Bollettino
Unione Matematica Italiana

Fausto De Mari

Groups Satisfying Weak Chain Conditions on Certain Non-Normal Subgroups

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 853–866.
Unione Matematica Italiana
<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_853_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 853-866

Groups Satisfying Weak Chain Conditions on Certain Non-Normal Subgroups

FAUSTO DE MARI

Sunto. – *In questo articolo si considerano gruppi risolubili (generalizzati) che verificano una condizione di catena debole sui sottogruppi che non hanno indice finito nella loro chiusura normale.*

Summary. – *In this paper, (generalized) soluble groups for which the set of all subgroups having infinite index in their normal closure satisfies some weak chain condition are described.*

1. – Introduction.

A subgroup H of a group G is said to be a *nearly normal* subgroup if it has finite index in its normal closure H^G. A theorem of B.H. Neumann [8] states that all subgroups of a group G are nearly normal if and only if its commutator subgroup G' is finite. The structure of groups which are rich of nearly normal subgroups has later been investigated by S. Franciosi and F. de Giovanni [6], who gave a complete description of non-periodic or locally finite groups for which the set of non-(nearly normal) subgroups satisfies the minimal condition: if the commutator subgroup of such a group G is infinite, then either G is a Černikov group or G contains a finite normal subgroup E such that $G/E = J/E \times D/E$ where J/E is a divisible abelian group satisfying the minimal condition and D/E is an infinite dihedral group. More recently, (generalized) soluble groups satisfying the maximal condition on non-(nearly normal) subgroups have been considered by A. Galoppo [7].

Let χ be a subgroup theoretical property. Recall that a group G satisfies the *weak minimal condition* on χ-subgroups if in any descending chain

$$X_1 > X_2 > \ldots > X_n > X_{n+1} > \ldots$$

of χ-subgroups of G all but finitely many indices $|X_a : X_{a+1}|$ are finite. The group G satisfies *weak maximal condition* on χ-subgroups if for any ascending chain

$$X_1 < X_2 < \ldots < X_n < X_{n+1} < \ldots$$

of χ-subgroups of G only finitely many of the indices $|X_a : X_{a+1}|$ are infinite. It was proved independently by R. Baer [2] and D.I. Zaičev [12], [13] that for soluble groups the weak minimal and the weak maximal conditions on subgroups are equivalent and characterizes *minimax groups*, i.e. groups having a finite series whose factors either satisfy the minimal or the maximal condition on subgroups. This result was there improved by Zaičev [14], proving that the class of soluble-by-finite minimax groups coincide with that of locally (soluble-by-finite) groups satisfying the weak double chain condition on subgroups.

An infinite collection $(X_n)_{n\in\mathbb{Z}}$ of subgroups of a group G such that

$$\ldots < X_{-n} < \ldots < X_{-1} < X_0 < X_1 < \ldots < X_n < \ldots$$

is called a *double chain* of subgroups, and if the index $|X_{n+1} : X_n|$ is infinite for any integer n, the double chain is called an ∞-*double chain*. If χ is a property pertaining to subgroups, a group G is said to satisfy the *double chain condition* (respectively, the *weak double chain condition*) on χ-subgroups if it has no double chains (respectively, ∞-double chains) consisting of χ-subgroups. In [4], G. Cutolo and L.A. Kurdachenko completely describe (generalized) soluble groups satisfying the weak double chain condition on subgroups which are not almost normal; where we recall that a subgroup H of a group G is called *almost normal* if it has finitely many conjugates or, equivalentely, if its normalizer $N_G(H)$ has finite index in G.

In this paper groups satisfying the weak double chain condition on non-(nearly normal) subgroups are considered; we will call $DC^{\infty}_{\overline{nn}}$*-groups* such groups, and the following result will be proved.

THEOREM A – *Let G be a group having an ascending series with locally (soluble-by-finite) factors. Then the following statements are equivalent:*

(*i*) *G satisfies the condition $DC^{\infty}_{\overline{nn}}$;*

(*ii*) *G satisfies the weak minimal condition on non-(nearly normal) subgroups;*

(*iii*) *G satisfies the weak maximal condition on non-(nearly normal) subgroups;*

(*iv*) *every subgroup of G which is not minimax is nearly normal;*

(*v*) *either G is a minimax group or G' is finite or G has a finite normal subgroup E such that G/E is the direct product of finitely many Prüfer groups and of a nilpotent torsion-free group N of class 2 and finite rank such that $N/Z(N)$ is finitely generated and N' is contained in all pure subgroups of $Z(N)$ which are not minimax.*

Of course, Theorem A gives some information about groups satisfying the double chain condition on non-(nearly normal) subgroups: we will refer to groups with this propety as $DC_{\overline{nn}}$*-groups*. It has been proved by T.S. Shores [10] that every locally radical group satisfying the double chain condition on subgroups either satisfies the minimal or the maximal condition, while F. De Mari and F. de Giovanni [5] have recently proved that a corresponding theorem holds for locally radical groups satisfying the double chain condition on non-normal subgroups. In this context, the following result will be proved here.

THEOREM B – *Let G be a $DC_{\bar{n}n}$-group having an ascending series with locally (soluble-by-finite) factors. Then G satisfies either the minimal or the maximal condition on non-(nearly normal) subgroups.*

Most of our notation is standard and can be found in [9].

2. – Proof of Theorem A.

Let G be any group. If H and K are nearly normal subgroups of G, it is clear that $H \cap K$ and $\langle H, K \rangle$ are likewise nearly normal. Moreover, if E is any finite normal subgroup of G, a subgroup X of G is nearly normal if and only if XE/E is nearly normal in G/E; in particular, the group G satisfies one of the conditions $DC_{\bar{n}n}$ and $DC_{\bar{n}n}^{\infty}$ if and only if G/E satisfies the same property.

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having finitely many conjugates, and G is said to be an *FC-group* if it concides with its FC-centre. Our first lemma is well-known and proves that any cyclic nearly normal subgroup of a group G is contained in the FC-centre of G.

LEMMA 2.1 ([6], Lemma 2.1). – *Let G be a group and let x be an element of G such that the index $|\langle x \rangle^G : \langle x \rangle|$ is finite. Then x has finitely many conjugates in G.*

It has been proved by B.H. Neumann [8] that any FC-group which is not finite-by-abelian must contains certain 'special' subgroups; such subgroups play an essential role in the proof of the following result.

THEOREM 2.2. – *Let G be a $DC_{\bar{n}n}^{\infty}$-group. If G is an FC-group, then G' is finite.*

PROOF. – Assume by contradiction that G' is infinite, so that there exist two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of elements of G satisfying the following relations

$$c_n = [a_n, b_n] \neq 1, \; c_n \neq c_m \quad \text{and} \quad [a_n, a_m] = [b_n, b_m] = [a_n, b_m] = 1$$

for all $n, m \in \mathbb{N}$ with $n \neq m$ (see [8], Lemma 11.1). For every subset X of $\mathbb{N}$, put

$$A(X) = \langle a_n | n \in X \rangle \quad \text{and} \quad B(X) = \langle b_n | n \in X \rangle.$$

Let $X \subset Y \subseteq \mathbb{N}$. If $n \in Y \setminus X$, then b_n centralizes $A(X)$ but not $A(Y)$ and hence $A(X) < A(Y)$; moreover, if $Y \setminus X$ is infinite there are infinitely many subgroups between $A(X)$ and $A(Y)$ and so the index $|A(Y) : A(X)|$ is infinite. Similary, $B(X) < B(Y)$ and also the index $|B(Y) : B(X)|$ is infinite whenever the set $Y \setminus X$ is infinite.

Let $(X_n)_{n\in\mathbb{Z}}$ be a chain of subsets of $\mathbb{N}$ such that $X_n \subseteq X_{n+1}$ and $X_{n+1} \setminus X_n$ is infinite for all $n \in \mathbb{Z}$. Assume that for each integer n, there is a set $\bar{X}_n$ such that $X_n \subset \bar{X}_n \subset X_{n+1}$ and $A(\bar{X}_n)$ is not nearly normal in G. Then

$$\ldots < A(\bar{X}_{-2n}) < \ldots < A(\bar{X}_{-2}) < A(\bar{X}_0) < A(\bar{X}_2) < \ldots < A(\bar{X}_{2n}) < \ldots$$

is an ∞-double chain of non-(nearly normal) subgroups of G. This contradiction shows that there exists an integer k such that $A(X)$ is a nearly normal subgroup of G for each subset X with $X_k \subset X \subset X_{k+1}$. Consider now a collection $(Y_n)_{n\in\mathbb{Z}}$ of subsets such that $X_k \subset Y_n \subseteq Y_{n+1} \subset X_{k+1}$ and $Y_{n+1} \setminus Y_n$ is infinite for all n. Then

$$\ldots < B(Y_{-2}) < B(Y_{-1}) < B(Y_0) < B(Y_1) < B(Y_2) < \ldots$$

is an ∞-double chain and hence the condition $DC^{\infty}_{\bar{n}\bar{n}}$ yields that there exists an integer m such that $B = B(Y_m)$ is nearly normal in G. By the choice of k, also $A = A(Y_m)$ is a nearly normal subgroup of G.

Put $C = \langle c_n | n \in Y_m \rangle$, so that $C \leq A^G \cap B^G$ and hence the indices $|AC : A|$ and $|BC : B|$ are finite. Thus there exists an infinite subset I of Y_m such that $Ac_i = Ac_j$ and $Bc_i = Bc_j$ for all $i, j \in I$. If $i \neq j$ are elements of I, then $c_i \in Ac_j \subseteq C_G(a_i)$ and so $[c_i, A] = \{1\}$. Similary, we obtain that $[c_i, B] = \{1\}$ for all $i \in I$. Write $G_1 = \langle a_n, b_n | n \in I \rangle$. Then $G_1' = \langle c_n | n \in I \rangle$ is an infinite central subgroup of G_1, and so there exists a section G^* of G_1 containing two sequences of elements $(a^*_n)_{n\in\mathbb{N}}$ and $(b^*_n)_{n\in\mathbb{N}}$ satisfying the relations

$$c^*_n = [a^*_n, b^*_n] \neq 1,\ [a^*_n, a^*_m] = [b^*_n, b^*_m] = [a^*_n, b^*_m] = 1$$

and

$$c^*_n (c^*_m)^{-1} \notin \langle a^*_n | n \in \mathbb{N} \rangle$$

for all $n, m \in \mathbb{N}$ with $n \neq m$ (see [8], Lemma 11.6, Lemma 11.7, Lemma 8.5 and Lemma 9.6). If now for any subset X of $\mathbb{N}$ we put $A^*(X) = \langle a^*_n | n \in X \rangle$, as before it can be shown that there exists a subset S of $\mathbb{N}$ for which $A^*(S)$ is a nearly normal subgroup of G^*, a contradiction since $A^*(S)$ is not nearly normal in $\langle a^*_n, b^*_n | n \in S \rangle$ (see [8], Lemma 8.4). The theorem is proved. □

LEMMA 2.3. – *Let G be a $DC^{\infty}_{\bar{n}\bar{n}}$-group and let H/K be a section of G which is the direct product of infinitely many non-trivial subgroups. Then H and K are nearly normal subgroups of G.*

PROOF. – Let $(H_n)_{n\in\mathbb{Z}}$ be a countably infinite collection of subgroups of H properly containing K such that

$$H/K = \operatorname*{Dr}_{n\in\mathbb{Z}} H_n/K,$$

and put

$$\check{H} = \langle H_n | n \leq 0\rangle \quad \text{ and } \quad \hat{H} = \langle H_n | n > 0\rangle.$$

Consider a collection $(X_n)_{n\in\mathbb{Z}}$ of subsets of $\mathbb{N}$ such that $X_n \subset X_{n+1}$ and $X_{n+1} \setminus X_n$ is infinite for all $n \in \mathbb{Z}$, and for any integer i let

$$\hat{H}_i = \langle H_j | j \in X_i\rangle.$$

Since $\hat{H}_i$ is a subgroup of infinite index of $\hat{H}_{i+1}$ for each i, we have that

$$\ldots < \check{H}\hat{H}_{-2} < \check{H}\hat{H}_{-1} < \check{H}\hat{H}_0 < \check{H}\hat{H}_1 < \check{H}\hat{H}_2 < \ldots$$

is an ∞-double chain. It follows from the weak chain condition that there exists an integer k such that $\check{H}\hat{H}_k$ is a nearly normal subgroup of G. In a similar way it can be proved that $\check{H}$ contains a subgroup L such that $L\hat{H}$ is nearly normal in G. Therefore also

$$H = \langle \check{H}\hat{H}_k, L\hat{H}\rangle$$

is a nearly normal subgroup of G. The same argument proves that both subgroups $\check{H}$ and $\hat{H}$ are nearly normal in G, so that also

$$K = \check{H} \cap \hat{H}$$

is a nearly normal subgroup of G. □

Let A be any abelian group. Recall that the *torsion-free rank* of A is the cardinality $r_0(A)$ of a maximal independent set of elements of infinite order of A, and for any prime p the *p-rank* of A is the cardinality $r_p(A)$ of a maximal independent set of elements of A whose order is a power of p. Moreover, the *rank* of A is $r_0(A) + \max_p r_p(A)$, while the *total rank* of A is $r_0(A) + \sum_p r_p(A)$. Clearly, an abelian group has finite total rank if and only if it is the direct product of finitely many cyclic and quasicyclic groups and a torsion-free abelian group of finite rank. An arbitrary group G is said to have *finite rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated with at most r elements.

Theorem 2.2 and Lemma 2.3 can be used to restrict the total rank of abelian subgroups of groups satisfying $DC^{\infty}_{\overline{nn}}$.

LEMMA 2.4. – *Let G be a $DC^{\infty}_{\overline{nn}}$-group containing a subgroup A which is the direct product of infinitely many non-trivial cyclic subgroups. Then G' is finite.*

PROOF. – By Theorem 2.2, it is enough to show that G is an FC-group; moreover, without loss of generality we may suppose that A is the direct product of a countably infinite collection of non-trivial cyclic groups. Assume by contradiction that G contains an element x with infinitely many conjugates. Clearly, it can be assumed that $\langle x\rangle \cap A = \{1\}$; since A is a nearly normal subgroup of G by Lemma 2.3, the subgroup $A^G \cap \langle x\rangle$ is finite. Put $A = B \times C$, where both B and C are direct product of a countably infinite collection of non-trivial cyclic groups. Again Lemma 2.3 yields that B and C are nearly normal subgroups of G, so that $B^G \cap C^G$ is finite and hence $\langle x\rangle$ has finite index in $\langle x\rangle B^G \cap \langle x\rangle C^G$. Write

$$B = \operatorname*{Dr}_{n\in\mathbb{N}} \langle b_n\rangle,$$

and consider a chain $\{X_n | n \in \mathbb{Z}\}$ of subsets of $\mathbb{N}$ such that $X_n \subset X_{n+1}$ and $X_{n+1} \setminus X_n$ is infinite for all $n \in \mathbb{Z}$. For any integer n, put

$$B_n = \operatorname*{Dr}_{m\in X_n} \langle b_m\rangle.$$

Then each B_n is a nearly normal subgroup of G, and the index $|B_{n+1} : B_n|$ is infinite. As $\langle x\rangle \cap A^G$ is finite, it follows that

$$\ldots < \langle x\rangle B_{-2}^G < \langle x\rangle B_{-1}^G < \langle x\rangle B_0^G < \langle x\rangle B_1^G < \langle x\rangle B_2^G < \ldots$$

is an ∞-double chain of G, and hence $\langle x\rangle B_k^G$ is a nearly normal subgroup of G for some integer k. Similary, there exists a subgroup D of C such that $\langle x\rangle D^G$ is nearly normal in G. Therefore

$$E = \langle x\rangle B_k^G \cap \langle x\rangle D^G$$

is a nearly normal subgroup of G. Since $\langle x\rangle$ has finite index in E, it follows that $\langle x\rangle$ is a nearly normal subgroup of G. This contradiction proves the lemma. $\square$

The following result proves in particular that any locally finite group in the class $DC^{\infty}_{\bar{n}n}$ satisfies the minimal condition on non-(nearly normal) subgroups.

COROLLARY 2.5. – *Let G be a locally finite $DC^{\infty}_{\bar{n}n}$-group. Then either G is a Černikov group or G' is finite.*

PROOF. – Assume that G is not a Černikov group. Then G contains an abelian subgroup with infinite socle (see [11]) and hence G' is finite by Lemma 2.4. $\square$

Recall that a group is said to be an $\mathfrak{S}_1$-*group* if it has a finite series whose factors are either torsion-free abelian of finite rank or abelian with the minimal condition. Moreover, an $\mathfrak{S}_1\mathfrak{F}$-group is a group containing a subgroup of finite index which is an $\mathfrak{S}_1$-group. Clearly, any $\mathfrak{S}_1\mathfrak{F}$-group has finite rank and soluble-by-finite minimax groups belong to the class $\mathfrak{S}_1\mathfrak{F}$. Recall also that if G is a soluble-by-finite minimax group, the finite residual J of G is the direct product of finitely many Prüfer subgroups and G/J does not contain periodic infinite subgroups; moreover, the Fitting subgroup F/J of G/J is nilpotent and G/F is an abelian-by-finite group satisfying the maximal condition (see [9] Part 2, Theorem 10.33).

We need the following result. It proves in particular that $G/Z(G)$ is polycyclic-by-finite, provided G is an $\mathfrak{S}_1\mathfrak{F}$-group whose commutator subgroup is polycyclic-by-finite.

LEMMA 2.6. – ([4], Lemma 6) *Let G be a group. If G' is polycyclic-by-finite and $Z_2(G)/Z(G)$ has finite p-rank for every prime p, then $G/Z(G)$ is polycyclic-by-finite.*

LEMMA 2.7. – *Let G be a $DC^{\infty}_{\overline{nn}}$-group containing an abelian non-minimax subgroup. If G' is infinite, then G is an $\mathfrak{S}_1\mathfrak{F}$-group and $G/Z(G)$ is polycyclic-by-finite.*

PROOF. – Let A be an abelian non-minimax subgroup of G, and let B a free subgroup of A such that A/B is periodic. Then B is finitely generated by Lemma 2.4, so that A/B does not satisfy the minimal condition and its socle is the direct product of infinitely many cyclic subgroups. It follows from Lemma 2.3 that the index $|B^G : B|$ is finite. Thus also the periodic group AB^G/B^G has infinite socle and another application of Lemma 2.4 yields that $G'B^G/B^G$ is finite. Therefore G' is polycyclic-by-finite and G is soluble-by-finite. Again by Lemma 2.4, all abelian subgroups of G have finite total rank, so that G is an $\mathfrak{S}_1\mathfrak{F}$-group (see [3]) and hence $G/Z(G)$ is polycyclic-by-finite by Lemma 2.6. □

The following result proves that any non-minimax subgroup of a soluble-by-finite $DC^{\infty}_{\overline{nn}}$-group is nearly normal; in particular, it turns out that for soluble-by-finite groups the weak double chain condition and the weak minimal condition on the set of all subgroups which are not nearly normal are equivalent.

COROLLARY 2.8. – *Let G be a soluble-by-finite $DC^{\infty}_{\overline{nn}}$-group and let H be a subgroup of G which is not minimax. If G' is infinite, then H^G/H_G is finite.*

PROOF. – Since G is soluble-by-finite, H contains an abelian non-minimax subgroup (see [9] Part 2, Theorem 10.35) and hence $G/Z(G)$ is polycyclic-by-finite by Lemma 2.7; in particular, H/H_G is polycyclic-by-finite. As G' is polycyclic-by-finite (see [9] Part 1, p. 115), the factor group H/H' is not minimax. On the other hand, Lemma 2.7 also yields that G is an $\mathfrak{S}_1\mathfrak{F}$-group, so that H must have a periodic abelian homomorphic image with infinitely many non-trivial primary components, and hence the index $|H^G : H|$ is finite by Lemma 2.3. If L is the core of H in H^G, it follows that also L has finite index in H^G and so H^G/L_G has finite exponent. As L_G is contained in H_G and H^G/H_G is polycyclic-by-finite, we obtain that H^G/H_G is finite. □

LEMMA 2.9. – *Let G be a non-periodic $DC^{\infty}_{\bar{n}n}$-group whose infinite cyclic subgroups are nearly normal. Then G is finite-by-abelian-by-finite.*

PROOF. – Let x be any element of infinite order of G and let y be any element of $C_G(x)$. Since $\langle x, y\rangle = \langle x, xy\rangle$ is generated by elements of infinite order, it is contained in the FC-centre of G. Thus $C_G(x)$ is an FC-group. On the other hand, the subgroup $C_G(x)$ has finite index in G and hence it follows from Theorem 2.2 that G is finite-by-abelian-by-finite. □

LEMMA 2.10. – *Let G be a locally (soluble-by-finite) $DC^{\infty}_{\bar{n}n}$-group and let H be a subgroup of G such that G/H^G is not soluble-by-finite. Then there exists a subgroup K of G containing H such that the indices $|K : H|$ and $|K^G : K|$ are infinite.*

PROOF. – As G/H^G is not soluble-by-finite, it follows from Corollary 2.5 and Lemma 2.9 that there exists an element of infinite order xH^G of G/H^G such that $K = \langle x\rangle H^G$ is not nearly normal in G. Clearly, H is contained in K and the index $|K : H|$ is infinite. □

LEMMA 2.11. – *Let G be a $DC^{\infty}_{\bar{n}n}$-group with an ascending series whose factors are locally (soluble-by-finite). Then G is soluble-by-finite.*

PROOF. – Clearly, it can be assumed that G' is infinite; moreover, by Lemma 2.7 we may also suppose that all abelian subgroups of G are minimax so that, in particular, all soluble-by-finite subgroups of G are minimax (see [9] Part 2, Theorem 10.35).

Suppose first that G is locally (soluble-by-finite) but not soluble-by-finite. Then G' is not minimax and the result of D.I. Zaičev quoted in the introduction yields that G' contains an ∞-double chain of subgroups

$$\ldots < X_{-2} < X_{-1} < X_0 < X_1 < X_2 < \ldots$$

Since G satisfies the condition $DC^{\infty}_{\bar{n}n}$, there exists an integer n such that $Y_0 = X_n$ in nearly normal in G. Clearly, Y_0 and G'/Y_0^G are not minimax; in particular, Y_0 is not soluble-by-finite. Assume for a contradiction that $\bar{G} = G/Y_0^G$ is soluble-by-finite. As $\bar{G}'$ is not minimax, $\bar{G}/Z(\bar{G})$ cannot be polycyclic-by-finite (see [9] Part 1, p.115) and hence all abelian subgroups of $\bar{G}$ are minimax by Lemma 2.7. Therefore $\bar{G}$ is minimax and this contradiction proves that both Y_0 and G/Y_0^G are not soluble-by-finite. The above argument shows that there exists nearly normal non-minimax subgroups Y_{-1} and Y_1 of G with $Y_{-1} < Y_0 \leq Y_0^G < Y_1$ and such that both $Y_0/Y_{-1}^{Y_0}$, Y_1/Y_0^G and G/Y_1^G are not soluble-by-finite. In particular, also $Y_1/Y_0^{Y_1}$ is not soluble-by-finite. Iterating this method we may construct a double chain $(Y_n)_{n\in\mathbb{Z}}$ of nearly normal subgroups of G such that $Y_{n+1}/Y_n^{Y_{n+1}}$ is not soluble-by-finite for each n, and so it follows form Lemma 2.10 that G has an ∞-double chain of non-(nearly normal) subgroups. This contradiction shows that G is soluble-by-finite whenever it is locally (soluble-by-finite).

In the general case, by the first part of the proof, G has an ascending series whose factors are either abelian or finite; since all abelian subgroups of G are minimax, it follows that G is a soluble-by-finite (minimax) group (see [1]). $\square$

Let G be a group containing a finite normal subgroup E such that G/E is the direct product of finitely many Prüfer groups and a torsion-free nilpotent group N of class 2 and finite rank such that $N/Z(N)$ is finitely generated and N' is contained in all pure subgroups of $Z(N)$ which are not minimax. Then it has been proved in Theorem 12 of [4] that G satisfies the weak maximal condition on non-(almost normal) subgroups and the same proof actually shows that G satisfies also the weak maximal condition on non-(nearly normal) subgroups. Therefore the results obtained in this section together with Theorem 12 of [4] shows that in order to prove Theorem A it is enough to prove the following result.

THEOREM 2.12. – *Let G be a group with an ascending series whose factors are locally (soluble-by-finite) factors and suppose that G' is infinite. Then G is a $DC^{\infty}_{\overline{nn}}$-group if and only if it satisfies the weak double chain condition on non-(almost normal) subgroups.*

PROOF. – Let $(H_n)_{n\in\mathbb{Z}}$ be an ∞-double chain of G and let H be any element of this chain; clearly, H does not satisfy the weak minimal condition. If G is a $DC^{\infty}_{\overline{nn}}$-group, then G is soluble-by-finite by Lemma 2.11 so that H is not minimax and hence it is an almost normal subgroup of G by Corollary 2.8; thus G satisfies the weak double chain condition on non-(almost normal) subgroups.

Conversely, suppose that G satisfies the weak double chain condition on non-(almost normal) subgroups. Then again G is soluble-by-finite by Proposition 8 of [4] and hence H is not minimax; moreover, G/H_G is central-by-finite (see [4], Lemma 9). Therefore H is nearly normal in G and hence G is a $DC^{\infty}_{\overline{nn}}$-group. $\square$

It should be observed that the condition on the subgroup N', that appear in the statement of Theorem A, gives further information on the structure of a soluble-by-finite group G satisfying the condition $DC^{\infty}_{\overline{nn}}$. In fact, if G neither is minimax nor finite-by-abelian, then $Z(G)$ is an extension of a minimax group by a torsion-free abelian group of rank 1 (see [4], Proposition 12).

3. – Proof of Theorem B.

The first lemma of this section is an easy consequence of the imposition of the double chain condition on non-(nearly normal) subgroups.

LEMMA 3.1. – *Let G be a $DC_{\bar{n}n}$-group and let x be an element of infinite order of G with infinitely many conjugates. Then every ascending chain of non-(nearly normal) subgroups of G containing $\langle x\rangle$ is finite. Moreover, $\langle x\rangle^G$ satisfies the maximal condition on normal subgroups and $G/\langle x\rangle^G$ satisfies the maximal condition on non-(nearly normal) subgroups.*

PROOF. – Since the subgroup $\langle x\rangle$ is not nearly normal in G, there exists a sequence $k_1, k_2, \dots$ of positive integer such that each $\langle x^{k_n}\rangle$ is not nearly normal in G and

$$\langle x\rangle > \langle x^{k_1}\rangle > \langle x^{k_2}\rangle > \dots > \langle x^{k_n}\rangle > \dots$$

Thus every ascending chain of non-(nearly normal) subgroups of G containing $\langle x\rangle$ must be finite and so $G/\langle x\rangle^G$ satisfies the maximal condition on non-(nearly normal) subgroups. If X is any subgroup such that $\langle x\rangle \leq X < \langle x\rangle^G$, then $X^G = \langle x\rangle^G$ and hence the interval $[\langle x\rangle^G/\langle x\rangle]$ satisfies the maximal condition. Therefore $\langle x\rangle^G$ satisfies the maximal condition on normal subgroup. □

COROLLARY 3.2. – *Let G be a $DC_{\bar{n}n}$-group and let x be an element of infinite order of G. If x normalizes some infinite Černikov subgroup of G, then $\langle x\rangle$ is a nearly normal subgroup of G.*

PROOF. – Let D be a divisible abelian non-trivial p-subgroup of G satisfying the minimal condition and such that $D^x = D$ (where p is a prime). Then the chain

$$\langle x\rangle < \langle x\rangle\Omega_1(D) < \dots < \langle x\rangle\Omega_n(D) < \langle x\rangle\Omega_{n+1}(D) < \dots$$

is infinite and hence the statement follows immediately from Lemma 3.1. □

LEMMA 3.3. – *Let G be a $DC_{\bar{n}n}$-group and let X be a subgroup of $Z(G)$ which neither is periodic nor finitely generated. Then $G'X/X$ is finite.*

PROOF. – Let g be any element of G such that the subgroup $\langle g\rangle$ is not nearly normal. Suppose first that g has infinite order. Since $\langle g\rangle X$ is not finitely generated, there exist infinitely many elements $x_1, x_2, \dots$ of X such that the chain

$$\langle g, x_1\rangle < \dots < \langle g, x_1, \dots, x_n\rangle < \langle g, x_1, \dots, x_n, x_{n+1}\rangle < \dots$$

is infinite, and it follows from Lemma 3.1 that the subgroup $\langle g, x_1, \dots, x_r\rangle$ is nearly normal in G for some positive integer r. Thus $\langle g\rangle X = \langle g, x_1, \dots, x_r\rangle X$ is likewise nearly normal in G. Assume now that g has finite order and let h be an element of infinite order of X. Then $\langle g\rangle X = \langle gh\rangle X$ is a nearly normal subgroup of G by the first part of the proof. Therefore G/X is an FC-group by Lemma 2.1 and so $G'X/X$ is finite by Theorem 2.2. □

LEMMA 3.4. – *Let G be a soluble-by-finite $DC_{\bar{nn}}$-group which is not minimax. Then G satisfies the maximal condition on non-(nearly normal) subgroups.*

PROOF. – Clearly we may suppose that G' is infinite. It follows from Theorem A that G is a group of finite rank containing a finite normal subgroup E such that G/E is the direct product of finitely many Prüfer subgroup and of a torsion-free nilpotent subgroup N of class 2 such that $N/Z(N)$ is finitely generated. Replacing G by G/E, it can be assumed that G is nilpotent. Since G is generated by its elements of infinite order, application of Lemma 2.1 and Theorem 2.2 yields that G contains an infinite cyclic subgroup which is not nearly normal. Thus $G = N$ by Corollary 3.2. If X is any subgroup of $Z(G)$ which is not finitely generated, $G'X/X$ is finite by Lemma 3.3. Therefore G satisfies the maximal condition on non-(nearly normal) subgroups (see [7]). □

If follows from Lemma 3.4 and Lemma 2.11 that in order to prove Theorem B we only have to consider soluble-by-finite minimax $DC_{\bar{nn}}$-groups.

LEMMA 3.5. – *Let G be a finite-by-nilpotent $DC_{\bar{nn}}$-group. Then G either satisfies the minimal or the maximal condition on non-(nearly normal) subgroups.*

PROOF. – Assume for a contradiction that the statement is false, so that G is minimax by Lemma 3.4. Let N be a finite normal subgroup of G such that G/N is nilpotent. Clearly, G/N is also a counterexample and so we may suppose that G is nilpotent. Let F be the FC-centre of G. Then F' is finite by Theorem 2.2 and replacing G by G/F' it can be assumed that F is abelian. Moreover, Corollary 2.5 shows that G is not periodic, so that there exists an element of infinite order in $G \setminus F$. In particular, G must be residually finite by Corollary 3.2. Let $(H_n)_{n\in\mathbb{N}}$ an infinite ascending chain of non-(nearly normal) subgroups of G and put

$$H = \bigcup_{n\in\mathbb{N}} H_n.$$

Then H is not periodic and hence by Lemma 3.1 it is contained in F. As G satisfies the double chain condition on non-(nearly normal) subgroups, the set of all non-(nearly normal) subgroups of G contained in H_1 has a minimal element K. Then K cannot be generated by two proper subgroups, which is impossible since G is residually finite. This contradiction completes the proof. □

COROLLARY 3.6. – *Let G be a $DC_{\bar{n}n}$-group whose centre $Z(G)$ neither is periodic nor finitely generated. Then G either satisfies the minimal or the maximal condition on non-(nearly normal) subgroups.*

PROOF. – Since $G'Z(G)/Z(G)$ is finite by Lemma 3.3, the group G is finite-by-nilpotent and so the statement follows from Lemma 3.5. □

LEMMA 3.7. – *Let G be a minimax $DC_{\bar{n}n}$-group whose infinite cyclic subgroups are nearly normal. If G is not periodic, then $G/Z(G)$ is polycyclic-by-finite.*

PROOF. – By Lemma 2.6 it is enough to prove that G' is polycyclic-by-finite, so that Lemma 2.9 allows us to assume that G contains an abelian normal subgroup A of finite index. In particular, G locally satisfies the maximal condition on subgroups and so we may suppose that A is not finitely generated. Since all infinite cyclic subgroups of G are nearly normal, G contains an infinite cyclic normal subgroup X. Let g be any element of finite order of G. As all subgroups of A have finitely many conjugates in G and $XA\langle g\rangle$ is not finitely generated, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of finitely generated G-invariant subgroups of A such that $XA_n\langle g\rangle < XA_{n+1}\langle g\rangle$ for each n. The $DC_{\bar{n}n}$-condition applied to the double chain

$$\ldots < \langle g\rangle X^{2^{n+1}} < \langle g\rangle X^{2^n} < \ldots < \langle g\rangle X \leq \langle g\rangle XA < \ldots < \langle g\rangle XA_n < \ldots$$

yields that there exists a finitely generated normal subgroup E of G such that the index $|\langle g\rangle^G E : \langle g\rangle E|$ is finite. In particular, $\langle g\rangle^G$ is polycyclic-by-finite. As G/A is finite, it follows that G contains a polycyclic-by-finite normal subgroup B such that $G = AB$. Thus G' is contained in B, and hence G' is polycyclic-by-finite. □

LEMMA 3.8. – *Let G be a soluble-by-finite non-periodic minimax $DC_{\bar{n}n}$-group. If G is residually finite, then it either satisfies the minimal or the maximal condition on non-(nearly normal) subgroups.*

PROOF. – Can obviously be assumed that G does not satisfy the maximal condition on subgroups. If all infinite cyclic subgroups of G are nearly normal, then the statement follows from Lemma 3.7 and Corollary 3.6. Thus we may suppose that G contains an element x of infinite order with infinitely many conjugates. In particular, $G/\langle x\rangle^G$ satisfies the maximal condition on non-(nearly normal) subgroups by Lemma 3.1 and hence $G'\langle x\rangle^G/\langle x\rangle^G$ is polycyclic-by-finite (see [7], Lemma 2.6). Let F be the Fitting subgroup of G. As G/F is polycyclic-by-finite, F cannot be finitely generated. Moreover, $\langle x\rangle^G$ satisfies the maximal condition on normal subgroups by Lemma 3.1. If x belongs to F, then $\langle x\rangle^G$ satisfies he maximal condition on subgroups (see [9] Part 1, Theorem 5.37) and hence G' is polycyclic-by-finite; thus $G/Z(G)$ is polycyclic-by-finite by Lemma 2.6

and the statement follows again from Corollary 3.6. Assume that x does not belong to F. The above argument allows to suppose that F is contained in the FC-centre of G; in particular, G locally satisfies the maximal condition on subgroups. Then $F\langle x\rangle$ is not finitely generated and it follows from Lemma 3.1 that there exists a finitely generated subgroup E of F such that $\langle x, E\rangle$ is nearly normal in G. Therefore $\langle x\rangle^G$ satisfies the maximal condition on subgroups, so that G' is polycyclic-by-finite also in this case, and the statement follows as before. □

We can now prove Theorem B.

PROOF OF THEOREM B – It follows from Lemma 2.11 that G is soluble-by-finite. Moreover, by Lemma 3.4 and Corollary 2.5, it can be assumed that G is a non-periodic minimax group. By Lemma 3.8 we may also suppose that the finite residual J of G is not trivial. Thus all infinite cyclic subgroups of G are nearly normal by Corollary 3.2 and so Lemma 3.7 yields that $G/Z(G)$ is polycyclic-by-finite; in particular, J is contained in $Z(G)$. Moreover, by Corollary 3.6 it can be assumed that $Z(G)$ is periodic, so that $Z(G)/J$ is finite and G/J is polycyclic-by-finite. Assume for a contradiction that

$$H_1 > H_2 > \ldots > H_n > \ldots$$

is an infinite descending chain of non-(nearly normal) subgroups of G, and let M be a maximal non-(nearly normal) subgroup of G containing H_1. Then J must be contained in M and so G/J has infinite commutator subgroup.

Let H/J be any infinite subgroup of G/J. As G/J is polycyclic-by-finite, H is not periodic and there exists a finitely generated subgroup K such that $H = JK$. Since all infinite cyclic subgroups of G are nearly normal, H contains an infinite cyclic normal subgroup $\langle x\rangle$ of G. Let y be any element of K such that $\langle y\rangle$ is not nearly normal in G; then y has finite order. If p is any prime in $\pi(J)$ and n is the smallest positive integer such that $\langle x, y\rangle \cap J_p$ is contained in $\Omega_n(J_p)$, then

$$\ldots < \langle x^{2^{n+1}}, y\rangle < \langle x^{2^n}, y\rangle < \ldots < \langle x, y\rangle \leq \langle x, y\rangle\Omega_n(J_p) < \langle x, y\rangle\Omega_{n+1}(J_p) < \ldots$$

is a double chain; it follows from the condition $DC_{\overline{nn}}$ that there exists a subgroup X_y of $J\langle x\rangle$ such that $\langle y, X_y\rangle$ is a nearly normal subgroup of G. Thus $\langle x, y\rangle J = \langle y, X_y\rangle J\langle x\rangle$ is likewise nearly normal. As K is finitely generated, $H = JK = JK\langle x\rangle$ is nearly normal in G. Therefore all infinite subgroups of G/J are nearly normal and so G/J is cyclic-by-finite since it is neither periodic nor finite-by-abelian (see [6], Theorem 2.2). Let L be a finitely generated subgroup of G such that $G = JL$. As $J \leq Z(G)$ and $J \cap L$ is finite, the subgroup L is an infinite cyclic-by-finite normal subgroup of G and L' is infinite. Then there exists a finite characteristic subgroup E of L containing $J \cap L$ such that L/E is infinite dihedral. Therefore $G/E = JE/E \times L/E$ and hence G satisfies the minimal condition on non-(nearly normal) subgroups (see [6], Theorem 2.14). This contradiction completes the proof. □

REFERENCES

[1] B. AMBERG, *Fast-Polyminimaxgruppen*, Math. Ann., **175** (1968), 44-49.
[2] R. BAER, *Polyminimaxgruppen*, Math. Ann., **175** (1968), 1-43.
[3] V. S. CARIN, *On soluble groups of type A_3*, Mat. Sb., **54** (1961), 489-499.
[4] G. CUTOLO - L. A. KURDACHENKO, *Weak chain conditions for non-almost normal subgroups*, Proceedings of *Groups St. Andrews in Galway*, Cambridge University Press (1993), 120-130.
[5] F. DE MARI - F. DE GIOVANNI, *Double chain conditions for infinite groups*, Ricerche Mat., **54** (2005), 59-70.
[6] S. FRANCIOSI - F. DE GIOVANNI, *Groups satisfying the minimal condition on certain non-normal subgroups*, Proceedings of *Groups Korea 1994*, de Gruyter (1995), 107-118.
[7] A. GALOPPO, *Groups satisfying the maximal condition on non-nearly normal subgroups*, Ricerche Mat., **49** (2000), 213-220.
[8] B. H. NEUMANN, *Groups with finite classes of conjugate subgroups*, Math Z., **63** (1955), 76-96.
[9] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin (1972).
[10] T. S. SHORES, *A chain condition for groups*, Rocky Mountain J. Math., **3** (1973), 83-89.
[11] V. P. ŠUNKOV, *Locally finite groups with a minimality condition for abelian subgroups*, Algebra and Logic, **9** (1970), 350-370.
[12] D. I. ZAIČEV, *Groups satisfying a weak minimum condition*, Soviet Math. Dokl., **9** (1968), 194-197.
[13] D. I. ZAIČEV, *On groups which satisfy a weak minimality condition*, Math. USSR Sb., **7** (1969), 315-322.
[14] D. I. ZAIČEV, *Theory of minimax groups*, Ukrainian Math. J., **23** (1971), 536-542.

Dipartimento di Matematica e Applicazioni Università di Napoli Federico II,
Complesso Universitario Monte S. Angelo, Via Cintia I - 80126 Napoli (Italy)
e-mail: fausto.demari@dma.unina.it

Pervenuta in Redazione
il 12 aprile 2006