Patrizia Di Gironimo

Quasiharmonic Fields: a Higher Integrability Result


Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_843_0>
Quasiharmonic Fields: a Higher Integrability Result

Patrizia Di Gironimo

Sunto. – In questo lavoro si studia il grado di integrabilità dei campi quasiharmonici. Questi campi sono connesi con lo studio dell’equazione \( \text{div}(A(x)\nabla u(x)) = 0 \), dove la matrice simmetrica \( A(x) \) soddisfa la condizione

\[
|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)(A(x)\xi, \xi).
\]

La funzione non negativa \( \mathcal{K}(x) \) appartiene alla classe esponenziale, cioè esiste \( \beta > 0 \) tale che \( \exp(\beta \mathcal{K}(x)) \) è integrabile. Si dimostra che il gradiente di una soluzione locale dell’equazione appartiene agli spazi di Zygmund \( L^2_{\text{loc}} \log^{a-1}L \), \( 0 < a = a(\beta) \). Inoltre si prova come il grado di migliore regolarità dipenda da \( \beta \).

Summary. – In this paper we study the degree of integrability of quasiharmonic fields. These fields are connected with the study of the equation \( \text{div}(A(x)\nabla u(x)) = 0 \), where the symmetric matrix \( A(x) \) satisfies the condition

\[
|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)(A(x)\xi, \xi).
\]

The nonnegative function \( \mathcal{K}(x) \) belongs to the exponential class, i.e. \( \exp(\beta \mathcal{K}(x)) \) is integrable for some \( \beta > 0 \). We prove that the gradient of a local solution of the equation belongs to the Zygmund spaces \( L^2_{\text{loc}} \log^{a-1}L \), \( 0 < a = a(\beta) \). Moreover we show exactly how the degree of improved regularity depends on \( \beta \).

1. – Introduction.

Let \( \Omega \subset \mathbb{R}^n \) be a connected open set.

If \( B : \Omega \to \mathbb{R}^n \), \( E : \Omega \to \mathbb{R}^n \) are integrable vector fields on \( \Omega \) such that

\[
\text{div} B = \sum_{i=1}^n \frac{\partial B_i}{\partial x_i} = 0
\]

(1.1)

\[
\text{curl} E = \left( \frac{\partial E_j}{\partial x_i} - \frac{\partial E_i}{\partial x_j} \right)_{i,j=1,...,n} = 0,
\]

in the sense of distributions, the scalar product \( \langle B, E \rangle \) is referred to as a div-curl product.

In this paper we shall study the degree of integrability of a class of div-curl
fields \((B, E)\) which are coupled by the distortion inequality

\[ |B|^2 + |E|^2 \leq \mathcal{K}(x)(B, E) \quad \text{a.e. in } \Omega \]

where \(1 \leq \mathcal{K}(x) < \infty\) is a measurable function in \(\Omega\).

A div-curl field \((B, E)\) satisfying (1.2) is called a quasiharmonic field.

An example of quasiharmonic fields grew out of the study of the equation

\[ \text{div}(A(x) \nabla u(x)) = 0 \]

where the symmetric matrix \(A(x) \in \mathbb{R}^{n \times n}\) satisfies the condition

\[ \frac{1}{K(x)} |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq K(x)|\xi|^2 \]

and \(K(x), K(x) \geq 1\) a.e., is a measurable function on \(\Omega\).

It is well-known (see [IS2]) that it is possible to express (1.4), equivalently, by using just one inequality

\[ |\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x) \langle A(x)\xi, \xi \rangle \]

for almost every \(x \in \Omega\) and all \(\xi \in \mathbb{R}^n\), with \(\mathcal{K}(x) = K(x) + \frac{1}{K(x)}\).

There are two vector fields associated with a solution of the equation (1.3). The first one, denoted by \(E = \nabla u(x)\), is curl free, while the second \(B = A(x)\nabla u(x)\) is divergence free. The condition (1.5) shows that the pair \(\mathcal{F} = [B, E]\) is a quasiharmonic field.

Throughout this paper we shall assume that

\[ \langle B, E \rangle \in L^1_{\text{loc}}(\Omega). \]  

The function \(\mathcal{K}(x)\) in (1.5) belongs to the exponential class \(\text{Exp} (\Omega)\), defined via the Orlicz function \(P(t) = e^t - 1\). Precisely, we assume that

\[ \int_{\Omega} e^{\beta \mathcal{K}(x)} \, dx < +\infty \]

for some \(\beta > 0\).

By assumptions (1.2), (1.6) and (1.7) we deduce that \(B\) and \(E\) belong to the Orlicz-Zygmund spaces \(L^2_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n)\) (see [RR]).

It is our goal here to investigate the degree of integrability of a quasiharmonic field, under the assumptions (1.6) and (1.7).

Our theorem not only shows a higher integrability, but it also indicates exactly how the degree of improved regularity depends on \(\beta\).

**Theorem 1.1.** Let \((B, E)\) a div-curl field verifying (1.2) and (1.6). Assume that the distortion \(\mathcal{K}(x) \geq 1\) satisfies (1.7) for some \(\beta > 0\). Then there exists \(a = c(n)\beta > 0\) such that \(B \in L^2_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n)\), \(E \in L^2_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n)\).
As a consequence we deduce a higher integrability result for the gradient of “finite energy” solutions of the equations (1.3) verifying (1.5)(see Prop.3.3).

Recently regularity results for quasiharmonic fields have been investigated in [IS$_2$], [IMMP], [M]. The aim of the previous paper is to establish regularity results for $B$ and $E$ without fixing $\beta$ in (1.7). There the result states a higher integrability of $B$ and $E$, provided $\beta$ is sufficiently large.

In [MM], assuming that $K(x)^{1/2}, \gamma > 1$, belongs to the exponential class, the authors prove that $B$ and $E$ belong to $L^2_{\text{loc}}\log^{a} L$ for any $a > 0$.

When $K(x)$ is bounded higher integrability results of quasiharmonic fields have been investigate in [IS$_2$]. (See also the references therein).

Recently a result similar to Theorem 1 has been obtained by [FKZ] for mappings of finite distortion.

2. – Preliminary results.

Define $L^s\log^a L(\Omega), 1 \leq s < + \infty, a \in R$ as the Orlicz-Zygmund space generated by $\phi(t) = t^a \log^a (e + t)$, at least for sufficiently large values of $t$, i.e. the space of all measurable functions $f$ on $\Omega$ such that

$$
\|f\|_\phi = \inf \{\lambda > 0 : \int_{\Omega} \phi \left( \frac{|f|}{\lambda} \right) \, dx \leq 1 \}.
$$

(2.1)

Let us recall that for $a \geq 0$ the non linear functional

$$
[f]_{s,a} = \left( \int_{\Omega} |f|^a \log^a \left( e + \frac{|f|}{\|f\|_s} \right) \right)^{\frac{1}{a}}
$$

is comparable with the Luxemburg norm defined by (2.1).

A central ingredient in our arguments is the classical Hardy-Littlewood maximal function. Recall that, give a function $g \in L^1_{\text{loc}}(\Omega)$, we define the Hardy-Littlewood maximal function $Mg$ of $g$ by

$$
Mg(x) = \sup_{r > 0} \frac{1}{|B_r|} \int_{B_r} |g(y)| \, dy,
$$

for every ball $B_r$ of $\Omega$ containing the given point $x \in \Omega$.

The following proposition is classical. The proof involves Vitali’s covering lemma and the Calderon-Zygmund decomposition, [S].

**Proposition 2.1.** – Let $h \in L^1(R^n)$. For any $t > 0$, we have

$$
\frac{1}{2^n t} \int_{|h| > t} |h(x)| \, dx \leq \frac{1}{|h|} \int_{|h| > 1/2} |h(x)| \, dx.
$$

$$
\frac{1}{2^n t} \int_{|h| > t} |h(x)| \, dx \leq \frac{2}{t} \int_{|h| > 1/2} |h(x)| \, dx.
$$

\text{(2.1)}

$$
\frac{1}{2^n t} \int_{|h| > t} |h(x)| \, dx \leq \frac{1}{|h|} \int_{|h| > 1/2} |h(x)| \, dx.
$$

The next Lemma is crucial to establish Theorem 1.1. For a proof see [IS$_1$], [GIM], [MM].

**Lemma 2.2.** Let $\langle B, E \rangle$ be a nonnegative div-curl product such that $B \in L^p \log^{-1} L(\Omega, R^n)$, $E \in L^q \log^{-1} L(\Omega, R^n)$ with $1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $0 < \sigma < 1$

$$\int_{\sigma B} \langle B, E \rangle \, dx \leq c \left( \int_{B \rho} |B|^r \, dx \right)^{\frac{1}{r}} \left( \int_{B \rho} |E|^s \, dx \right)^{\frac{1}{s}}$$

where $B \rho = B(x, \rho) \subset \subset \Omega$, $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}$, $1 \leq r \leq p$, $1 \leq s \leq q$ and $c = c(n, p, q)$.

3. **Proof of Theorem 1.1.**

Now we fix a ball $B_0 = B(x_0, r_0) \subset \subset \Omega$ and assume that

$$\int_{B_0} \langle B, E \rangle \, dx = 1 \quad (3.1)$$

for homogeneity property, under our assumptions this condition is not restrictive.

Define the following auxiliary functions

$$h_1(x) = d^n(x) \langle B, E \rangle \quad (3.2)$$

$$h_2(x) = d(x) (|B|^\frac{s}{n} + |E|^\frac{s}{n}) \quad (3.3)$$

$$h_3(x) = \chi_{B_0}(x) \quad (3.4)$$

where $d(x) = dist(x, R^n \setminus B_0)$ and $\chi_E$ is the characteristic function of the set $E$.

The following Lemma will be useful to establish Theorem 1.1.

**Lemma 3.1.** If (3.1) holds, then we have

$$\left( \int_{B} h_1(x) \, dx \right)^{\frac{1}{n}} \leq c(n) \left( \int_{2B} h_2^3(x) \, dx \right)^{\frac{1}{3}} + c(n) \left( \int_{2B} h_3(x) \, dx \right)^{\frac{1}{3}} \quad (3.5)$$

for all balls $B \subset R^n$, where $h_1(x), h_2(x), h_3(x)$ are defined by (3.2)-(3.4) and

$$q = \frac{n^2}{n + 1}.$$ 

**Proof.** We need to prove it only if $B$ intersects $B_0$, otherwise one can easily see that (3.5) is trivial. We split the proof of (3.5) in two cases, precisely when $3B$ is or is not contained in $B_0$. 

CASE 1: $3B \subset B_0$. By a geometric consideration we have that

$$\max_{x \in B} d(x) \leq 4 \min_{x \in 2B} d(x).$$

Applying Lemma 2.2

$$\left( \int_B h_1(x) dx \right)^{\frac{1}{n}} \leq \max_{x \in B} d(x) \left( \int_B \langle B, E \rangle dx \right)^{\frac{1}{n}}$$

$$\leq c(n) \min_{2B} d(x) \left( \int_{2B} |B|^{\frac{n}{2n+1}} dx \right)^{\frac{n+1}{n}} \left( \int_{2B} |E|^{\frac{n}{2n+1}} dx \right)^{\frac{n+1}{2n}}$$

$$\leq c(n) \min_{2B} d(x) \left[ \left( \int_{2B} |B|^{\frac{n^2}{2n+1}} dx \right)^{\frac{n+1}{n}} + \left( \int_{2B} |E|^{\frac{n^2}{2n+1}} dx \right)^{\frac{n+1}{2n}} \right]$$

$$\leq c(n) \left\{ \int_{2B} d(x) \left[ |B|^{\frac{2}{n+1}} + |E|^{\frac{2}{n+1}} \right]^{\frac{n+1}{n}} dx \right\}^{\frac{n}{n+1}}$$

$$= c(n) \left( \int_{2B} h_2(x) dx \right)^{\frac{1}{n}}$$

CASE 2: $3B \not\subset B_0, B \cap B_0 \neq \emptyset$. We have that

$$\max_{x \in B} d(x) \leq \max_{x \in 2B} d(x) \leq c(n)|2B \cap B_0|^{\frac{1}{n}}.$$

By using (3.1), we conclude that

$$\left( \int_B h_1(x) dx \right)^{\frac{1}{n}} \leq \max_{B} d(x) \left( \frac{1}{|B|} \int_{B \cap B_0} \langle B, E \rangle dx \right)^{\frac{1}{n}}$$

$$\leq c(n) \left( \frac{|2B \cap B_0|}{|B|} \right)^{\frac{n}{n+1}} \left( \int_{B_0} \langle B, E \rangle dx \right)^{\frac{1}{n}}$$

$$\leq c(n) \left( \frac{1}{|2B|} \int_{2B} h_3(x) dx \right)^{\frac{1}{n}}.$$

Combining these two cases we get the inequality (3.5). 

**Proof of Theorem 1.1.** – According to Lemma (3.1), we observe that (3.5) is true for all balls $B \subset \mathbb{R}^n$. So the following point-wise inequality for the maximal functions yields

$$[M(h_1(y))]^{\frac{1}{n}} \leq c(n)[M(h_2^2(y))]^{\frac{1}{n}} + c(n)[M(h_3(y))]^{\frac{1}{n}}, \quad \forall y \in \mathbb{R}^n$$
from which, for \( \lambda > 0 \), we also deduce that

\[
\left| \{ x \in \mathbb{R}^n / M(h_1)(x) > \lambda^n \} \right| \leq \left| \{ x \in \mathbb{R}^n / c(n)M(h_2^q)(x) > \lambda^q \} \right| + \left| \{ x \in \mathbb{R}^n / c(n)M(h_3)(x) > \lambda^n \} \right|.
\]

The definition of \( h_3 \) implies that \( M(h_3)(x) \leq 1 \) in \( \mathbb{R}^n \), then the set \( \{ x \in \mathbb{R}^n / M(h_2)(x) > \lambda^n \} \) is empty for \( \lambda > \lambda_1 = \lambda_1(n) \). Hence

\[
\left| \{ x \in \mathbb{R}^n / M(h_1)(x) > \lambda^n \} \right| \leq \left| \{ x \in \mathbb{R}^n / c(n)M(h_2^q)(x) > \lambda^q \} \right|
\]

for all \( \lambda > \lambda_1 \). We use Proposition 2.1 to deduce

\[
(3.6) \quad \int_{h_1 > \lambda^n} h_1(x)dx \leq c(n)\lambda^{n-q} \int_{c(n)h_2 > \lambda} h_2^q(x)dx
\]

for all \( \lambda > \lambda_1 \). We may assume that the constant \( c(n) \) in (3.6) is bigger than one.

Let us define the function

\[
\psi(\lambda) = \frac{n - q}{a} \log^a \lambda + \log^{a-1} \lambda,
\]

where \( q = \frac{n^2}{n+1} \) as above, and \( a \) is a positive constant that will be fixed in equation (3.9) below.

Observe that

\[
\phi(\lambda) = \frac{d}{d\lambda} \psi(\lambda) = \frac{n - q}{\lambda} \log^a \lambda + \frac{a - 1}{\lambda} \log^{a-2} \lambda > 0
\]

for all \( \lambda > \lambda_2 = \exp\left(\frac{n+1}{n}\right) \) and that

\[
\lambda^{n-q} \phi(\lambda) = \frac{d}{d\lambda} (\lambda^{n-q} \log^a \lambda).
\]

So we can multiply both sides of (3.6) by \( \phi(\lambda) \), integrated with \( \lambda \) over (\( \lambda_0, j \)), for \( j \) large and \( \lambda_0 = \max(\lambda_1, \lambda_2) \). Changing the order of the integration we get

\[
\int_{j^* > h_1 > \lambda_0^n} h_1^1(x)dx \int_{\lambda_0}^{h_1^1} \phi(\lambda)d\lambda \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^q(x)dx \int_{\lambda_0}^{h_2^q} \lambda^{n-q} \phi(\lambda)d\lambda,
\]

that is,

\[
\int_{j^* > h_1 > \lambda_0^n} (\psi(h_1^1(x)) - \psi(\lambda_0))h_1(x)dx \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^q(x)\log^{a-1}(c(n)h_2(x))dx.
\]
Then it follows easily from (3.1) that

\[
\frac{1}{a} \int_{j^a > h_1 > j_0^a} h_1(x) \log^a h_1^2(x) dx \leq c(n) \int_{j > c(n) h_2 > j_0} h_2^n(x) \log^{a-1} (c(n) h_2(x)) dx \\
+ c(n, a) |\mathcal{B}_0|
\]

(3.7)

where \( c(n) \geq 1 \).

In the remaining part of the proof, by using the distortion inequality, we want to absorb the right hand side of (3.7) on the left.

Using the following elementary inequality (see [FKZ])

\[
ab \log^{a-1} (C(n) (ab)^\frac{1}{2}) \leq \frac{C(n)}{\beta} a \log^a (a^\frac{1}{2}) + C(a, \beta, n) \exp(\beta b)
\]

with \( a = h_1(x), b = K(x) \), we deduce by (1.2) that

\[
\frac{1}{a} \int_{j^a > h_1 > j_0^a} h_1(x) \log^a h_1^2(x) dx \\
\leq \frac{c(n)}{\beta} \int_{j^a > h_1 > j_0^a} h_1(x) \log^a h_1^2(x) dx + c(n, a, \beta) \int_{\mathcal{B}_0} \exp(\beta K(x)) dx \\
+ c(n, a) |\mathcal{B}_0|
\]

(3.8)

\[
\leq \frac{c(n)}{\beta} \int_{j^a > h_1 > j_0^a} h_1(x) \log^a h_1^2(x) dx + c(n, a, \beta) \int_{\mathcal{B}_0} \exp(\beta K(x)) dx.
\]

By (3.8), setting

(3.9)

\[
a = \frac{\beta}{2c(n)}
\]

we obtain

(3.10)

\[
\int_{j^a > h_1 > j_0^a} h_1(x) \log^a h_1^2(x) dx \leq c(n, \beta) \int_{\mathcal{B}_0} \exp(\beta K(x)) dx.
\]

By letting \( j \rightarrow +\infty \), in both sides of (3.10), and using the monotone convergence theorem we deduce that

\[
\int_{\mathcal{B}_0} d^n(x) \langle B, E \rangle \log^a (e + d^n(x) \langle B, E \rangle) dx \leq c(n, \beta) \int_{\mathcal{B}_0} \exp(\beta K(x)) dx.
\]

Finally, noticing that in \( \sigma \mathcal{B}_0 = \mathcal{B}(x_0, \sigma r_0) \) with \( 0 < \sigma < 1 \) we have \( d^n(x) \geq (1 - \sigma)^n r_0^n \geq c(n, \sigma) |\mathcal{B}_0| \), and taking into account the normalization (3.1), we have
the inequality
\[
\int_{B_0} (B,E) \log^\alpha \left( e + \frac{(B,E)}{\int_{B_0} (B,E) dx} \right) dx 
\leq c(u, \beta, \sigma) \left( \int_{B_0} \exp(\beta K(x)) dx \right) \left( \int_{B_0} (B,E) dx \right)
\]
which concludes the proof, by means of (1.2).

At this point we apply this result to the study of equation (1.3).

Note that (1.3) is the Euler-Lagrange equation of the variational integral
\[
\mathcal{E}[u] = \int_\Omega < A(x) \nabla u, \nabla u > dx.
\]
We deal with solution of (1.3) having “finite energy”, namely \( \mathcal{E}[u] \) is finite.

If \( u \in W^{1,1}_{loc}(\Omega) \) is a local solution of (1.3), we set \( B = A \nabla u \) and \( E = \nabla u \) so that \( \text{div} B = 0 \) and \( \text{curl} E = 0 \). Let us remark that \( (B,E) \) is locally integrable on \( \Omega \), since \( u \) is a local solution of (1.3) with “finite energy”. By assumptions (1.5), (1.7) the gradient of a finite energy solution belongs to the Orlicz-Zygmund space \( L^2_{loc} \log^{\alpha - 1} L(\Omega, R^n) \). From Theorem 1.1 we deduce the following

\textbf{Proposition 3.3.} – Let \( u \) be a local solution of (1.3) with “finite energy”. Assume that the distortion \( K(x) \geq 1 \) satisfies (1.7) for some \( \beta > 0 \). Then there exists \( a = c(n)\beta > 0 \) such that \( |\nabla u| \in L^2_{loc} \log^{\alpha - 1} L(\Omega, R^n) \), \( |A \nabla u| \in L^2_{loc} \log^{\alpha - 1} L(\Omega, R^n) \).

\textbf{REFERENCES}


Dipartimento di Matematica e Informatica,
Università di Salerno, Via Ponte Don Melillo-84084 Fisciano (SA)
E-mail: pdigironimo@unisa.it

Pervenuta in Redazione
il 28 gennaio 2006 e in forma rivista il 10 ottobre 2007