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CHU WENCHANG

**Sunto.** – *La famosa identità di Jacobi riguardante il prodotto triplo viene esaminata grazie alle due dimostrazioni più semplici dovute a Cauchy (1843) e Gauss (1866). Applicando il principio di induzione ed il metodo di differenze finite, lo stesso spirito ci conduce alla riconferma delle due forme finite dell’identità di prodotto quintuplo.*

**Summary.** – *The simplest proof of Jacobi’s triple product identity originally due to Cauchy (1843) and Gauss (1866) is reviewed. In the same spirit, we prove by means of induction principle and finite difference method, a finite form of the quintuple product identity. Similarly, the induction principle will be used to give a new proof of another algebraic identity due to Guo and Zeng (2005), which can be considered as another finite form of the quintuple product identity.*

1. – Jacobi’s Triple Product Identity.

The celebrated Jacobi triple product identity states that

\[(1) \quad [q, x, q/x; q]_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q^{(2k)} x^k \quad \text{for} \quad |q| < 1\]

where the \(q\)-shifted factorial is defined by

\[(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1} x) \quad \text{for} \quad n = 1, 2, \cdots\]

with the following abbreviated multiple parameter notation

\[[a, \beta, \cdots; \gamma; q]_\infty = (a; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty.\]

There are several algebraic and combinatorial proofs (see [2, 10, 17, 22, 29], for examples). Here we present the simplest proof by using only the \(q\)-binomial theorem, which is originally due to Cauchy (1843) and Gauss (1866). However, it has not been well noticed up to now (cf. [4, P 497]).
Recall that the $q$-binomial theorem reads as

$$\tag{2} (x; q)_{m} = \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} q^{\binom{k}{2}} x^{k} \quad \text{where} \quad \binom{m}{k} = \frac{(q; q)_{m}}{(q; q)_{k}(q; q)_{m-k}}.$$ 

This can easily be established by induction principle on $m$.

Now replacing $m$ and $x$ by $m + n$ and $xq^{-m}$ respectively, and then noting the relation

$$\tag{3} (q^{-m}x; q)_{m+n} = (q^{-m}x; q)_{m} (x; q)_{n} = (-1)^{m} q^{\binom{1+m}{2}} x^{m} (q/x; q)_{m} (x; q)_{n}$$

we can reformulate the $q$-binomial theorem as

$$(x; q)_{n}(q/x; q)_{m} = \sum_{k=0}^{m+n} (-1)^{k-m} \binom{m+n}{k} q^{\binom{k-m}{2}} x^{k-m}$$

which becomes, under summation index substitution $k \rightarrow m + k$, the following finite form of the Jacobi triple product identity

$$\tag{4} (x; q)_{n}(q/x; q)_{m} = \sum_{k=-m}^{n} (-1)^{k} \binom{m+n}{m+k} q^{\binom{k}{2}} x^{k}.$$ 

Letting $m, n \rightarrow \infty$ in (4), we get (1) immediately in view of limiting relation

$$\binom{m+n}{m+k} = \frac{(q; q)_{m+n}}{(q; q)_{m+k}(q; q)_{n-k}} \xrightarrow{m,n \rightarrow \infty} \frac{1}{(q; q)_{\infty}} \quad \text{where} \quad |q| < 1.$$ 

2. The Quintuple Product Identity.

Just like the derivation of Jacobi’s triple product identity from the $q$-binomial theorem, we will show that the quintuple product identity is the limiting form of the following algebraic identity.

THEOREM [Finite form of the quintuple product identity: Chen-Chu-Gu [9]. For a natural number $m$ and a variable $x$, there holds an algebraic identity:

$$\tag{5} \sum_{k=0}^{m} (1 + xq^{k} \binom{m}{k} \frac{(x; q)_{m+1}}{(q^{k+2}; q)_{m+1}} x^{k} q^{2k} \equiv 1.$$ 

Performing parameter replacements $m \rightarrow m + n$, $x \rightarrow -q^{-m}x$ and $k \rightarrow k + m$ and then simplifying the result through (3), we may restate the algebraic identity displayed in the theorem as the following finite bilateral
identity
\[
\sum_{k=-m}^{n} (1 - xq^k) \left[ \frac{m + n}{m + k} \right] (-x; q)_{1+n}(-q/x; q)_m \frac{x^{2k}q^{2k+\frac{2}{+k}}}{(x^2; q)_{1+n+k}(q/x^2; q)_{m-k}} = 1.
\]

Letting \( m, n \to \infty \) in this equation and applying the relation
\[
(q; q)_{\infty} \frac{(x^2; q)_{\infty}(q/x^2; q)_{\infty}}{(-x; q)_{\infty}(-q/x; q)_{\infty}} = [q, x, q/x; q]_{\infty}[qx^2, q/x^2; q^2]_{\infty}
\]
we derive the famous quintuple product identity
\[
\sum_{k=-\infty}^{\infty} (1 - xq^k)q^{3\left(\frac{n}{4}\right)}(qx^3)^k = [q, x, q/x; q]_{\infty}[qx^2, q/x^2; q^2]_{\infty} \quad \text{for} \quad |q| < 1.
\]

3. – Inductive Proof of the Theorem.

In terms of basic hypergeometric series, the identity (5) can be derived as the limiting case \( M \to \infty \) of the terminating very-well poised \( _6\phi_5 \)-series identity (cf. [19, II-21]):
\[
_6\phi_5 \left[ \begin{array}{cc} x^2, qx, -qx, q^{-m}, x \end{array} x, -x, q^{1-m}x^2 \right]_{\infty, M} \frac{q^{1+m}x}{M} = \frac{(qx^2; q)_{m}(qx/M; q)_{m}}{(qx; q)_{m}(qx^2/M; q)_{m}}.
\]

For those who are not familiar with basic hypergeometric series, we offer a very elementary proof of (5) based on induction principle on \( m \).

When \( m = 0 \), it is trivial to see that (5) is true. Suppose that (5) holds for a natural number \( m \). Then we need to check it also for \( m + 1 \).

Replacing \( x \) by \( qx \) and then \( k \) by \( k - 1 \), we can restate (5) as
\[
x \equiv \sum_{k=1}^{1+m} (1 + xq^k) \left[ \frac{m}{k - 1} \right] \frac{(qx; q)_{m+1}}{(q^{1+k}x^2; q)_{m+1}} x^k q^{k^2-k}.
\]

Then the linear combination
\[
\frac{1 - xq^{m+1}}{1 - x^2 q^{m+1}} \text{Eq(5)} + \frac{(1 - x)q^{m+1}}{1 - x^2 q^{m+1}} \text{Eq(9)}
\]
leads us to the following
\[
1 \equiv \frac{1 - xq^{m+1}}{1 - x^2 q^{m+1}} + \frac{(1 - x)xq^{m+1}}{1 - x^2 q^{m+1}}
\]
\[
= \sum_{k=0}^{1+m} (1 + xq^k) \left[ \frac{1 + m}{k} \right] \frac{(x; q)_{m+2}}{(q^{k}x^2; q)_{m+2}} x^k q^k
\]
\[
\times \frac{(1 - q^{1+m-k})(1 - x^2 q^{1+m-k}) + q^{1+m-k}(1 - q^k)(1 - x^2 q^k)}{(1 - q^{m+1})(1 - x^2 q^{m+1})}
\]
which corresponds to the case \( m + 1 \) of (5) on account of the fact that the last rational-factor reduces to one. According to induction principle, this confirms (5).

4. – Constructive Proof of the Theorem.

For a natural number \( m \) and a variable \( x \), we will investigate the finite sum \( \Omega_m(x) \) given by

\[
\Omega_m(x) := \sum_{k=0}^{m} \frac{(x; q)_{m+1}}{(q^k x^2; q)_{m+1}} x^k q^{k^2}.
\]

Let \( A_k \) and \( B_k \) be two sequences defined respectively by

\[
A_k := \frac{(-1)^k x^k q^{k+1}}{1 - xq^k} \quad \text{and} \quad B_k := (1 - xq^k)^{1-x^2q^{2k}}.
\]

In view of the boundary condition \( B_{-1} = B_m = 0 \) and the finite differences

\[
A_k - A_{k+1} = (-1)^k \frac{1 - x^2 q^{2k+1}}{(1 - xq^k)(1 - xq^{k+1})} x^k q^{k+1}
\]

\[
B_k - B_{k-1} = (-1)^k \frac{1 - x^2 q^{2k}}{(1 - xq^{k+1})(1 - xq^k)} x^k q^{k+1}
\]

we can reformulate the \( \Omega \)-sum defined in (10) as follows:

\[
\Omega_m(x) = \sum_{k=0}^{m} A_k \{ B_k - B_{k-1} \} = \sum_{k=0}^{m} A_k B_k - \sum_{k=1}^{m} A_k B_{k-1}
\]

\[
= \sum_{k=0}^{m-1} A_k B_k - \sum_{k=0}^{m-1} A_{k+1} B_k = \sum_{k=0}^{m-1} B_k \{ A_k - A_{k+1} \}
\]

which leads us to the following relation:

\[
\Omega_m(x) = \sum_{k=0}^{m-1} \frac{1 - x^2 q^{2k+1}}{(1 - xq^k)(1 - xq^{k+1})} \left[ \frac{(x; q)_{m+1}}{(q^{k+1} x^2; q)_{m+1}} x^k q^{k^2+k} \right]
\]

From this expression, we can derive the following interesting result.

**Lemmas** [Recurrence relation]. For the \( \Omega \)-function defined in (10), there holds the recursion

\[
\Omega_m(x) = \Omega_{m-1}(qx) \quad \text{where} \quad m = 1, 2, \ldots.
\]
PROOF. – We further define two sequences $C_k$ and $D_k$ respectively by

$$C_k := \frac{(q^{m-k}; q)_k (x; q)_{m+1}}{(x^2; q)_{k+m+1}} x^{2k} q^{k^2+k}$$

$$D_k := \frac{(x^2; q)_{k+1}}{(q; q)_k} \frac{x^{-k}}{(1-x)(1-xq^{k+1})}.$$  

In view of the boundary condition $C_m = D_{-1} = 0$ and the finite differences

$$C_k - C_{k+1} = (1 - x^2 q^{2k+2}) \frac{(q^{m-k}; q)_k (x; q)_{m+1}}{(x^2; q)_{k+m+2}} x^{2k} q^{k^2+k}$$

$$D_k - D_{k-1} = \frac{1 - x^2 q^{2k+1}}{(1-xq^k)(1-xq^{k+1})} \frac{(x^2; q)_{k+1}}{(q; q)_k} x^{-k}$$

we can manipulate $\Omega$-sum displayed in (11) as follows:

$$\Omega_m(x) = \sum_{k=0}^{m-1} C_k \{ D_k - D_{k-1} \} = \sum_{k=0}^{m-1} C_k D_k - \sum_{k=1}^{m-1} C_k D_{k-1}$$

$$= \sum_{k=0}^{m-1} C_k D_k - \sum_{k=0}^{m-1} C_{k+1} D_k = \sum_{k=0}^{m-1} D_k \{ C_k - C_{k+1} \}$$

$$= \sum_{k=0}^{m-1} \left\{ 1 + xq^{k+1} \right\} \binom{m-1}{k} (q^k x; q)_m (q^{2k+2}; q)_m x^{k^2+k}.$$  

This leads us to the recursion stated in the lemma.  

Iterating for $m$-times the recurrence relation:

$$\Omega_m(x) = \Omega_{m-1}(qx)$$

we find the following algebraic identity:

$$\Omega_m(x) = \Omega_{m-1}(qx) = \Omega_{m-2}(q^2x) \cdots = \cdots \Omega_0(q^m x) = 1$$

(13)

which leads us immediately to the algebraic identity stated in the Theorem.

The informed reader will notice that the procedure just employed is the so-called “Abel’s lemma on summation by parts”. This method has been shown powerful to evaluate classical and basic hypergeometric series. The interested reader may refer to Chu and Jia [11, 12, 13, 14] for more details and further developments.
5. – Another Finite Form of the Quintuple Product Identity.

In a recent paper [30], Guo and Zeng found another finite form of the quintuple product identity. Its reduced case \( m = 2m \) has appeared in Paule [24, Eq 27].

**Proposition (Guo and Zeng [30, Theorem 9.1])** For a natural number \( m \) and a variable \( x \), there holds an algebraic identity:

\[
\sum_{k=0}^{m} \left( 1 - x^{2} q^{1+2k} \right) \binom{m}{k} \frac{(qx; q)_{m}}{(q^{1+k} q^{2}; q)_{m+1}} x^{k} q^{k^{2}} = 1. \tag{14}
\]

In terms of basic hypergeometric series, the identity (14) can be derived as the limiting case \( M \to \infty \) of the terminating very-well poised \( \phi_{5} \)-series identity (cf. [19, II-21]):

\[
\phi_{5} \left[ q^{x^{2}}, q^{3/2} x^{2}, -q^{3/2} x^{2}, -q^{m}, q^{x}, M \left| q^{1+m} x^{2} / M \right. \right] = \frac{(q^{2} x^{2}; q)_{m}(q x^{2}/ M; q)_{m}}{(q x^{2}; q)_{m}(q^{2} x^{2}/ M; q)_{m}}.
\]

Analogously, an inductive proof of (14) can be reproduced as follows.

When \( m = 0 \), it is trivial to see that (14) is true. Suppose that (14) holds for a natural number \( m \). Then we need to check it also for \( m + 1 \).

Replacing \( x \) by \( qx \) and then \( k \) by \( k - 1 \), we can restate (14) as

\[
x \equiv \sum_{k=1}^{1+m} \left( 1 - x^{2} q^{1+2k} \right) \binom{m}{k-1} \frac{(q^{2} x^{2}; q)_{m}}{(q^{2+k} q^{2}; q)_{m+1}} x^{k} q^{k^{2} - k}. \tag{15}
\]

Then the linear combination

\[
\frac{1 - x q^{m+1}}{1 - x^{2} q^{m+2}} \text{Eq}(14) + \frac{1 - x q^{m+1}}{1 - x^{2} q^{m+2}} \text{Eq}(15)
\]

leads us to the following

\[
1 \equiv \frac{1 - x q^{m+1}}{1 - x^{2} q^{m+2}} + \frac{(1 - x q)x q^{m+1}}{1 - x^{2} q^{m+2}}
\]

\[
= \sum_{k=0}^{1+m} \left( 1 - x^{2} q^{1+2k} \right) \binom{1+m}{k} \frac{(qx; q)_{m+1}}{(q^{1+k} q^{2}; q)_{m+2}} x^{k} q^{k^{2}}
\]

\[
\times \frac{(1 - q^{1+m-k})(1 - x^{2} q^{2+m+k}) + q^{1+m-k}(1 - q^{k})(1 - x^{2} q^{1+k})}{(1 - q^{m+1})(1 - x^{2} q^{m+2})}
\]
which corresponds to the case $m + 1$ of (14) on account of the fact that the last rational-factor reduces to one. According to induction principle, this confirms (14).

Similar to the last section, a constructive proof of (14) can be provided either. We leave it to the reader as an exercise.

**Remark** For the historical note about the quintuple product identity, the reader can refer to [8]. More comprehensive coverage has been provided recently by Cooper [15]. Compared with the known proofs of this identity due to Watson [27, 28] based on functional equations and elliptic functions, Atkin and Swinnerton-Dyer [5] via function theoretic methods, Gordan [20] through functional equations, Carlitz and Subbarao [8] by multiplying two triple products as well as Paule [24] by the WZ-method, the proof presented in this paper is much simpler and more elementary, which requires only some high school algebra.

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