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Quasiharmonic Fields: a Higher Integrability Result


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<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_843_0>
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Sunto. – In questo lavoro si studia il grado di integrabilitá dei campi quasiarmonici. Questi campi sono connessi con lo studio dell’equazione \( \text{div}(A(x)\nabla u(x)) = 0 \), dove la matrice simmetrica \( A(x) \) soddisfa la condizione

\[
|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)(A(x)\xi, \xi).
\]

La funzione non negativa \( \mathcal{K}(x) \) appartiene alla classe esponenziale, cioè esiste \( \beta > 0 \) tale che \( \exp(\beta \mathcal{K}(x)) \) è integrabile. Si dimostra che il gradiente di una soluzione locale dell’equazione appartiene agli spazi di Zygmund \( L_{\text{loc}}^2 \log^{a-1} L \), \( 0 < a = a(\beta) \). Inoltre si prova come il grado di migliore regolarità dipende da \( \beta \).

Summary. – In this paper we study the degree of integrability of quasiharmonic fields. These fields are connected with the study of the equation \( \text{div}(A(x)\nabla u(x)) = 0 \), where the symmetric matrix \( A(x) \) satisfies the condition

\[
|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)(A(x)\xi, \xi).
\]

The nonnegative function \( \mathcal{K}(x) \) belongs to the exponential class, i.e. \( \exp(\beta \mathcal{K}(x)) \) is integrable for some \( \beta > 0 \). We prove that the gradient of a local solution of the equation belongs to the Zygmund spaces \( L_{\text{loc}}^2 \log^{a-1} L \), \( 0 < a = a(\beta) \). Moreover we show exactly how the degree of improved regularity depends on \( \beta \).

1. – Introduction.

Let \( \Omega \subset \mathbb{R}^n \) be a connected open set.

If \( B : \Omega \rightarrow \mathbb{R}^n, E : \Omega \rightarrow \mathbb{R}^n \) are integrable vector fields on \( \Omega \) such that

\[
\text{div} B = \sum_{i=1}^{n} \frac{\partial B_i}{\partial x_i} = 0
\]

(1.1)

\[
\text{curl} E = \left( \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right)_{i,j=1,...,n} = 0,
\]

in the sense of distributions, the scalar product \( \langle B, E \rangle \) is referred to as a div-curl product.

In this paper we shall study the degree of integrability of a class of div-curl
fields \((B, E)\) which are coupled by the distortion inequality

\[
|B|^2 + |E|^2 \leq \mathcal{K}(x)(B, E) \quad \text{a.e. in } \Omega
\]

where \(1 \leq \mathcal{K}(x) < \infty\) is a measurable function in \(\Omega\).

A div-curl field \((B, E)\) satisfying (1.2) is called a quasiharmonic field.

An example of quasiharmonic fields grew out of the study of the equation

\[
\text{div}(A(x)\nabla u(x)) = 0
\]

where the symmetric matrix \(A(x) \in \mathbb{R}^{n \times n}\) satisfies the condition

\[
\frac{1}{K(x)} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq K(x)|\xi|^2
\]

and \(K(x), K(x) \geq 1\) a.e., is a measurable function on \(\Omega\).

It is well-known (see [IS2]) that it is possible to express (1.4), equivalently, by using just one inequality

\[
|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)|A(x)\xi, \xi|
\]

for almost every \(x \in \Omega\) and all \(\xi \in \mathbb{R}^n\), with \(\mathcal{K}(x) = K(x) + \frac{1}{K(x)}\).

There are two vector fields associated with a solution of the equation (1.3). The first one, denoted by \(E = \nabla u(x)\), is curl free, while the second \(B = A(x)\nabla u(x)\) is divergence free. The condition (1.5) shows that the pair \(\mathcal{F} = [B, E]\) is a quasiharmonic field.

Throughout this paper we shall assume that

\[
\langle B, E \rangle \in L^1_{\text{loc}}(\Omega).
\]

The function \(\mathcal{K}(x)\) in (1.5) belongs to the exponential class \(Exp(\Omega)\), defined via the Orlicz function \(P(t) = e^t - 1\). Precisely, we assume that

\[
\int_{\Omega} e^{\beta\mathcal{K}(x)} \, dx < +\infty
\]

for some \(\beta > 0\).

By assumptions (1.2), (1.6) and (1.7) we deduce that \(B\) and \(E\) belong to the Orlicz-Zygmund spaces \(L^2_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n)\) (see [RR]).

It is our goal here to investigate the degree of integrability of a quasiharmonic field, under the assumptions (1.6) and (1.7).

Our theorem not only shows a higher integrability, but it also indicates exactly how the degree of improved regularity depends on \(\beta\).

**Theorem 1.1.** -- Let \((B, E)\) a div-curl field verifying (1.2) and (1.6). Assume that the distortion \(\mathcal{K}(x) \geq 1\) satisfies (1.7) for some \(\beta > 0\). Then there exists \(a = c(n)\beta > 0\) such that \(B \in L^a_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n), E \in L^2_{\text{loc}} \log^{a-1} L(\Omega, \mathbb{R}^n)\).
As a consequence we deduce a higher integrability result for the gradient of “finite energy” solutions of the equations (1.3) verifying (1.5) (see Prop.3.3).

Recently regularity results for quasiharmonic fields have been investigated in [IS], [IMMP], [M]. The aim of the previous paper is to establish regularity results for $B$ and $E$ without fixing $\beta$ in (1.7). There the result states a higher integrability of $B$ and $E$, provided $\beta$ is sufficiently large.

In [MM], assuming that $\mathcal{K}(x)^{\gamma}, \gamma > 1$, belongs to the exponential class, the authors prove that $B$ and $E$ belong to $L_{\text{loc}}^{2}(\log^{a} L)$ for any $a > 0$.

When $\mathcal{K}(x)$ is bounded higher integrability results of quasiharmonic fields have been investigate in [IS]. (See also the references therein).

Recently a result similar to Theorem 1 has been obtained by [FKZ] for mappings of finite distortion.

2. – Preliminary results.

Define $L^{s}(\log^{a} L)(\Omega), 1 \leq s < + \infty, a \in R$ as the Orlicz-Zygmund space generated by $\phi(t) = t^{a} \log^{a}(e + t)$, at least for sufficiently large values of $t$, i.e. the space of all measurable functions $f$ on $\Omega$ such that

\begin{equation}
\|f\|_{\phi} = \inf \{ \lambda > 0 : \int_{\Omega} \phi\left(\frac{|f|}{\lambda}\right) dx \leq 1 \}.
\end{equation}

Let us recall that for $a \geq 0$ the non linear functional

\[ [f]_{s,a} = \left[ \int_{\Omega} |f|^{s} \log^{a} \left( e + \frac{|f|}{\|f\|_{s}} \right) \right]^{rac{1}{s}} \]

is comparable with the Luxemburg norm defined by (2.1).

A central ingredient in our arguments is the classical Hardy-Littlewood maximal function. Recall that, given a function $g \in L_{\text{loc}}^{1}(\Omega)$, we define the Hardy-Littlewood maximal function $Mg$ of $g$ by

\[ Mg(x) = \sup_{r > 0} \frac{1}{|B_{r}|} \int_{B_{r}} |g(y)| dy, \]

for every ball $B_{r}$ of $\Omega$ containing the given point $x \in \Omega$.

The following proposition is classical. The proof involves Vitali’s covering lemma and the Calderon-Zygmund decomposition, [S].

**Proposition 2.1.** – Let $h \in L^{1}(R^{n})$. For any $t > 0$, we have

\[ \frac{1}{2^{2n}t} \int_{|h| > t} |h(x)| dx \leq \{|x \in R^{n}/Mh(x) > t\}| \leq \frac{2 \cdot 5^{n}}{t} \int_{|h| \geq \frac{1}{2}} |h(x)| dx. \]
The next Lemma is crucial to establish Theorem 1.1. For a proof see [IS1], [GIM], [MM].

**Lemma 2.2.** Let \( \langle B, E \rangle \) be a nonnegative div-curl product such that \( B \in L^p \log^{-1} L(\Omega, R^n), E \in L^q \log^{-1} L(\Omega, R^n) \) with \( 1 < p, q < +\infty, \frac{1}{p} + \frac{1}{q} = 1 \). Then for any \( 0 < \sigma < 1 \)

\[
\int_{\sigma B_r} \langle B, E \rangle \, dx \leq c \left( \int_{B_r} |B|^r \, dx \right)^{\frac{1}{r}} \left( \int_{B_r} |E|^s \, dx \right)^{\frac{1}{s}}
\]

where \( B_r = B(x, r) \subset \Omega, \frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}, 1 \leq r \leq p, 1 \leq s \leq q \) and \( c = c(n, p, q) \).

3. – Proof of Theorem 1.1.

Now we fix a ball \( B_0 = B(x_0, r_0) \subset \Omega \) and assume that

\[
(3.1) \quad \int_{B_0} \langle B, E \rangle \, dx = 1
\]

for homogeneity property, under our assumptions this condition is not restrictive.

Define the following auxiliary functions

\[
(3.2) \quad h_1(x) = d^n(x) \langle B, E \rangle
\]

\[
(3.3) \quad h_2(x) = d(x)(|B|^{\frac{s}{n}} + |E|^{\frac{s}{n}})
\]

\[
(3.4) \quad h_3(x) = \chi_{B_0}(x)
\]

where \( d(x) = \text{dist}(x, R^n \setminus B_0) \) and \( \chi_E \) is the characteristic function of the set \( E \).

The following Lemma will be useful to establish Theorem 1.1.

**Lemma 3.1.** If (3.1) holds, then we have

\[
(3.5) \quad \left( \int_{B} h_1(x) \, dx \right)^{\frac{1}{n}} \leq c(n) \left( \int_{2B} h_2^s(x) \, dx \right)^{\frac{s}{n}} + c(n) \left( \int_{2B} h_3(x) \, dx \right)^{\frac{s}{n}}
\]

for all balls \( B \subset R^n \), where \( h_1(x), h_2(x), h_3(x) \) are defined by (3.2)-(3.4) and \( q = \frac{n^2}{n + 1} \).

**Proof.** We need to prove it only if \( B \) intersects \( B_0 \), otherwise one can easily see that (3.5) is trivial. We split the proof of (3.5) in two cases, precisely when \( 2B \) is or is not contained in \( B_0 \).
CASE 1: $3B \subset B_0$. By a geometric consideration we have that
\[
\max_{x \in B} d(x) \leq 4 \min_{x \in 2B} d(x).
\]
Applying Lemma 2.2
\[
\left( \int_B h_1(x) dx \right)^{\frac{1}{n}} \leq \max_B d(x) \left( \int_B (B, E) dx \right)^{\frac{1}{n}}
\]
\[
\leq c(n) \min_{2B} d(x) \left( \int_{2B} |B|^{\frac{n}{n+1}} dx \right)^{\frac{n+1}{2n}} \left( \int_{2B} |E|^{\frac{n}{n+1}} dx \right)^{\frac{n+1}{2n}}
\]
\[
\leq c(n) \min_{2B} d(x) \left[ \left( \int_{2B} |B|^{\frac{n}{n+1}} dx \right)^{\frac{n+1}{2n}} + \left( \int_{2B} |E|^{\frac{n}{n+1}} dx \right)^{\frac{n+1}{2n}} \right]
\]
\[
\leq c(n) \left( \int_{2B} d(x) \left( |B|^{\frac{n}{n+1}} + |E|^{\frac{n}{n+1}} \right)^{\frac{n+1}{n}} dx \right)^{\frac{1}{n}}
\]
\[
= c(n) \left( \int_{2B} h_2^\beta(x) dx \right)^{\frac{1}{n}}
\]
CASE 2: $3B \not\subset B_0, B \cap B_0 \neq \emptyset$. We have that
\[
\max_{x \in B} d(x) \leq \max_{x \in 2B} d(x) \leq c(n)|2B \cap B_0|^\frac{1}{n}.
\]
By using (3.1), we conclude that
\[
\left( \int_B h_1(x) dx \right)^{\frac{1}{n}} \leq \max_B d(x) \left( \frac{1}{|B| B \cap B_0} \int_{B \cap B_0} (B, E) dx \right)^{\frac{1}{n}}
\]
\[
\leq c(n) \left( \frac{|2B \cap B_0|}{|B|} \int_{B_0} (B, E) dx \right)^{\frac{1}{n}}
\]
\[
\leq c(n) \left( \frac{1}{|2B|} \int_{2B} h_2^\beta(x) dx \right)^{\frac{1}{n}}.
\]
Combining these two cases we get the inequality (3.5). \qed

PROOF OF THEOREM 1.1. – According to Lemma (3.1), we observe that (3.5) is true for all balls $B \subset R^n$. So the following point-wise inequality for the maximal functions yields
\[
[M(h_1(y))]^{\frac{1}{n}} \leq c(n)[M(h_2^\beta(y))]^{\frac{1}{n}} + c(n)[M(h_3(y))]^{\frac{1}{n}}, \quad \forall y \in R^n
\]
from which, for } \lambda > 0 \text{, we also deduce that}

\begin{align*}
|\{x \in \mathbb{R}^n / M(h_1)(x) > \lambda^n\}| & \leq |\{x \in \mathbb{R}^n / c(n)M(h_2^n)(x) > \lambda^q\}| \\
& + |\{x \in \mathbb{R}^n / c(n)M(h_3)(x) > \lambda^n\}|.
\end{align*}

The definition of } h_3 \text{ implies that } M(h_3)(x) \leq 1 \text{ in } \mathbb{R}^n, \text{ then the set } \\
\{x \in \mathbb{R}^n / M(h_3)(x) > \lambda^n\} \text{ is empty for } \lambda > \lambda_1 = \lambda_1(n). \text{ Hence}

\begin{align*}
|\{x \in \mathbb{R}^n / M(h_1)(x) > \lambda^n\}| & \leq |\{x \in \mathbb{R}^n / c(n)M(h_2^n)(x) > \lambda^q\}|
\end{align*}

for all } \lambda > \lambda_1. \text{ We use Proposition 2.1 to deduce}

\begin{align}
\int_{h_1 > \lambda^n} h_1(x)dx & \leq c(n)\lambda^{n-q} \int_{c(n)h_2 > \lambda^n} h_2^n(x)dx 
\end{align}

for all } \lambda > \lambda_1. \text{ We may assume that the constant } c(n) \text{ in (3.6) is bigger than one.}

Let us define the function

\[ \psi(\lambda) = \frac{n - q}{a} \log^a \lambda + \log^{a-1} \lambda, \]

where } q = \frac{n^2}{n + 1} \text{ as above, and } a \text{ is a positive constant that will be fixed in equation (3.9) below.}

Observe that

\[ \phi(\lambda) = \frac{d}{d\lambda}\psi(\lambda) = \frac{n - q}{\lambda} \log^{a-1} \lambda + \frac{a - 1}{\lambda} \log^{a-2} \lambda > 0 \]

for all } \lambda > \lambda_2 = \exp\left(\frac{n + 1}{n}\right) \text{ and that}

\[ \lambda^{n-q} \phi(\lambda) = \frac{d}{d\lambda}(\lambda^{n-q} \log^{a-1} \lambda). \]

So we can multiply both sides of (3.6) by } \phi(\lambda) \text{, integrated with } \lambda \text{ over } (\lambda_0, j), \text{ for}

\[ j \text{ large and } \lambda_0 = \max(\lambda_1, \lambda_2). \]

Changing the order of the integration we get

\begin{align*}
\int_{j^n > h_1 > \lambda_0^n} h_1^1(x)dx \int_{\lambda_0} \phi(\lambda)d\lambda & \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^n(x)dx \int_{\lambda_0} \lambda^{n-q} \phi(\lambda)d\lambda,
\end{align*}

that is,

\begin{align*}
\int_{j^n > h_1 > \lambda_0^n} (\psi(h_1^1(x)) - \psi(\lambda_0))h_1^1(x)dx & \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^n(x)\log^{a-1}(c(n)h_2(x))dx.
\end{align*}
Then it follows easily from (3.1) that

\[
\frac{1}{a} \int_{j^a > h_1 > \lambda_0^a} h_1(x) \log^a h_1^\frac{1}{a}(x) dx \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^a(x) \log^{a-1}(c(n)h_2(x)) dx
\]

\[
+ c(n, a)|B_0|
\]

where \(c(n) \geq 1\).

In the remaining part of the proof, by using the distortion inequality, we want to absorb the right hand side of (3.7) on the left.

Using the following elementary inequality (see [FKZ])

\[
ab \log^{a-1}(C(n)(ab)^{\frac{1}{a}}) \leq \frac{C(n)}{\beta} a \log^a (ab) + C(a, \beta, n) \exp(\beta b)
\]

with \(a = h_1(x), b = K(x)\), we deduce by (1.2) that

\[
\frac{1}{a} \int_{j^a > h_1 > \lambda_0^a} h_1(x) \log^a h_1^\frac{1}{a}(x) dx
\]

\[
\leq \frac{c(n)}{\beta} \int_{j^a > h_1 > \lambda_0^a} h_1(x) \log^a h_1^\frac{1}{a}(x) dx + c(n, a, \beta) \int_{B_0} \exp(\beta K(x)) dx
\]

\[
+ c(n, a)|B_0|
\]

\[
\leq \frac{c(n)}{\beta} \int_{j^a > h_1 > \lambda_0^a} h_1(x) \log^a h_1^\frac{1}{a}(x) dx + c(n, a, \beta) \int_{B_0} \exp(\beta K(x)) dx.
\]

By (3.8), setting

\[
a = \frac{\beta}{2c(n)}
\]

we obtain

\[
\int_{j^a > h_1 > \lambda_0^a} h_1(x) \log^a h_1^\frac{1}{a}(x) dx \leq c(n, \beta) \int_{B_0} \exp(\beta K(x)) dx.
\]

By letting \(j \rightarrow +\infty\), in both sides of (3.10), and using the monotone convergence theorem we deduce that

\[
\int_{B_0} d^n(x) \langle B, E \rangle \log^a(e + d^n(x) \langle B, E \rangle) dx \leq c(n, \beta) \int_{B_0} \exp(\beta K(x)) dx.
\]

Finally, noticing that in \(\sigma B_0 = B(x_0, \sigma r_0)\) with \(0 < \sigma < 1\) we have \(d^n(x) \geq (1 - \sigma)^n r_0^n \geq c(n, \sigma)|B_0|\), and taking into account the normalization (3.1), we have
the inequality
\[
\int_{\sigma B_0} (B, E) \log^a \left( e + \frac{(B, E)}{\int_{B_0} (B, E) \, dx} \right) \, dx \\
\leq c(n, \beta, \sigma) \left( \int_{B_0} \exp(\beta K(x)) \, dx \right) \left( \int_{B_0} (B, E) \, dx \right)
\]
which concludes the proof, by means of (1.2).

At this point we apply this result to the study of equation (1.3).
Note that (1.3) is the Euler-Lagrange equation of the variational integral
\[
\mathcal{E}[u] = \int_{\Omega} < A(x) \nabla u, \nabla u > \, dx.
\]
We deal with solution of (1.3) having “finite energy”, namely \( \mathcal{E}[u] \) is finite.
If \( u \in W_{loc}^{1,1}(\Omega) \) is a local solution of (1.3), we set \( B = A \nabla u \) and \( E = \nabla u \) so that \( \text{div} B = 0 \) and \( \text{curl} E = 0 \). Let us remark that \( (B, E) \) is locally integrable on \( \Omega \), since \( u \) is a local solution of (1.3) with “finite energy”. By assumptions (1.5), (1.7) the gradient of a finite energy solution belongs to the Orlicz-Zygmund space \( L^2_{\text{loc}} \log^{a-1} L(\Omega, R^n) \). From Theorem 1.1 we deduce the following

**Proposition 3.3.** – Let \( u \) be a local solution of (1.3) with “finite energy”. Assume that the distortion \( K(x) \geq 1 \) satisfies (1.7) for some \( \beta > 0 \). Then there exists \( a = c(n)\beta > 0 \) such that \( |\nabla u| \in L^2_{\text{loc}} \log^{a-1} L(\Omega, R^n) \), \( |A \nabla u| \in L^2_{\text{loc}} \log^{a-1} L(\Omega, R^n) \).

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*Civilemente in Redazione
il 28 gennaio 2006 e in forma rivista il 10 ottobre 2007*