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A Generalization of Quasi-Hamiltonian Groups

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Sunto. – *Iwasawa classifca i gruppi finiti G in cui tutti i sottogruppi V sono permutabili, ciò UV = VU per ogni sottogruppo U di G. Tali gruppi sono detti quasi-hamiltoniani.*

Noi classifichiamo i gruppi finiti in cui i sottogruppi non permutabili hanno tutti lo stesso ordine e quelli che hanno una sola classe di coniugio di sottogruppi non permutabili.

Summary. – *Iwasawa classifies finite groups G in which all subgroups V are permutable, that is UV = VU for all subgroups U of G. These groups are called quasi-hamiltonian.*

We classify the finite groups whose non-permutable subgroups have the same order and the ones which have a single conjugacy class of non-permutable subgroups.

Introduction.

The structure of groups whose subgroups are all normal (hamiltonian groups) has been completely described by R. Dedekind and R. Baer. A long series of papers has dealt with generalizations of this result; let me mention two of them. A first generalization studies groups which satisfy conditions on the numbers of non-normal subgroups. Brandl (see [1]) classifies groups in which non-normal subgroups are in a single conjugacy class.
G. Zappa (see [5] and [6]) classifies finite groups whose non-normal subgroups have the same order. In addition to the groups found by Brandl, Zappa finds only the p-groups described in Theorem 1 and 2 in [6].

A second generalization studies groups whose subgroups have a property close to being normal. Iwasawa (see [2]) classifies finite groups G in which all subgroups V are permutable, that is $UV = VU$ for all subgroups U of G. These groups are called quasi-hamiltonian. Our aim is to study finite groups whose non-permutable subgroups have the same order. This will also allow to classify the ones whose non-permutable subgroups are in the same conjugacy class.

1. – Preliminaries.

DEFINITION 1. – *A subgroup H of G is permutable in G if* $HK = KH$ *for all subgroups K of G. We will write HpermG.*

Such subgroups are also called quasinormal.

We list here a number of well known properties of permutable subgroups:

1. if $HpermG$, $K \leq G$ then $H \cap KpermK$;
2. if $H, KpermG$ then $HKpermG$;
3. if $HpermG$, $N \trianglelefteq G$ then $HN/NpermG/N$;
4. if $H \leq G$, $N \trianglelefteq G$, $N \leq H$ then $HN/NpermG$ if and only if $HpermG$;
5. if $HpermG$, $K \leq G$ and $(|H|, |K|) = 1$ then $K \leq N_G(H)$;
6. if H is a Sylow subgroup of G and $HpermG$ then $H \trianglelefteq G$;
7. if H is a maximal subgroup of G and $HpermG$ then $H \trianglelefteq G$;
8. if H is a cyclic permutable subgroup of G then each subgroup of H is permutable in G.

PROPOSITION 1.1. – *G is a finite non-nilpotent group whose non-permutable subgroups have the same order if and only if* $G = N \rtimes P$ *split extension where* $N \trianglelefteq G$ *is of prime order q, P is a cyclic p-group with* $p \neq q$ *and a generator of P acts on N as a nontrivial automorphism of order p.*

PROOF. – Assume first that the non-permutable subgroups of G have the same order. Since G is a finite non-nilpotent group, there exist a non-permutable Sylow p-subgroup P and a maximal non-permutable subgroup M of G. As non-permutable subgroups have the same order, $|M| = |P|$ and non-permutable subgroups are cyclic. It follows that P is a p-Sylow, maximal, non-permutable and cyclic subgroup.

Let N be the subgroup generated by all Sylow q-subgroups of G where q runs over all prime and $q \neq p$. These Sylow q-subgroups of G are permutable, as their order is different from $|P|$, and so they are normal. Set $g \in N$ an element of prime order q. $\langle g \rangle$ permutes with P, $P\langle g \rangle = G$ and so $N = \langle g \rangle$.

$\Phi(P) \trianglelefteq P$, it is permutable in G and then $g \in N_G(\Phi(P))$. It follows that $\Phi(P) \trianglelefteq G$ and $[N, \Phi(P)] \leq N \cap \Phi(P) = 1$.

Finally P and N do not commute, that is $[N, P] \neq 1$. Conversely, if G has the structure described in the statement, theorem in [1] proves that in G there is only a conjugacy class of non-permutable subgroup, with P as rapresentative. □

PROPOSITION 1.2. – *If G is a finite nilpotent group whose non-permutable subgroups have the same order then G is a p-group.*

PROOF. – Suppose G is not a $p-$group. Then $G = A \times B$ where A and B are nontrivial Hall-subgroups. The subgroups of G are $H \times K$ with $H \leq A$ and $K \leq B$. Let $H_1 \times K_1$, $H_2 \times K_2$ be subgroups of G such that $H_1H_2 \times K_1K_2 \neq H_2H_1 \times K_2K_1$. It follows that either $H_1H_2 \neq H_2H_1$ or $K_1K_2 \neq K_2K_1$

Suppose $H_1H_2 \neq H_2H_1$: $H_1 \times 1$ and $H_1 \times B$ are non-permutable in G but $|H_1 \times B| \neq |H_1|$, a contradiction. □

We are reduced to study p-groups. We indicate with $T(p^n)$ the class of finite non quasi-hamiltonian p-groups whose non-permutable subgroups have order p^n.

NOTATION:

$E(p^3)$ is the non abelian group of order p^3 and exponent p ($p \neq 2$);
$M(p^{n+1}) = \langle x, y : x^{p^n} = y^p = 1, x^y = x^{1+p^{n-1}} \rangle$;
$S_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{-1+2^{n-2}} \rangle$;

Q_{2^n} is the generalized quaternion group of order 2^n, D_{2^n} is the generalized diedral group of order 2^n and C_{p^n} is the cyclic group of order p^n. If A, B are non identity $p-$groups with cyclic centre, $A * B$ indicates a central product with central subgroups of order p amalgamated.

2. – The groups in $T(p)$.

PROPOSITION 2.1. – *Let G be a group in $T(p)$. Let A_1 and A_2 be subgroups of G of order p such that $A_1A_2 \neq A_2A_1$, and let N be a normal subgroup of G of order p. Then:*

1. *$\langle A_1, A_2 \rangle = A_1NA_2$ has order p^3 and is isomorphic to D_8 if $p = 2$, non abelian of exponent p if $p \neq 2$;*
2. *N is the only subgroup of order p which permutes with both A_1 and A_2. In particular N is the only normal subgroup of order p in G;*
3. *$A_1N \trianglelefteq G$.*

PROOF. – Let A_1 and A_2 be subgroups of G such that $|A_i| = p$, $A_i = \langle a_i \rangle$ for $i = 1, 2$ and $A_1A_2 \neq A_2A_1$, and let $N = \langle n \rangle$.

A_1N is a subgroup of G of order p^2 and so permutable. In particular A_1NA_2 is a subgroup of G of order p^3 and $A_1NA_2 = \langle A_1, A_2 \rangle$. As it contains non-permutable subgroups, we have $\langle A_1, A_2 \rangle \cong D_8$ if $p = 2$, $\langle A_1, A_2 \rangle \cong E(p^3)$ if $p \neq 2$, so that: $\langle A_1, A_2 \rangle = \langle a_1, a_2 : a_1^p = 1 = a_2^p, [a_1, a_2] = n \in Z(\langle A_1, A_2 \rangle), n^p = 1 \rangle$.

Let A_3 be a subgroup of G of order p such that $A_1 \neq A_3$ and $A_1A_3 = A_3A_1$. Having order p^2, A_1A_3 is a permutable subgroup. In particular the subgroup $A_1A_3A_2$ has order p^3 and then $A_1A_3A_2 = A_1NA_2$. Moreover A_1A_3 and A_1N are normal subgroups of $\langle A_1, A_2\rangle$ and they both contain all the coniugates of A_1 in $\langle A_1, A_2\rangle$. Then $A_1A_3 = A_1N$. Likewise if A_3 is a subgroup of G of order p such that $A_2 \neq A_3$ and $A_2A_3 = A_3A_2$ then $A_2N = A_3A_2$. In particular N is the only subgroup of order p which permutes with both A_1 and A_2 and then N is the only normal subgroup of order p in G.

We prove now that A_1N is normal in G. Let $x \in G$.

Suppose first $o(x) = p$. If $a_1x = xa_1$ then $A_1^x \leq A_1N$. If $a_1x \neq xa_1$ then $\langle a_1\rangle$ and $\langle x\rangle$ do not permute. As seen before $\langle A_1, x\rangle = A_1N\langle x\rangle$ has order p^3 and $A_1^x \leq A_1N$. In particular $A_1N \trianglelefteq \Omega_1(G)$.

Suppose now $o(x) = p^n$ where $n > 1$. $\langle x\rangle$ is permutable in G and we may assume that $A_1 \nleq \langle x\rangle$. Set $\langle y\rangle = \Omega_1(\langle x\rangle)$. We have $\langle a_1\rangle\langle y\rangle = \langle y\rangle\langle a_1\rangle$ and likewise $\langle a_2\rangle\langle y\rangle = \langle y\rangle\langle a_2\rangle$. It follows that $\langle y\rangle = N$. $\langle a_1\rangle\langle x\rangle$ is a group with a maximal cyclic subgroup, its order is p^{n+1} and $|\Omega_1(\langle a_1\rangle\langle x\rangle)| > p$. If $p \neq 2$, $\langle a_1\rangle\langle x\rangle$ is either abelian or isomorphic to $M(p^{n+1})$ and then $x^{a_1} \equiv x \ mod\langle y\rangle$. Hence, $(a_1)^x \in A_1N$. Suppose now $p = 2$. If $\langle a_1\rangle\langle x\rangle$ is isomorphic toD_8 then $x \in \Omega_1(G)$. Since $D_{2^{n+1}}$ and $S_{2^{n+1}}$ with $n \geq 3$ contain non-permutable subgroups of order 4, we have that $\langle a_1\rangle\langle x\rangle$ is either isomorphic to $M(p^{n+1})$ or abelian. Then $x^{a_1} \equiv x \ mod\langle y\rangle$ and $(a_1)^x \in A_1N$. □

THEOREM 2.2. – *Let G be a p-group. Then:*

1. *$G \in T(p)$ where $p \neq 2$ if and only if G is isomorphic to one of the following groups:*
   (*a*) $E(p^3)$;
   (*b*) $E(p^3) * C_{p^n}$.
2. *$G \in T(2)$ if and only if G is isomorphic to one of the following groups:*
   (*a*) D_8;
   (*b*) $D_8 * C_{2^n}$;
   (*c*) $D_8 * Q_8$.

PROOF. – Let A_1 and A_2 be subgroups of G of order p such that $A_1A_2 \neq A_2A_1$, and let N be the normal subgroup of G of order p.

By prop. 2.1, A_1 and A_2 have p conjugates in G.

$C_G(\langle A_1, A_2\rangle) = C_G(A_1) \cap C_G(A_2)$. Since $[G : C_G(A_i)] = p$ $(i = 1, 2)$, $[G : C_G(A_1) \cap C_G(A_2)] = p^2$. Set $H = \langle A_1, A_2\rangle$; $H \trianglelefteq G$. $H \cap (C_G(A_1) \cap C_G(A_2)) = Z(H) = N$ and then $G = H * C_G(H)$.

Moreover if $K \leq C_G(H)$, $|K| = p$, we have $KA_1 = A_1K$ and $KA_2 = A_2K$. Then $K = N$ and $C_G(H)$ is cyclic or generalized quaternion, but if $n \geq 4$ then Q_{2^n} contains non-permutable subgroups of order 4. Hence we get the groups of the proposition.

The groups listed above are in $T(p)$. In fact $E(p^3)$ and D_8 contain non-permutable subgroups of order p, and all subgroups of order different from p are normal as proved in Theorem 2 in [6]. □

3. – The groups in $T(p^n)$ with $n \geq 2$.

PROPOSITION 3.1. – *Let $G \in T(p^n)$ with $n \geq 2$ and $|\Omega_1(G)| = p$. Then G is the generalized quaternion group of order 16 and $G \in T(4)$.*

PROOF. – If $|\Omega_1(G)| = p$ then G is either cyclic or generalized quaternion. Q_8 and cyclic groups are hamiltonian and, if $n \geq 5$, Q_{2^n} contains non-permutable subgroups of different orders. $Q_{16} = \langle a, b : a^4 = 1, b^4 = a^2, b^a = b^{-1} \rangle$ is in $T(4)$. In fact $\langle a \rangle$ and $\langle ab \rangle$ are not permutable, whereas the subgroup of order 2 and the subgroups of order 8 are normal. □

PROPOSITION 3.2. – *Assume $n \geq 2$ and let G be in $T(p^n)$ with $|\Omega_1(G)| > p$. Let A_1 and A_2 be subgroups of order p^n such that $A_1A_2 \neq A_2A_1$. Then:*

1. *A_1 and A_2 are cyclic;*
2. *$|A_1 \cap A_2| = p^{n-1}$;*
3. *$\langle A_1, A_2 \rangle = A_1\langle t \rangle A_2$ for every $t \in \Omega_1(G) \setminus \Omega_1(A_1)$;*

Moreover $\Omega_1(G)$ has order p^2 and is elementary abelian.

PROOF. – Since subgroups of order p are permutable, $\Omega_1(G) = \{g \in G : g^p = 1\}$ and it is elementary abelian. A_1 and A_2 are cyclic because otherwise they would be product of permutable subgroups. Set $A_i = \langle a_i \rangle$ $(i = 1, 2)$. Having order p^{n-1}, $\langle a_i^p \rangle$ is permutable in G. We consider $\langle a_1^p \rangle\langle a_2 \rangle \leq G$ and $\langle a_2^p \rangle\langle a_1 \rangle \leq G$.

$\left(\langle a_1^p \rangle\langle a_2 \rangle\right)\left(\langle a_2^p \rangle\langle a_1 \rangle\right) = \langle a_2 \rangle\langle a_1 \rangle$ and $\left(\langle a_2^p \rangle\langle a_1 \rangle\right)\left(\langle a_1^p \rangle\langle a_2 \rangle\right) = \langle a_1 \rangle\langle a_2 \rangle$.

Hence $(\langle a_1^p \rangle\langle a_2 \rangle)(\langle a_2^p \rangle\langle a_1 \rangle) \neq (\langle a_2^p \rangle\langle a_1 \rangle)(\langle a_1^p \rangle\langle a_2 \rangle)$ and we get $\left|\langle a_1^p \rangle\langle a_2 \rangle\right| = p^n$, $\left|\langle a_2^p \rangle\langle a_1 \rangle\right| = p^n$, so that $\langle a_1^p \rangle \leq \langle a_2 \rangle$ and $\langle a_2^p \rangle \leq \langle a_1 \rangle$.

$\langle a_1 \rangle / \langle a_1{}^p \rangle$ and $\langle a_2 \rangle / \langle a_2{}^p \rangle$ have order p and, as seen in section 2, they generate a subgroup of order p^3, which gives $|\langle a_1, a_2 \rangle| = p^{n+2}$.

Since $|A_1 \cap \Omega_1(G)| = p$ and $|\Omega_1(G)| > p$, there exists $t \in \Omega_1(G)$, $t \notin A_1$.

Having order p^{n+1}, $A_1\langle t \rangle$ is permutable in G, and $A_2 \cap A_1\langle t \rangle = \langle a_1{}^p \rangle$. It follows that $|A_1\langle t \rangle A_2| = p^{n+2}$ and then $\langle A_1, A_2 \rangle = A_1\langle t \rangle A_2$. Furthermore $N_{\langle A_1, A_2 \rangle}(A_1) = A_1\langle t \rangle$.

Suppose now that there exists $s \in \Omega_1(G)$, $s \notin A_1\langle t \rangle$. As proved above, $\langle A_1, A_2 \rangle = A_1\langle s \rangle A_2$ and $N_{\langle A_1, A_2 \rangle}(A_1) = A_1\langle s \rangle$. Hence we get $A_1\langle s \rangle = A_1\langle t \rangle$ which contradicts our assumptions. □

With the following theorem, we complete the description of p-groups in $T(p^n)$ if $p \neq 2$. This reduces us to study 2-groups in $T(2^n)$ with $n \geq 2$.

THEOREM 3.3. – *Let G be p-group, $p \neq 2$. The following conditions are equivalent:*

1. $G \in T(p^n)$ *where* $n \geq 2$;
2. $G \in T(3^2)$;
3. $G = \langle a, c, b : a^9 = c^3 = 1, b^3 = a^3, ac = ca, a^b = ac, c^b = ca^{-3} \rangle$.

PROOF. – Let A_1 and A_2 be subgroups of G of order p^n such that $A_1A_2 \neq A_2A_1$. By prop. 3.2, $A_1 = \langle a_1 \rangle$, $A_2 = \langle a_2 \rangle$ and $\langle A_1, A_2 \rangle = A_1 \langle t \rangle A_2$ where $t \in \Omega_1(G) \setminus \Omega_1(A_1)$. Moreover we can assume ${a_1}^p = {a_2}^p$.

$A_i \langle t \rangle \leq G$ is either abelian or isomorphic to $M(p^{n+1})$ and $A_i \langle t \rangle \trianglelefteq \langle A_1, A_2 \rangle$ for $i = 1, 2$. So we get: $a_1^t = a_1^{1+hp^{n-1}}, a_2^t = a_2^{1+kp^{n-1}}, a_1^{a_2} = a_1^r t^s$ where $h, k \in \{1, \ldots, p\}$, $s \in \{1, \ldots, p-1\}$ and $r \equiv 1 \bmod (p)$; from $a_1^p = (a_1^p)^{a_2} = (a_1^r t^s)^p = a_1^{rp} t^{sp} = a_1^{rp}$, we have $r = 1 + jp^{n-1}$ and then: $a_1^t = a_1^{1+hp^{n-1}}$, $a_2^t = a_2^{1+kp^{n-1}}$, $a_1^{a_2} = a_1^{1+jp^{n-1}} t^s$. $\langle a_1, a_2 \rangle$ has class ≤ 3 and derived subgroup contained in $\langle a_1^{p^{n-1}}, t \rangle = \Omega_1(\langle A_1, A_2 \rangle)$.

If $p > 3$ we obtain a contradiction. In fact $\langle A_1, A_2 \rangle$ is regular, hence $(a_2 a_1^{-1})^p = a_2^p a_1^{-p} x^p$ for some $x \in \langle A_1, A_2 \rangle'$. So $a_2 a_1^{-1}$ has order p but $\langle a_2 a_1^{-1} \rangle$ does not normalize A_1. It follows that there are not groups in $T(p^n)$ if $p > 3$, $n \geq 2$.

Suppose now $p = 3$. Since $\langle a_1, a_2 \rangle / \langle a_1^{3^{n-1}} \rangle$ has class ≤ 2, it follows that $\langle a_1, a_2 \rangle / \langle a_1^{3^{n-1}} \rangle$ is regular and $(a_1 a_2^{-1})^3 \langle a_1^{3^{n-1}} \rangle = 1$.

If $n \geq 3$ we obtain a contradiction: $a_1 a_2^{-1}$ has order ≤ 9 but $\langle a_1 a_2^{-1} \rangle$ does not permute with A_2. Finally if $p = 3$ and $n = 2$, two non-permutable subgroups of order 9 generate a group of order 81 whose structure is partially described above: $H = \langle a_1, a_2 \rangle$, $a_1^3 = a_2^3$, $\Omega_1(H) = \langle a_1^3, t \rangle$. $\langle a_i, t \rangle$ is either abelian or isomorphic to $M(3^3)$. Since $[H : C_H(\Omega_1(H))] = 3$ we can choose $a_1 \in C_H(\Omega_1(H))$; further we may choose t such that $a_1^{a_2} = a_1 t$. $a_1 a_2$ does not normalize A_1. If $t^{a_2} = t a_2^3$ then $(a_2 a_1)^3 = 1$, a contradiction. So we have $t^{a_2} = t a_2^{-3}$ and this shows that H is as in 3. Conversely , it can be easily checked that G is in $T(3^2)$.

Suppose now that G is in $T(3^2)$ and contains H as a proper subgroup; we may also assume that $[G : H] = 3$. By theorem (4.12) in [4], $G = \langle b \rangle\ C_G(\Omega_1(G))$.

We shall prove that $C_G(\Omega_1(G)) = \langle a, c \rangle$. It will be enough to show that $C_G(\Omega_1(G))$ contains no elements of order 9 or 27 outside $\langle a, c \rangle$.

First we note that $a^3 \in Z(G)$: indeed $\Omega_1(G) \cap Z(G) \neq 1$ and $c \notin Z(\langle a, b \rangle)$.

Suppose $y \in C_G(\Omega_1(G)) \setminus \langle a, c \rangle$ of order 9. $\langle a \rangle$ and $\langle y \rangle$ permute. Otherwise we have a contradiction: $\langle a, y \rangle \cong \langle a, b \rangle$ but $\Omega_1(\langle y, a \rangle) = \langle a^3, c \rangle \leq Z(\langle a, y \rangle)$ whereas $\Omega_1(\langle b, a \rangle) \not\leq Z(\langle b, a \rangle)$.

If $y^3 \in \langle a^3 \rangle$ then $a^3 = y^{3k}, y^a = y^{1+3h}$ and $(ay^{-k})^3 = 1$, which gives $y \in \langle a, c \rangle$. Assume now $y^3 \notin \langle a^3 \rangle$, that is $y^3 = a^{3k} c$. Since $\langle b \rangle \cap \langle y \rangle = 1$, $\langle b \rangle$ permutes with $\langle y \rangle$ and $y^b = y^{1+3i} a^{3j}$. Now $ca^{-3} = c^b = (y^3)^b = y^3 = c$, a contradiction.

Suppose $y \in C_G(\Omega_1(G)) \setminus \langle a, c \rangle$ of order 27. As $y^3 \in \langle a, c \rangle$, $y^3 = ac^k$ and then $y^9 = a^3$. b normalizes $\langle y \rangle$ and from $(y^9)^b = y$ we get $y^b = y^{1+3i}$. But $a^b = (y^3 c^{-k})^b = y^{3+9i} c^{-k} a^{3k} = ac^k a^{3i} c^{-k} a^{3k} = a^{1+3i+3k} \in \langle a \rangle$, a contradiction. □

4. – Groups in $T(2^n)$ with $n \geq 2$: first results.

In view of prop. 3.1 and 3.2, we will assume that the groups G in $T(2^n)$ that we consider satisfy $|\Omega_1(G)| = 4$.

We will be interested in studying the following groups:

$T_1(n) = \langle a, b : a^4 = b^{2^n} = 1, a^b = a^3 \rangle$ $(n \geq 2)$ and

$T_2(n) = \langle a, b : a^8 = 1, a^4 = b^{2^{n-1}}, a^b = a^7 \rangle$ $(n \geq 3)$.

PROPOSITION 4.1. – *$T_1(n)$ for $n \geq 2$ is in $T(2^n)$.*

PROOF. – $Z(T_1(n)) = \langle a^2, b^2 \rangle$ and the square of every element of $Z(T_1(n))$ is in $\langle b^4 \rangle$. The elements of $T_1(n)$ are z, az_1, abz_2, bz_3 where $z, z_i \in Z(T_1(n))$. Since $\langle abz_2 \rangle$ and $\langle bz_3 \rangle$ have order 2^n, we have to prove that $\langle az_1 \rangle$ permutes with both $\langle bz_2 \rangle$ and $\langle abz_3 \rangle$.

$(az_1)(bz_3) = abz_1z_3 = a^2b^aa^3z_1z_3 = ba^2az_1z_3 = bz_3(az_1)^3z_1{}^{-2}$. Setting $z_3{}^2 = b^{4i}$ and $z_1{}^2 = b^{4j}$, we get: $(bz_3)^2 = b^{2(1+2i)}$ and there exists an integer r such that $az_1bz_3 = (bz_3)^r(az_1)^3$.

The same if we consider abz_2 instead of bz_3. □

PROPOSITION 4.2. – *$T_2(n)$ with $n \geq 3$ is in $T(2^n)$.*

PROOF. – One see easily that: $Z(T_2(n)) = \langle b^2 \rangle$, $[a^2, T_2(n)] = \langle a^4 \rangle$, $|T_2(n)| = 2^{n+2}$, and $T_2(n)/\langle a^4 \rangle \cong T_1(n-1)$. Moreover, for each $g \in T_2(n) \setminus \langle a, b^2 \rangle$ we have $\langle b^2 \rangle = \langle g^2 \rangle$, $|\langle g \rangle| = 2^n$. It follows that non-permutable subgroups of $T_2(n)$ containing $\langle a^4 \rangle$ have order 2^n by prop. 4.1.

A subgroup not containing $\langle a^4 \rangle$ is cyclic; the possibilities are: $\langle a^2b^{\pm 2^{n-2}} \rangle$ of order 2 and (if $n > 3$) $\langle ab^{\pm 2^{n-3}} \rangle$ of order 4. Now $a^2b^{\pm 2^{n-2}}$ normalizes every subgroup of $T_2(n)$. $\langle ab^{\pm 2^{n-3}} \rangle$ centralizes $\langle a, b^2 \rangle$ and , if $g \notin \langle a, b^2 \rangle$, we have $|g| = 2^n$, $\langle g \rangle \cap \langle ab^{\pm 2^{n-3}} \rangle = 1$, $|\langle g, ab^{\pm 2^{n-3}} \rangle| = |T_2(n)| = 2^{n+2}$, so that $\langle g \rangle$ and $\langle ab^{\pm 2^{n-3}} \rangle$ permute. □

PROPOSITION 4.3. – *Let $G \in T(2^n)$ with $n \geq 2$. Two non-permutable subgroups of order 2^n generate a group isomorphic to one of $T_1(n)$ $(n \geq 2)$, $T_2(n)$ $(n > 2)$.*

PROOF. – Let A_1 and A_2 be subgroups of G of order 2^n such that $A_1A_2 \neq A_2A_1$. By prop. 3.2, $A_1 = \langle a_1 \rangle, A_2 = \langle a_2 \rangle$ and $\langle A_1, A_2 \rangle = A_1\langle t \rangle A_2$ where $t \in \Omega_1(G) \setminus \Omega_1(A_1)$. Moreover we can suppose $a_1{}^2 = a_2{}^2$.

$\langle a_i, t \rangle$ $(i = 1, 2)$ has a maximal cyclic subgroup and then it is either abelian or (if $n \geq 3$) isomorphic to $M(2^{n+1})$. Then $a_i^t = a_i$ or (if $n \geq 3$) $a_i^t = a_i^{1+2^{n-1}}$ for $i = 1, 2$. Moreover $\langle a_1, t \rangle \trianglelefteq \langle a_1, a_2 \rangle$ and then $a_1^{a_2} = a_1^j t$ with j odd.

Hence the possibilities are:

1. $a_1t = ta_1$, $a_2t = ta_2$.
   From $a_1^2 = (a_1^2)^{a_2}$, we have $2j \equiv 2 \mod(2^n)$ which gives $a_1^{a_2} = a_1^{1+h2^{n-1}}t$. Then we may choose t such that $a_1^{a_2} = a_1t$,$a_1t = ta_1$ and $a_2t = ta_2$. Setting $a = a_1a_2^{-1}$ and $b = a_2$, we get the group $T_1(n)$.
2. (if $n \geq 3$) $a_1^t = a_1$, $a_2^t = a_2^{1+2^{n-1}}$.
   As seen above, we may choose t such that $a_1^{a_2} = a_1t$, $a_1^t = a_1$, $a_2^t = a_2^{1+2^{n-1}}$. Now $(a_2^2)^{a_1} = a_2ta_2t = a_2a_2^{1+2^{n-1}} \neq a_2^2$, a contradiction.
3. (if $n \geq 3$) $a_2^t = a_2$, $a_1^t = a_1^{1+2^{n-1}}$.
   As seen above, we may choose t such that $a_2^{a_1} = a_2t$, $a_2^t = a_2$, $a_1^t = a_1^{1+2^{n-1}}$. Now $(a_1^2)^{a_2} = a_1ta_1t = a_1a_1^{1+2^{n-1}} \neq a_1^2$, a contradiction.
4. (if $n \geq 3$) $a_1^t = a_1^{1+2^{n-1}}$, $a_2^t = a_2^{1+2^{n-1}}$.
   From $a_1^2 = (a_1^2)^{a_2}$, we have $2 \equiv 2j + 2^{n-1} \mod(2^n)$ which gives $a_1^{a_2} = a_1^{1+j\cdot 2^{n-2}}t$ where $j = 1, 3$. Hence we may choose t such that $a_1^{a_2} = a_1^{1+2^{n-2}}t$. Setting $a = a_1a_2^{-1}$ and $b = a_2$, we get the group $T_2(n)$.

□

5. – The groups in $T(4)$.

THEOREM 5.1. – *There is no group $G \in T(4)$ having exponent > 4.*

PROOF. – By prop. 4.3, a group $G \in T(4)$ contains a subgroup isomorphic to $T = \langle a, b : a^4 = b^4 = 1, a^b = a^3 \rangle$. Since $exp(G) > 4$, we can suppose $G = T\langle z \rangle$ where $o(z) = 8$. We note that $\Omega_1(G) = \Omega_1(T)$.

We first prove that $\langle z \rangle \trianglelefteq G$.

Since $\langle z \rangle permG$, every element of order 2 normalizes $\langle z \rangle$.

Let t be an element of T of order 4. If $t^2 = z^4$ then $t \in N_G(\langle z \rangle)$. Suppose that $t^2 \neq z^4$. Then $\langle z, t^2 \rangle \trianglelefteq \langle z, t \rangle$ but if $z^t = z^it^2$, we have a contradiction: $(tz)^2 = (z^i)^{t^2}z^i \in \langle z \rangle$, $|\langle tz, z \rangle| = 16$ whereas $|\langle t, z \rangle| = 32$.

It follows that $T \leq N_G(\langle z \rangle)$.

Suppose now $z^2 \in T$. Since $\langle ba^{2i}b^{2j} \rangle$ and $\langle baa^{2i}b^{2j} \rangle$ are not normal in T, we have $z^2 \in a\langle a^2, b^2 \rangle$. From $(z^2)^a = z^6$, we get $z^b = z^{3+4k}$ and $(bz)^2 = y^2z^{3+4k}z = y^2z^{4(1+k)}$. Then bz has order 4 and $|\langle bz, b \rangle| \leq 16$, a contradiction because $|\langle b, z \rangle| = 32$. It follows that $z^2 \notin T$.

If $z^4 = b^2$ then $G/\langle z \rangle \cong D_8$ and $\langle z, b \rangle$ is not permutable in G.

Suppose $z^4 = a^2$. Since $|\langle z, a \rangle| = 16$ and $G = \langle z, a \rangle\langle b \rangle$, we get $\langle z, a \rangle \cap \langle b \rangle = 1$, and $|\Omega_1(\langle z, a \rangle)| = 2$. Then $\langle a, z \rangle$ is either cyclic or a generalized quaternion subgroup of order 16. Suppose that $\langle a, z \rangle$ is cyclic. Then z^2 is in $\langle a \rangle$ and we have a contradiction. Suppose now that $\langle z, a \rangle \cong Q_{16}$. In this case the element az has order 4 and $\langle az \rangle \cap \langle b \rangle = 1$. Then $\langle az, b \rangle$ has order 16 but it does not permute with

$\langle a \rangle$, a contradiction. Assume now $z^4 = a^2b^2$. If t is an element of T of order 4, then $[z,t] \in \langle z^2 \rangle$. If $[z,t] = z^{2k}$ with k odd then $o(tz) \leq 4$ and $|\langle tz, t \rangle| \leq 16$ a contradiction because $|\langle t, z \rangle| = 32$. Hence, $[z,b] \in \langle z^4 \rangle$, $[z,a] \in z^4$. Now az has order 8, $(az)^4 = \langle z^4 \rangle$ but $[az,b] \notin \langle az \rangle$, a contradiction. □

OBSERVATION 1. – A finite 2-group of exponent 4 has derived subgroup contained in $\Omega_1(G)$. In particular the derived subgroup of $G \in T(4)$ has order 2 or 4.

PROPOSITION 5.2. – *Let G be a group of exponent* 4 *with* $|G| = 32$, $|G'| = 2$. *Then G is in $T(4)$ if and only if*

$$G \cong \langle a, b, c : c^4 = a^4 = 1, a^2 = b^2, ca = ac, bc = cb, b^a = b^3 \rangle = M \cong Q_8 \times C_4.$$

PROOF. – Since $|G| = 32$, $|G'| = 2$ and $T = \langle a, b : a^4 = b^4 = 1, b^a = b^3 \rangle \leq G$, $G' = \langle b^2 \rangle$. Every element in $G \setminus T$ has order 4. Let $c \in G \setminus T$. Now $c^2 \in \Omega_1(G) = \langle a^2, b^2 \rangle \leq Z(G)$, $[c,a] \in \langle b^2 \rangle$, $[c,b] \in \langle b^2 \rangle$ so that $[c,a] = b^{2h}, [c,b] = b^{2k}$.

c acts on T as $a^k b^h$ and then, replacing c with $c(a^k b^h)$, we can suppose $c \in Z(G)$. We can have neither $c^2 = a^2$ ($(ac)^2 = 1$) nor $c^2 = b^2$ ($(bc)^2 = 1$).

Then, $G = \langle a, b, c : a^4 = b^4 = 1, c^2 = a^2b^2, b^a = b^3, ac = ca, bc = cb \rangle$.

Replacing a with ac, we get the presentation of the proposition.

Conversely, in the group

$$\langle a, b, c : c^4 = a^4 = 1, a^2 = b^2, ca = ac, bc = cb, b^a = b^3 \rangle$$

the subgroups $\langle ac \rangle$ and $\langle bc \rangle$ are non-permutable subgroups of order 4. Theorem 2 of [6] proves that subgroups of M of order different from 4 are normal, hence permutable.

PROPOSITION 5.3. – *Let G be a group of exponent* 4 *with* $|G| = 32$, $|G'| = 4$. *Then G is in $T(4)$ if and only if*

$$G \cong \langle a, b, c : a^4 = b^4 = 1, b^2 = c^2, ca = ac, c^b = ca^2, b^a = b^3 \rangle = R.$$

PROOF. – Since $|G| = 32$, $|G'| = 4$, $T = \langle a, b : a^4 = b^4 = 1, b^a = b^3 \rangle$ is contained in G and $G' \leq \Omega_1(G) = \Omega_1(T)$, we have $G' = \langle b^2, a^2 \rangle \leq Z(G)$. Let $c \in G \setminus T$. Then, $c^2 \in \Omega_1(G) = \langle a^2, b^2 \rangle$ and $c^2 \neq 1$. Since $[c,a] \in \langle a^2, b^2 \rangle$, $[c,b] \in \langle a^2, b^2 \rangle$ we have $[c,a] = a^{2i}b^{2j}$, $[c,b] = a^{2h}b^{2k}$ and then $[cb^ja^k, a] = a^{2i}$, $[cb^ja^k, b] = a^{2h}$. Replacing c with cb^ja^k, we get $a^c = a^{1+2^i}$ and $b^c = a^{2h}b$. Since $a^2 \in G'$, either i or h has to be odd. If they are both odd, $(ab)^c = ab$ and we replace a with ab. So, the possibilities are:

- $a^c = a$, $b^c = a^2b$. It can be neither $c^2 = a^2$ ($(ac)^2 = 1$) nor $c^2 = a^2b^2$ ($(cb)^2 = 1$). It follows that $c^2 = b^2$ and $G \cong R$.
- $a^c = a^{-1}$, $b^c = b$. It can be neither $c^2 = b^2$ ($(bc)^2 = 1$) nor $c^2 = a^2$ $\langle\langle c \rangle \trianglelefteq G$, $G/\langle c \rangle \cong D_8$ and then $\langle a, c \rangle$ is non permutable in G). It follows that $c^2 = a^2b^2$. Replacing $a' = c$, $b' = a$, $c' = bc$, we get $G \cong R$.

The subgroups $\langle ab\rangle$ and $\langle a\rangle$ of order 4 of R are non-permutable subgroups. Theorem 2 of [6] proves that subgroups of R of order different from 4 are normal, hence permutable. □

OBSERVATION 2. – M and R are the only groups in $T(4)$ of order 32. Moreover in R there are neither central elements of order 4 nor subgroups of order 8 isomorphic to the quaternion group.

PROPOSITION 5.4. – *M is not contained in a group $G \in T(2^2)$ of order ≥ 64. In particular a group G in $T(2^2)$ with $|G'| = 2$ has order ≤ 32.*

PROOF. – Suppose that there exists a group $G \in T(2^2)$ of order 64 containing M. $G = \langle a, b, c, d\rangle$ where $d \notin M$. Since $G' \leq \Omega_1(G) = \langle b^2, c^2\rangle$ the possibilities are:

1. $|G'| = 2$. Then $G' = \langle b^2\rangle$ and $[a, d] = b^{2h}$, $[b, d] = b^{2k}$, $[c, d] = b^{2r}$. Replacing d with $d(b^h a^k)$, we get $[a, d] = 1$, $[b, d] = 1$, $[c, d] = b^{2r}$. Moreover $d \notin Z(G)$: for each $w \in \Omega_1(G)$ there is $t \in M$ such that $t^2 = w$ and so if $d^2 = w$ then $(dt)^2 = 1$. Hence, we get: $[a, d] = 1$, $[b, d] = 1$, $[c, d] = b^2$. It can be neither $d^2 = a^2$ $((da)^2 = 1)$ nor $d^2 = a^2c^2$ $((dc)^2 = d^2ca^2c = 1)$, and if $d^2 = c^2$ then $(dac)^2 = d^2aca^2ac = 1$.
2. $|G'| = 4$. Then $G' = \langle b^2, c^2\rangle$ and $[a, d] = b^{2h}c^{2k}$, $[b, d] = b^{2i}c^{2j}$, $[c, d] = b^{2r}c^{2s}$ where $h, k, i, j, r, s \in 0, 1$. Since $[a, db^h a^i] = c^{2k}$, $[b, db^h a^i] = c^{2j}$ and $[c, db^h a^i] = b^{2r}c^{2s}$, replacing d with $db^h a^i$, we get $[a, d] = c^{2k}$, $[b, d] = c^{2j}$, $[c, d] = b^{2r}c^{2s}$. If $[a, d] = c^2$, we have $d \notin N_G\langle a, b\rangle$ and so $d^2 = a^{2l}c^2$. Now $(da)^2 = d^2ac^2a \in \langle a^2\rangle$, hence $da \in N_G(\langle a, b\rangle)$, a contradiction. It follows that $[d, a] = 1$. Likewise we prove that $[b, d] = 1$. We can not have $d^2 = a^2$ because in this case $(da)^2 = 1$, a contradiction. Moreover, since $c^2 \in G'$, we get $[d, c] = c^2a^{2i}$. Hence, we have the following cases:
   (a) $d^2 = c^2$. If $[c, d] = c^2b^2$, we have that the groups $\langle dc, ac\rangle \cong Q_8$ and $\langle bd\rangle$ do not permute. If $[c, d] = c^2$, we have that the groups $\langle ac, db\rangle \cong Q_8$ and $\langle ad\rangle$ do not permute;
   (b) $d^2 = a^2c^2$. If $[c, d] = c^2$, $\langle dab, c\rangle \cong Q_8$ does not permute with $\langle db\rangle$. If $[c, d] = a^2c^2$ then $(dc)^2 = d^2ca^2c^2c = c^2$ and, replacing d with dc, we are in the previous case.

In each case we reached a contradiction and then M is not contained in a group $G \in T(2^2)$ of order > 32 □

OBSERVATION 3. – Let $G \in T(4)$ of order ≥ 64 and let K be a subgroup of order 32 of G. By prop. 5.4, if K contains non-permutable subgroups then $K \cong R$. If K is quasi-hamiltonian then it should be either abelian or isomorphic to $Q_8 \times E$ where E is elementary abelian, but in both cases we should have $|\Omega_1(K)| > 4$.

Hence, a subgroup of order 32 of $G \in T(4)$ of order ≥ 64 is isomorphic to R.

At this point, we note that we are in a situation already considered by Zappa in [5]. The argument of lemma 7 and prop. 3 of [5] allow to prove the following propositions:

PROPOSITION 5.5. – *Let* $G \in T(2^2)$ *with* $|G| = 64$. *Then:*

$$G \cong V = \langle a, b, c, d : a^4 = b^4 = 1, b^2 = c^2, d^2 = a^2, ca = ac, c^b = ca^2,$$
$$b^a = b^3, db = bd, a^d = aa^2b^2, c^d = cb^2 \rangle.$$

PROPOSITION 5.6. – *If* $G \in T(4)$ *then* $|G| \leq 64$.

5.1 – *The groups in* $T(2^n)$, $n > 2$.

OBSERVATION 4. – Let $G \in T(2^n)$, $T_1(n) \leq G$. By prop.3.2, $|\Omega_1(G)| = 4$ and $\Omega_1(G) = \Omega_1(T_1(n)) = \left\langle a^2, b^{2^{n-1}} \right\rangle$. Let K be a normal subgroup of G containing $\Omega_1(G)$. G/K is quasi-hamiltonian, and so if $u, v \in G$, $o(uK) = 2$ and $o(vK) \leq 4$, we get $[v, u] \in K$.

We always take $K = Z(T_1(n))$.

THEOREM 5.7. – *If* $n > 2$ *there is no group* G *in* $T(2^n)$ *such that* $|G| > 2^{n+2}$ *and* $T_1(n) \leq G$.

PROOF. – Suppose $G \in T(2^n)$, $T_1(n) \leq G$. We may assume that $[G : T_1(n)] = 2$.

The subgroups generated by elements of order 2 or 4 are permutable and so $\Omega_2(G)$ is abelian or isomorphic to $Q_8 \times E$ where E is elementary abelian. Since $\Omega_2(T_1(n))$ is the direct product of two cyclic groups of order 4, we get that $\Omega_2(G)$ is abelian.

Let $z \in G, z \notin T_1(n)$ of order 4. Since every element of $\Omega_1(T_1(n))$ is a square in $T_1(n)$, we have $z^2 = t^2$ and $(zt)^2 = 1$, a contradiction. It follows that $\Omega_2(G) = \Omega_2(T_1(n))$.

Let $z \in G \setminus T_1(n)$ of order 2^m where $m \leq n$ and $z^2 \in T_1(n)$. The elements of order $\leq 2^{n-1}$ are ab^{2i}, a^3b^{2i}, a^2b^{2i} and b^{2i} (i an integer).

If $z^2 = b^{2i}$ then $\langle z \rangle$ and $\langle b \rangle$ permute. Otherwise, since $ba^2 = a^2b$ we would get $\langle b, z \rangle \cong T_1(n)$, $\Omega_2(\langle z, b \rangle) = \Omega_2(G) = \Omega_2(T_1(n))$,$a \in \langle z, b \rangle$ and then $T_1(n) = \langle z, b \rangle$, a contradiction. $\langle z, b \rangle$ is either abelian or isomorphic to $M(2^n)$. In both cases $o(zb^{-i}) \leq 4$, a contradiction.

If $z^2 = ab^{2i}$ then, by obs.4, we get $[b, z] \in \langle z^4, b^2 \rangle$ so that $b^z = b^{1+2j}z^{4i}$. Now $ba^2 = b^{z^2} = (b^{1+2j}z^{4i})^{1+2j}z^{4i} = b^{1+2h}z^{8i(1+2j)} \in \langle b \rangle$, a contradiction.

If $z^2 = a^3b^{2i}$, replacing z with z^{-1}, we are in the previous case.

If $z^2 = a^2b^{2i}$ then, by obs.4, we get $[a, z] = a^{2r}b^{2s}$ and $[b, z] = a^{2h}b^{2k}$. If $[a, z] = b^{2s}$ then $(za)^2 = z^2ab^{2s}a = z^2a^2b^{2s} \in \langle b^2 \rangle$ and, replacing z with za, we are in a previous case. If $[b, z] = a^2b^{2k}$ then $(zb)^2 = z^2a^2b^{1+2k}b = a^2b^{2i}a^2b^{1+2k}b \in \langle b^2 \rangle$ and, replacing z with zb, we are in a previous case.

Finally if $a^z = a^3b^{2s}$ and $b^z = b^{1+2k}$, $(zab)^2 \in \langle b^2 \rangle$ and, replacing z with zab, we are again in a previous case.

Suppose now $z \in G \setminus T_1(n)$ of order 2^{n+1}. Since $G \in T(2^n)$, $\langle z \rangle$perm G and it can not contain a non-permutable subgroup. Now all the elements of order 2^n in $T_1(n)$ generate non-permutable subgroups and so this case is not possible. □

THEOREM 5.8. – *There is no group G in $T(2^n)$ such that $T_2(n) \leq G$.*

PROOF. – Suppose first $n = 3$, and let $G \in T(8)$, $T_2(3) \leq G$, $[G : T_2(3)] = 2$. $G/\Omega_1(G)$ is quasi-hamiltonian, $T_2(3)/\Omega_1(T_2(3)) \cong Q_8$ and then $G/\Omega_1(G) \cong Q_8 \times C_2$. This proves that elements outside $T_2(3)$ have order ≤ 4.

Let $z \in G \setminus T_2(3)$ of order 4.

If $z^2 = a^4$ then $\langle z, a \rangle$ is either abelian or isomorphic to $M(2^4)$. In both cases $(a^2z)^2 = 1$, a contradiction.

Suppose $z^2 = a^2b^2$. Since $(z^2)^b = z^6$ and $\langle b, z^2 \rangle \trianglelefteq \langle z, b \rangle$, it can be neither $b^z = y^{1+2i}$ nor $b^z = b^{1+4k}z^2$. Hence we get $b^z = b^{3+4k}z^2$. $(zb)^2 = z^2b^{3+4k}z^2b = a^6b^{3+4k}a^2b = b^{7+4k}b = b^{4k}$ and so $|\langle zb, b \rangle| \leq 16$, a contradiction because $|\langle b, z \rangle| = 32$.

Finally if $z^2 = a^2b^6$, replacing b with b^{-1}, we are in the previous case.

Assume now $n > 3$ and let $G \in T(2^n)$, $T_2(n) \leq G$. Since $\Omega_1(T_2(n)) \cap Z(T_2(n)) = \langle a^4 \rangle$, $\Omega_1(T_2(n)) \cap Z(G) \neq 1$, we have $\Omega_1(T_2(n)) \cap Z(G) = \langle a^4 \rangle$. The subgroups of $G/\langle a^4 \rangle$ are $H/\langle a^4 \rangle$ where $\langle a^4 \rangle \leq H \leq G$ and $b\langle a^4 \rangle$ is non-permutable in $G/\langle a^4 \rangle$. Let $H/\langle a^4 \rangle$ and $K/\langle a^4 \rangle$ be subgroups such that $H/\langle a^4 \rangle K/\langle a^4 \rangle \neq K/\langle a^4 \rangle H/\langle a^4 \rangle$. H and K are non-permutable cyclic subgroups of G of order 2^n and $\langle a^4 \rangle \leq H$. It follows that $|H/\langle a^4 \rangle| = 2^{n-1} = |b\langle a^4 \rangle|$, $G/\langle a^4 \rangle \in T(2^{n-1})$ and $T_1(n-1) \cong T_2(n)/\langle a^4 \rangle \leq G/\langle a^4 \rangle$. By theorem 5.7, $T_2(n)/\langle a^4 \rangle = G/\langle a^4 \rangle$ and then we have $T_2(n) = G$. □

6. – Conclusions.

The task of classifying finite p–groups in $T(p^n)$ is now completed. Our results are collected in the following theorem:

THEOREM 6.1. – *The following conditions are equivalent:*

- *The group G is in $T(p^n)$;*
- *G is isomorphic to one of the following groups:*

1. $\langle a, b : a^4 = 1, b^4 = a^2, b^a = b^{-1} \rangle$ *where* $p = 2$ *and* $n = 2$;
2. $\langle a, b, c : a^p = b^p = c^p = 1, [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$ *where* $p \geq 3$ *and* $n = 1$;
3. $\langle a, b, d : a^p = b^p = d^{p^m} = 1, [a, b] = d^{p^{m-1}}, [a, d] = 1, [b, d] = 1 \rangle$ *where* $p \geq 3$, $n = 1$ *and* $m > 1$;
4. $\langle a, c, b : a^9 = c^3 = 1, a^3 = b^3, ac = ca, a^b = ac, c^b = ca^{-3} \rangle$ *where* $p = 3$ *and* $n = 2$;
5. $\langle a, b : b^4 = 1 = a^2, b^a = b^{-1} \rangle$ *where* $p = 2$ *and* $n = 1$;
6. $\langle a, b, c : b^4 = a^2 = 1, c^{2^{m-1}} = b^2, b^a = b^{-1}, bc = cb, ac = ca \rangle$ *where* $p = 2$, $n = 1$ *and* $m > 1$;
7. $\left\langle \begin{array}{ll} a, b, c, d : & b^4 = a^2, b^2 = c^2 = d^2, b^a = b^{-1}, c^d = c^{-1}, \\ & bc = cb, ac = ca, bd = db, ad = da \end{array} \right\rangle$ *where* $p = 2$ *and* $n = 1$;
8. $\langle a, b, c : a^4 = c^4 = 1, a^2 = b^2, b^a = b^3, a^c = a, b^c = b \rangle$ *where* $p = 2$ *and* $n = 2$;
9. $\langle a, b, c : a^4 = b^4 = 1, c^2 = b^2, b^a = b^3, ac = ca, b^c = ba^2 \rangle$ *where* $p = 2$ *and* $n = 2$;
10. $\left\langle \begin{array}{ll} a, b, c, d : & a^4 = b^4 = 1, b^2 = c^2, d^2 = a^2, ca = ac, c^b = ca^2, \\ & b^a = b^3, db = bd, a^d = aa^2b^2, c^d = cb^2 \end{array} \right\rangle$ *where* $p = 2$ *and* $n = 2$;
11. $\langle a, b : a^4 = b^{2^n} = 1, a^b = a^3 \rangle$ *where* $p = 2$ *and* $n \geq 2$;
12. $\langle a, b : a^8 = 1, a^4 = b^{2^{n-1}}, a^b = a^7 \rangle$ *where* $p = 2$ *and* $n \geq 3$.

Brandl [1] classified the finite groups in which the non normal subgroups are in a single conjugacy class. We can use the list given above to solve the analogous problem for non-permutable subgroups.

PROPOSITION 6.2. – *The group* $G = N \rtimes P$ *in prop.* 1.1 *has only a conjugacy class of non-permutable subgroup.*

The groups listed in theorem 6.1 *have at least two conjugacy classes of non-permutable subgroups.*

PROOF. – Let $G = N \rtimes P$ be a split extension where $N \trianglelefteq G$ is of prime order q, P is a cyclic p-group with $p \neq q$ and a generator of P acts on N as a nontrivial automorphism of order p. Then G has only a conjugacy class of non-permutable subgroup, whose rapresentative P has q conjugates.

In groups 1, 5, 6, 7, 9 and 10, listed in theorem 6.1, the non-permutable subgroups $\langle a \rangle$ and $\langle ab \rangle$ are not conjugated. In fact $N_G(\langle a \rangle)$ is maximal in G and

$\langle ab \rangle \not\leq N_G(\langle a \rangle)$. In groups 2, 3, 4, 11 and 12, listed in theorem 6.1, the non-permutable subgroups $\langle a \rangle$ and $\langle b \rangle$ are not conjugated. $N_G(\langle a \rangle)$ is a maximal subgroup of G and $\langle b \rangle \not\leq N_G(\langle a \rangle)$. In group 8 of theorem 6.1, non-permutable subgroups $\langle ac \rangle$ and $\langle bc \rangle$ are not conjugated. In fact $N_G(\langle a \rangle) = \langle ac, c \rangle$ is maximal in G and $\langle bc \rangle \not\leq \langle ac, c \rangle$. □

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