \(X - G\) is a closed set such that \((X - G) \cap A = \emptyset\), by Theorem 4.10, there exist disjoint open sets \(U\) and \(V\) and \(I\) and \(J\) in \(\mathcal{I}\) such that \(A - I \subset U\) and \((X - G) - J \subset V\). \((X - G) - J \subset V \Rightarrow (X - V) \subset G \cup J\) and so \(\text{cl}(U) \subset G \cup J\). \(\text{cl}(U)\) is the required closed set such that \(A - I \subset \text{cl}(U) \subset G \cup J\). Hence the proof.

If \(\mathcal{I} = \{\emptyset\}\) in the above Theorem 4.12, we have the following well known result. If \(\mathcal{I} = \mathcal{N}\) in the above Theorem 4.12, we have Corollary 4.14.

Corollary 4.13. – Let \((X, \tau)\) be a regular space and \(A\) be a compact subset of \(X\). If \(G\) is an open set containing \(A\), then there exists a closed set \(F\) such that \(A \subset F \subset G\).

Corollary 4.14. – Let \((X, \tau, \mathcal{N})\) be \(\mathcal{N}\)-regular and \(A\) be an \(\mathcal{N}\)-compact subset of \(X\). If \(G\) is an open set containing \(A\), then there exist a closed set \(F\) and nowhere dense sets \(I\) and \(J\) such that \(A - I \subset F \subset G \cup J\).

In general, a topology larger than a regular topology need not be regular. But for ideal spaces, the following Theorem 4.15 shows that if an ideal space \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-regular, then \((X, \tau^*, \mathcal{I})\) is always \(\mathcal{I}\)-regular.

Theorem 4.15. – If an ideal space \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-regular, then \((X, \tau^*, \mathcal{I})\) is \(\mathcal{I}\)-regular.

Proof. – Let \(F\) be \(\tau^*\)-closed and \(p \notin F\). Since \(X - F\) is a \(\tau^*\)-open set containing \(p\), there exists \(U \in \tau\) and \(I \in \mathcal{I}\) such that \(p \in U - I \subset X - F\). Since \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-regular, there exists an open set \(V\) such that \(p \in V\) and \(\text{cl}(V) - U \in \mathcal{I}\), by Theorem 4.1(b).

\[
\text{cl}(V) - U \in \mathcal{I} \Rightarrow \text{cl}(V) - U = J \in \mathcal{I} \Rightarrow \text{cl}(V) - J \subset U \Rightarrow \text{cl}(V) - J \subset (X - F) \cup I
\]

\[
\Rightarrow \text{cl}(V) - (X - F) \subset J \cup I \in \mathcal{I} \Rightarrow \text{cl}(V) \cap F \in \mathcal{I} \Rightarrow \text{cl}^*(V) \cap F \in \mathcal{I}.
\]

Hence \((X, \tau^*, \mathcal{I})\) is \(\mathcal{I}\)-regular, by Theorem 4.1(d).

The following Example 4.16 shows that the converse of the above Theorem 4.15 is not true but is true, if the ideal is codense, as shown by Theorem 4.17 below.

Example 4.16. – Let \(X = \{a, b, c\}\), \(\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\) and \(\mathcal{I} = \{\emptyset, \{b\}\}\). Then \((X, \tau, \mathcal{I})\) is not \(\mathcal{I}\)-regular. Since \(\tau^* = \wp(X)\), \((X, \tau^*, \mathcal{I})\) is \(\mathcal{I}\)-regular.
Theorem 4.17. – If \((X, \tau, I)\) is an ideal space such that \((X, \tau^*, I)\) is \(I\)-regular and \(I\) is codense, then \((X, \tau, I)\) is \(I\)-regular.

Proof. – Let \(A\) be closed and \(x \notin A\). Since \(A\) is \(\tau^*\)-closed, by Corollary 4.2(c), there exists a \(\tau^*\)-open set \(V\) of \(x\) such that \(V^* \cap A \in I\). Since \(V\) is a \(\tau^*\)-open set containing \(x\), there exists \(U \in \tau\) and \(I \in I\) such that \(x \in U \setminus I \subset V\). Now \(U \setminus I \subset V \Rightarrow U^* \subset V^*\) and so \(U^* \cap A \subset V^* \cap A \in I\). By Corollary 4.2(c), \((X, \tau, I)\) is \(I\)-regular.

The ideal space \((X, \tau, I)\) in Example 4.16 is a \(T_2\) mod \(I\) space which is not \(I\)-regular. The ideal space \((X, \tau, I)\) in the following Example 4.18 is \(I\)-regular which is not \(T_2\) mod \(I\). Hence \(I\)-regular and \(T_2\) mod \(I\) are independent concepts.

Example 4.18. – Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}\) and \(I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}\). Then \((X, \tau, I)\) is an \(I\)-regular space which is not \(T_2\) mod \(I\).

5. – \(I\)-Hausdorff and quasi \(I\)-Hausdorff Spaces.

A subset \(A\) of an ideal space \((X, \tau, I)\) is \(I\)-open [8] if \(A \subset \text{int}(A^*)\). The family of all \(I\)-open sets is denoted by \(\text{IO}(X, \tau, I)\) or \(\text{IO}(X, \tau)\) or \(\text{IO}(X)\). A subset \(A\) of an ideal space \((X, \tau, I)\) is quasi \(I\)-open [1] if \(A \subset \text{cl}(\text{int}(A^*))\). Every \(I\)-open set is a quasi \(I\)-open set but the converse implication need not be true. The family of all quasi \(I\)-open sets is denoted by \(\text{QIO}(X, \tau)\). A space \((X, \tau, I)\) is called \(I\)-Hausdorff [2] if for each pair of distinct points, there exist disjoint \(I\)-open sets containing the points. A space \((X, \tau, I)\) is called quasi \(I\)-Hausdorff [14] if for each pair of distinct points, there exist disjoint quasi \(I\)-open sets containing the points. Every \(I\)-Hausdorff space is quasi \(I\)-Hausdorff [14]. The following examples show that \(T_2\) mod \(I\) and \(I\)-Hausdorff (resp. quasi \(I\)-Hausdorff) are independent concepts.

Example 5.1. – Let \(X = \{a, b, c\}\), \(\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\) and \(I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}\). Then \(\text{QIO}(X, \tau) = \text{IO}(X, \tau) = \{\emptyset, \{a\}\}\). \((X, \tau, I)\) is \(T_2\) mod \(I\) but not \(I\)-Hausdorff.

Example 5.2. – Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{a, b\}\}\) and \(I = \{\emptyset, \{d\}\}\). Then \(\text{QIO}(X, \tau) = \text{IO}(X, \tau)
\begin{align*}
&= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}.
\end{align*}
Then \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-Hausdorff but for \(a\) and \(b\), there exist no open sets \(U\) and \(V\) such that \(U \cap V \in \mathcal{I}\) and so \((X, \tau, \mathcal{I})\) is not \(T_2\) mod \(\mathcal{I}\).

Theorem 3.1 of [14] gives a characterization of quasi \(\mathcal{I}\)-Hausdorff spaces in which it is assumed that (i) if \(U\) and \(V\) are disjoint quasi \(\mathcal{I}\)-open sets, then \(\text{cl}(U) \cap V = \emptyset\) and (ii) every open set is a quasi \(\mathcal{I}\)-open set. In Example 5.1 above, \(\{a, b\}\) is an open set but not a quasi \(\mathcal{I}\)-open set and so condition (ii) is false. The following Example 5.3 shows that the condition (i) is also false.

**Example 5.3.** – Consider the ideal space \((X, \tau, \mathcal{I})\) of Example 5.2. \(\{a\}\) and \(\{b\}\) are disjoint quasi \(\mathcal{I}\)-open sets containing \(a\) and \(b\), respectively, such that \(\text{cl}\{\{a\}\} = \text{cl}\{\{b\}\} = X\) and so \(\text{cl}\{\{a\}\} \cap \{b\} \neq \emptyset\).

The following Theorem 5.4 gives a sufficient condition for an ideal space \((X, \tau, \mathcal{I})\) to be a quasi \(\mathcal{I}\)-Hausdorff space, if \(\mathcal{I}\) is codense.

**Theorem 5.4.** – Let \((X, \tau, \mathcal{I})\) be an ideal space where \(\mathcal{I}\) be codense. Then \((X, \tau, \mathcal{I})\) is quasi \(\mathcal{I}\)-Hausdorff, if for every pair of distinct points \(x\) and \(y \in X\), there exists a quasi \(\mathcal{I}\)-open set \(U\) such that \(x \in U \subset U^* \subset X - \{y\}\).

**Proof.** – Suppose the condition holds. Let \(x\) and \(y \in X\) such that \(x \neq y\). Then there exists a quasi \(\mathcal{I}\)-open set \(U\) such that \(x \in U \subset U^* \subset X - \{y\}\) and so \(y \in X - U^*\). Let \(V = X - U^*\). Then \(V\) is open. Since \(\mathcal{I}\) is codense, \(V \subset V^*\) and so \(U \cap V = \emptyset\). Since \(\text{cl}(\text{int}(V^*)) \supset \text{cl}(\text{int}(V)) = \text{cl}(V) \supset V\), \(V\) is quasi \(\mathcal{I}\)-open. Hence \((X, \tau, \mathcal{I})\) is quasi \(\mathcal{I}\)-Hausdorff.

The following Example 5.5 shows that the converse of the above theorem need not be true.

**Example 5.5.** – Consider the ideal space \((X, \tau, \mathcal{I})\) of Example 5.2. \((X, \tau, \mathcal{I})\) is quasi \(\mathcal{I}\)-Hausdorff. For the points \(a\) and \(b \in X\), there exists no quasi \(\mathcal{I}\)-open set \(U\) such that \(a \in U \subset U^* \subset X - \{b\}\).

**References**


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