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A Generalization of Quasi-Hamiltonian Groups


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Sunto. – Iwasawa classifica i gruppi finiti $G$ in cui tutti i sottogruppi $V$ sono permutabili, cioè $UV = VU$ per ogni sottogruppo $U$ di $G$. Tali gruppi sono detti quasi-hamiltoniani.

Noi classifichiamo i gruppi finiti in cui i sottogruppi non permutabili hanno tutti lo stesso ordine e quelli che hanno una sola classe di coniugio di sottogruppi non permutabili.

Summary. – Iwasawa classifies finite groups $G$ in which all subgroups $V$ are permutable, that is $UV = VU$ for all subgroups $U$ of $G$. These groups are called quasi-hamiltonian.

We classify the finite groups whose non-permutable subgroups have the same order and the ones which have a single conjugacy class of non-permutable subgroups.

Introduction.

The structure of groups whose subgroups are all normal (hamiltonian groups) has been completely described by R. Dedekind and R. Baer. A long series of papers has dealt with generalizations of this result; let me mention two of them.

A first generalization studies groups which satisfy conditions on the numbers of non-normal subgroups. Brandl (see [1]) classifies groups in which non-normal subgroups are in a single conjugacy class.

G. Zappa (see [5] and [6]) classifies finite groups whose non-normal subgroups have the same order. In addition to the groups found by Brandl, Zappa finds only the $p$-groups described in Theorem 1 and 2 in [6].

A second generalization studies groups whose subgroups have a property close to being normal. Iwasawa (see [2]) classifies finite groups $G$ in which all subgroups $V$ are permutable, that is $UV = VU$ for all subgroups $U$ of $G$. These groups are called quasi-hamiltonian. Our aim is to study finite groups whose non-permutable subgroups have the same order. This will also allow to classify the ones whose non-permutable subgroups are in the same conjugacy class.
1. – Preliminaries.

**Definition 1.** – A subgroup $H$ of $G$ is permutable in $G$ if $HK = KH$ for all subgroups $K$ of $G$. We will write $H \operatorname{perm} G$.

Such subgroups are also called quasinormal.

We list here a number of well known properties of permutable subgroups:

1. if $H \operatorname{perm} G$, $K \triangleleft G$ then $H \cap K \operatorname{perm} K$;
2. if $H, K \operatorname{perm} G$ then $HK \operatorname{perm} G$;
3. if $H \operatorname{perm} G$, $N \triangleleft G$ then $HN/N \operatorname{perm} G/N$;
4. if $H \leq G$, $N \triangleleft G$, $N \leq H$ then $HN/N \operatorname{perm} G$ if and only if $H \operatorname{perm} G$;
5. if $H \operatorname{perm} G$, $K \leq G$ and $(|H|, |K|) = 1$ then $K \leq N_G(H)$;
6. if $H$ is a Sylow subgroup of $G$ and $H \operatorname{perm} G$ then $H \triangleleft G$;
7. if $H$ is a maximal subgroup of $G$ and $H \operatorname{perm} G$ then $H \triangleleft G$;
8. if $H$ is a cyclic permutable subgroup of $G$ then each subgroup of $H$ is permutable in $G$.

**Proposition 1.1.** – $G$ is a finite non-nilpotent group whose non-permutable subgroups have the same order if and only if $G = N \times P$ split extension where $N \triangleleft G$ is of prime order $q$, $P$ is a cyclic $p$-group with $p \neq q$ and a generator of $P$ acts on $N$ as a nontrivial automorphism of order $p$.

**Proof.** – Assume first that the non-permutable subgroups of $G$ have the same order. Since $G$ is a finite non-nilpotent group, there exist a non-permutable Sylow $p$-subgroup $P$ and a maximal non-permutable subgroup $M$ of $G$. As non-permutable subgroups have the same order, $|M| = |P|$ and non-permutable subgroups are cyclic. It follows that $P$ is a $p$-Sylow, maximal, non-permutable and cyclic subgroup.

Let $N$ be the subgroup generated by all Sylow $q$-subgroups of $G$ where $q$ runs over all prime and $q \neq p$. These Sylow $q$-subgroups of $G$ are permutable, as their order is different from $|P|$, and so they are normal. Set $g \in N$ an element of prime order $q$. $(g)$ permutes with $P, P \langle g \rangle = G$ and so $N = \langle g \rangle$.

$\Phi(P) \triangleleft P$, it is permutable in $G$ and then $g \in N_G(\Phi(P))$. It follows that $\Phi(P) \triangleleft G$ and $[N, \Phi(P)] \leq N \cap \Phi(P) = 1$.

Finally $P$ and $N$ do not commute, that is $[N, P] \neq 1$. Conversely, if $G$ has the structure described in the statement, theorem in [1] proves that in $G$ there is only a conjugacy class of non-permutable subgroup, with $P$ as representative. 

**Proposition 1.2.** – If $G$ is a finite nilpotent group whose non-permutable subgroups have the same order then $G$ is a $p$-group.
Proof. – Suppose $G$ is not a $p$–group. Then $G = A \times B$ where $A$ and $B$ are nontrivial Hall-subgroups. The subgroups of $G$ are $H \times K$ with $H \leq A$ and $K \leq B$. Let $H_1 \times K_1, H_2 \times K_2$ be subgroups of $G$ such that $H_1 H_2 \times K_1 K_2 \neq H_2 H_1 \times K_2 K_1$. It follows that either $H_1 H_2 \neq H_2 H_1$ or $K_1 K_2 \neq K_2 K_1$.

Suppose $H_1 H_2 \neq H_2 H_1$: $H_1 \times 1$ and $H_1 \times B$ are non-permutable in $G$ but $|H_1 \times B| \neq |H_1|$, a contradiction. \hfill \Box

We are reduced to study $p$-groups. We indicate with $T(p^n)$ the class of finite non quasi-hamiltonian $p$-groups whose non-permutable subgroups have order $p^n$.

**Notation:**

$E(p^3)$ is the non abelian group of order $p^3$ and exponent $p$ ($p \neq 2$);

$M(p^{n+1}) = \langle x, y : x^{p^n} = y^p = 1, x^y = x^{1+p^{n-1}} \rangle$;

$S_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{-1+2^{n-2}} \rangle$;

$Q_{2^n}$ is the generalized quaternion group of order $2^n$, $D_{2^n}$ is the generalized dihedral group of order $2^n$ and $C_{p^n}$ is the cyclic group of order $p^n$. If $A, B$ are non identity $p$–groups with cyclic centre, $A \ast B$ indicates a central product with central subgroups of order $p$ amalgamated.

2. – The groups in $T(p)$.

**Proposition 2.1.** – Let $G$ be a group in $T(p)$. Let $A_1$ and $A_2$ be subgroups of $G$ of order $p$ such that $A_1 A_2 \neq A_2 A_1$, and let $N$ be a normal subgroup of $G$ of order $p$. Then:

1. $\langle A_1, A_2 \rangle = A_1 NA_2$ has order $p^3$ and is isomorphic to $D_8$ if $p = 2$, non abelian of exponent $p$ if $p \neq 2$;

2. $N$ is the only subgroup of order $p$ which permutes with both $A_1$ and $A_2$.

In particular $N$ is the only normal subgroup of order $p$ in $G$;

3. $A_1 N \leq G$.

Proof. – Let $A_1$ and $A_2$ be subgroups of $G$ such that $|A_i| = p$, $A_i = \langle a_i \rangle$ for $i = 1, 2$ and $A_1 A_2 \neq A_2 A_1$, and let $N = \langle n \rangle$.

$A_1 N$ is a subgroup of $G$ of order $p^2$ and so permutable. In particular $A_1 NA_2$ is a subgroup of $G$ of order $p^3$ and $A_1 NA_2 = \langle A_1, A_2 \rangle$. As it contains non-permutable subgroups, we have $\langle A_1, A_2 \rangle \cong D_8$ if $p = 2$, $\langle A_1, A_2 \rangle \cong E(p^3)$ if $p \neq 2$, so that: $\langle A_1, A_2 \rangle = \langle a_1, a_2 : a_1^p = 1 = a_2^p, [a_1, a_2] = n \in Z(\langle A_1, A_2 \rangle), n^p = 1 \rangle$.

Let $A_3$ be a subgroup of $G$ of order $p$ such that $A_1 \neq A_3$ and $A_1 A_3 = A_3 A_1$. Having order $p^2$, $A_1 A_3$ is a permutable subgroup. In particular the subgroup $A_1 A_3 A_2$ has order $p^3$ and then $A_1 A_3 A_2 = A_1 NA_2$. Moreover $A_1 A_3$ and $A_1 N$ are
normal subgroups of \( \langle A_1, A_2 \rangle \) and they both contain all the conjugates of \( A_1 \) in \( \langle A_1, A_2 \rangle \). Then \( A_1A_3 = A_1N \). Likewise if \( A_3 \) is a subgroup of \( G \) of order \( p \) such that \( A_2 \neq A_3 \) and \( A_2A_3 = A_3A_2 \) then \( A_2N = A_3A_2 \). In particular \( N \) is the only subgroup of order \( p \) which permutes with both \( A_1 \) and \( A_2 \) and then \( N \) is the only normal subgroup of order \( p \) in \( G \).

We prove now that \( A_1N \) is normal in \( G \). Let \( x \in G \).

Suppose first \( o(x) = p \). If \( a_1x = xa_1 \) then \( A_1^x \leq A_1N \). If \( a_1x \neq xa_1 \) then \( \langle a_1 \rangle \) and \( \langle x \rangle \) do not permute. As seen before \( \langle A_1, x \rangle = A_1N \langle x \rangle \) has order \( p^3 \) and \( A_1^x \leq A_1N \). In particular \( A_1N \trianglelefteq \Omega_1(G) \).

Suppose now \( o(x) = p^n \) where \( n > 1 \). \( \langle x \rangle \) is permutable in \( G \) and we may assume that \( A_1 \ntrianglelefteq \langle x \rangle \). Set \( \langle y \rangle = \Omega_1(\langle x \rangle) \). We have \( \langle a_1 \rangle \langle y \rangle = \langle y \rangle \langle a_1 \rangle \) and likewise \( \langle a_2 \rangle \langle y \rangle = \langle y \rangle \langle a_2 \rangle \). It follows that \( \langle y \rangle = N \). \( \langle a_1 \rangle \langle x \rangle \) is a group with a maximal cyclic subgroup, its order is \( p^{n+1} \) and \( |\Omega_1(\langle a_1 \rangle \langle x \rangle)| > p \). If \( p 
eq 2 \), \( \langle a_1 \rangle \langle x \rangle \) is either abelian or isomorphic to \( M(p^{n+1}) \) and then \( x^{a_1} \equiv x \mod \langle y \rangle \). Hence, \( \langle a_1 \rangle^x \in A_1N \).

Suppose now \( p = 2 \). If \( \langle a_1 \rangle \langle x \rangle \) is isomorphic to \( D_8 \) then \( x \in \Omega_1(G) \). Since \( D_{2n+1} \) and \( S_{2n+1} \) with \( n \geq 3 \) contain non-permutable subgroups of order \( 4 \), we have that \( \langle a_1 \rangle \langle x \rangle \) is either isomorphic to \( M(p^{n+1}) \) or abelian. Then \( x^{a_1} \equiv x \mod \langle y \rangle \) and \( \langle a_1 \rangle^x \in A_1N \).

\[ \square \]

**Theorem 2.2.** – Let \( G \) be a \( p \)-group. Then:

1. \( G \in T(p) \) where \( p \neq 2 \) if and only if \( G \) is isomorphic to one of the following groups:
   
   (a) \( E(p^3) \);
   
   (b) \( E(p^3) \ast C_{p^*} \).

2. \( G \in T(2) \) if and only if \( G \) is isomorphic to one of the following groups:
   
   (a) \( D_8 \);
   
   (b) \( D_8 \ast C_{2^*} \);
   
   (c) \( D_8 \ast Q_8 \).

**Proof.** – Let \( A_1 \) and \( A_2 \) be subgroups of \( G \) of order \( p \) such that \( A_1A_2 \neq A_2A_1 \), and let \( N \) be the normal subgroup of \( G \) of order \( p \).

By prop. 2.1, \( A_1 \) and \( A_2 \) have \( p \) conjugates in \( G \).

\( C_G(\langle A_1, A_2 \rangle) = C_G(A_1) \cap C_G(A_2) \). Since \( |G : C_G(A_i)| = p \) \((i = 1, 2)\), \( |G : C_G(A_1) \cap C_G(A_2)| = p^2 \).

Set \( H = \langle A_1, A_2 \rangle \); \( H \trianglelefteq G \). \( H \cap (C_G(A_1) \cap C_G(A_2)) = Z(H) = N \) and then \( G = H \ast C_G(H) \).

Moreover if \( K \leq C_G(H) \), \( |K| = p \), we have \( KA_1 = A_1K \) and \( KA_2 = A_2K \). Then \( K = N \) and \( C_G(H) \) is cyclic or generalized quaternion, but if \( n \geq 4 \) then \( Q_{2^n} \) contains non-permutable subgroups of order \( 4 \). Hence we get the groups of the proposition.

The groups listed above are in \( T(p) \). In fact \( E(p^3) \) and \( D_8 \) contain non-permutable subgroups of order \( p \), and all subgroups of order different from \( p \) are normal as proved in Theorem 2 in [6].

\[ \square \]
3. – The groups in $T(p^n)$ with $n \geq 2$.

**Proposition 3.1.** – Let $G \in T(p^n)$ with $n \geq 2$ and $|\Omega_1(G)| = p$.

Then $G$ is the generalized quaternion group of order 16 and $G \in T(4)$.

**Proof.** – If $|\Omega_1(G)| = p$ then $G$ is either cyclic or generalized quaternion. $Q_8$ and cyclic groups are hamiltonian and, if $n \geq 5$, $Q_2^a$ contains non-permutable subgroups of different orders. $Q_{16} = \langle a, b : a^4 = 1, b^4 = a^2, b^a = b^{-1} \rangle$ is in $T(4)$. In fact $\langle a \rangle$ and $\langle ab \rangle$ are not permutable, whereas the subgroup of order 2 and the subgroups of order 8 are normal. □

**Proposition 3.2.** – Assume $n \geq 2$ and let $G$ be in $T(p^n)$ with $|\Omega_1(G)| > p$. Let $A_1$ and $A_2$ be subgroups of order $p^n$ such that $A_1A_2 \neq A_2A_1$. Then:

1. $A_1$ and $A_2$ are cyclic;
2. $|A_1 \cap A_2| = p^{n-1}$;
3. $\langle A_1, A_2 \rangle = A_1 \langle t \rangle A_2$ for every $t \in \Omega_1(G) \setminus \Omega_1(A_1)$;

Moreover $\Omega_1(G)$ has order $p^2$ and is elementary abelian.

**Proof.** – Since subgroups of order $p$ are permutable, $\Omega_1(G) = \{g \in G : g^p = 1 \}$ and it is elementary abelian. $A_1$ and $A_2$ are cyclic because otherwise they would be product of permutable subgroups. Set $A_i = \langle a_i \rangle$ ($i = 1, 2$). Having order $p^{n-1}$, $\langle a_i^{p^n} \rangle$ is permutable in $G$. We consider $\langle a_1^{p^n} \rangle a_2 \leq G$ and $\langle a_2^{p^n} \rangle a_1 \leq G$.

$\langle a_1^{p^n} \rangle \langle a_2^{p^n} \rangle \langle a_1 \rangle = \langle a_2 \rangle \langle a_1 \rangle$ and $\langle a_1^{p^n} \rangle \langle a_1 \rangle \langle a_2^{p^n} \rangle \langle a_2 \rangle = \langle a_1 \rangle \langle a_2 \rangle$.

Hence $\langle a_1^{p^n} \rangle \langle a_2^{p^n} \rangle \langle a_1 \rangle \neq \langle a_1^{p^n} \rangle \langle a_1 \rangle \langle a_2^{p^n} \rangle \langle a_2 \rangle$ and we get $|\langle a_1^{p^n} \rangle \langle a_2 \rangle| = p^n$, $|\langle a_1 \rangle \langle a_2 \rangle| = p^3$, so that $|a_1^{p^n}| \leq |a_2|$ and $|a_2^{p^n}| \leq |a_1|$.

$\langle a_1 \rangle / \langle a_1^{p^n} \rangle$ and $\langle a_2 \rangle / \langle a_2^{p^n} \rangle$ have order $p$ and, as seen in section 2, they generate a subgroup of order $p^3$, which gives $|\langle a_1 \rangle \langle a_2 \rangle| = p^{n+2}$.

Since $|A_1 \cap \Omega_1(G)| = p$ and $|\Omega_1(G)| > p$, there exists $t \in \Omega_1(G)$, $t \notin A_1$.

Having order $p^{n+1}$, $A_1 \langle t \rangle$ is permutable in $G$, and $A_2 \cap A_1 \langle t \rangle = \langle a_1^{p^n} \rangle$. It follows that $|A_1 \langle t \rangle A_2| = p^{n+2}$ and then $\langle A_1, A_2 \rangle = A_1 \langle t \rangle A_2$. Furthermore $N_{\langle A_1, A_2 \rangle}(A_1) = A_1 \langle t \rangle$.

Suppose now that there exists $s \in \Omega_1(G)$, $s \notin A_1 \langle t \rangle$. As proved above, $\langle A_1, A_2 \rangle = A_1 \langle s \rangle A_2$ and $N_{\langle A_1, A_2 \rangle}(A_1) = A_1 \langle s \rangle$. Hence we get $A_1 \langle s \rangle = A_1 \langle t \rangle$ which contradicts our assumptions. □

With the following theorem, we complete the description of $p$–groups in $T(p^n)$ if $p \neq 2$. This reduces us to study 2–groups in $T(2^n)$ with $n \geq 2$.

**Theorem 3.3.** – Let $G$ be $p$–group, $p \neq 2$. The following conditions are equivalent:

1. $G$ is elementary abelian;
2. $G$ is a direct product of cyclic $p$–groups;
3. $G$ is a direct product of distinct cyclic $p$–groups.

**Proof.** – If $G$ is elementary abelian, then $G$ is a direct product of cyclic $p$–groups.

Conversely, if $G$ is a direct product of cyclic $p$–groups, then $G$ is elementary abelian.

Finally, if $G$ is a direct product of distinct cyclic $p$–groups, then $G$ is elementary abelian.

□
1. \( G \in T(p^n) \) where \( n \geq 2; \\
2. \( G \in T(3^2) \\
3. G = \langle a, c, b : a^9 = c^3 = 1, b^3 = a^3, ac = ca, a^b = ac, c^b = ca^{-3} \rangle. \\

**Proof.** Let \( A_1 \) and \( A_2 \) be subgroups of \( G \) of order \( p^n \) such that \( A_1A_2 \neq A_2A_1 \). By prop. 3.2, \( A_1 = \langle a_1 \rangle, A_2 = \langle a_2 \rangle \) and \( \langle A_1, A_2 \rangle = A_1(t)A_2 \) where \( t \in \Omega_1(G) \setminus \Omega_1(A_1) \). Moreover we can assume \( a_1^p = a_2^p \).

\( A_i(t) \leq G \) is either abelian or isomorphic to \( M(p^{n+1}) \) and \( A_i(t) \cong \langle A_1, A_2 \rangle \) for \( i = 1, 2 \). So we get: 
\[ a_1^t = a_1^{1+hp^{-1}}, \quad a_2^t = a_2^{1+hp^{-1}}, \quad a_1^{t^s} = a_1^{a_1 t^s} \text{ where } h, k \in \{1, \ldots , p \}, \]
\( s \in \{1, \ldots , p-1 \} \) and \( r \equiv 1 \mod(p) \); from \( a_1^p = (a_1^t)^{a_1} = (a_1^r)^t = a_1^{tp} \) we have \( r = 1 + j p^{-1} \) and then: 
\[ a_1^t = a_1^{1+hp^{-1}}, \quad a_2^t = a_2^{1+hp^{-1}}, \quad \langle a_1, a_2 \rangle \text{ has class } \leq 3 \text{ and derived subgroup contained in } \langle a_1^{p^{-1}}, t \rangle = \Omega_1(\langle A_1, A_2 \rangle). \]

If \( p > 3 \) we obtain a contradiction. In fact \( \langle A_1, A_2 \rangle \) is regular, hence \( (a_2a_1^{-1})^p = a_2^p a_1^{-1} t^p \) for some \( x \in \langle A_1, A_2 \rangle \). So \( a_2a_1^{-1} \) has order \( p \) but \( \langle a_2a_1^{-1} \rangle \) does not normalize \( A_1 \). It follows that there are no groups in \( T(p^n) \) if \( p > 3 \), \( n \geq 2. \)

Suppose now \( p = 3 \). Since \( \langle a_1, a_2 \rangle / \langle a_1^{3^{-1}} \rangle \) has class \( \leq 2 \), it follows that \( \langle a_1, a_2 \rangle / \langle a_1^{3^{-1}} \rangle \) is regular and \( a_1 a_2^{-1} a_1^{-3} \langle a_1^{3^{-1}} \rangle = 1. \)

If \( n \geq 3 \) we obtain a contradiction: \( a_1 a_2^{-1} \) has order \( \leq 9 \) but \( \langle a_1a_2^{-1} \rangle \) does not permute with \( A_2 \). Finally if \( p = 3 \) and \( n = 2 \), two non-permutable subgroups of order 9 generate a group of order 81 whose structure is described above: \( H = \langle a_1, a_2 \rangle, a_1^3 = a_2^3, \quad \Omega_1(H) = \langle a_1^3, t \rangle \). \( \langle a_1, t \rangle \) is either abelian or isomorphic to \( M(3^2) \). Since \( [H : C_H(\Omega_1(H))] = 3 \) we can choose \( a_1 \in C_H(\Omega_1(H)) \); further we may choose \( t \) such that \( a_1^{a_2} = a_1 t \). \( a_1 a_2 \) does not normalize \( A_1 \). If \( t_{a_2} = ta_2^{-3} \) then \( (a_2a_1^{-1})^3 = 1 \), a contradiction. So we have \( t_{a_2} = ta_2^{-3} \) and this shows that \( H \) is as in 3. Conversely, it can be easily checked that \( G \) is in \( T(3^2) \).

Suppose now that \( G \) is in \( T(3^2) \) and contains \( H \) as a proper subgroup; we may also assume that \( |G : H| = 3 \). By theorem (4.12) in [4], \( G = \langle b \rangle C_G(\Omega_1(G)) \).

We shall prove that \( C_G(\Omega_1(G)) = \langle a, c \rangle \). It will be enough to show that \( C_G(\Omega_1(G)) \) contains no elements of order 9 or 27 outside \( \langle a, c \rangle \).

First we note that \( a^3 \in Z(G) \); indeed \( \Omega_1(G) \cap Z(G) \neq 1 \) and \( c \not\in Z(\langle a, b \rangle) \).

Suppose \( y \in C_G(\Omega_1(G)) \setminus \langle a, c \rangle \) of order 9. \( \langle a \rangle \) and \( \langle y \rangle \) permute. Otherwise we have a contradiction: \( \langle a, y \rangle \cong \langle a, b \rangle \) but \( \Omega_1(\langle y, a \rangle) = \langle a^3, c \rangle \leq Z(\langle a, y \rangle) \) whereas \( \Omega_1(\langle b, a \rangle) \not\leq Z(\langle b, a \rangle) \).

If \( y^3 \in \langle a^3 \rangle \) then \( a^3 = y^{3k}, y^a = y^{1+3k} \) and \( (a y^{-k})^3 = 1 \), which gives \( y \in \langle a, c \rangle \).

Assume now \( y^3 \not\in \langle a^3 \rangle \), that is \( y^3 = a^{3k} \). Since \( \langle b \rangle \cap \langle y \rangle = 1, \langle b \rangle \) permutes with \( \langle y \rangle \) and \( y^b = y^{1+3i} a^{3j} \). Now \( c a^{-3} = c^b = (y^3)^b = y^3 = c \), a contradiction.

Suppose \( y \in C_G(\Omega_1(G)) \setminus \langle a, c \rangle \) of order 27. As \( y^3 \in \langle a, c \rangle, y^3 = ac^k \) and then \( y^9 = a^3. b \) normalizes \( \langle y \rangle \) and from \( (y^9)^b = y \) we get \( y^b = y^{1+3i} \). But \( a^b = (y^3c^{-k})^b = y^{3+3i}c^{-k} a^{3k} = ac^k a^{-3i} c^{-k} a^{3k} = a^{1+3i+3k} \in \langle a \rangle \), a contradiction. \( \square \)
4. – Groups in $T(2^n)$ with $n \geq 2$: first results.

In view of prop. 3.1 and 3.2, we will assume that the groups $G$ in $T(2^n)$ that we consider satisfy $|\Omega_1(G)| = 4$.

We will be interested in studying the following groups:

$T_1(n) = \langle a, b : a^4 = b^{2^n} = 1, a^b = a^3 \rangle \ (n \geq 2)$ and

$T_2(n) = \langle a, b : a^8 = 1, a^4 = b^{2^{n-1}}, a^b = a^7 \rangle \ (n \geq 3)$.

**Proposition 4.1.** – $T_1(n)$ for $n \geq 2$ is in $T(2^n)$.

**Proof.** – $Z(T_1(n)) = \langle a^2, b^2 \rangle$ and the square of every element of $Z(T_1(n))$ is in $\langle b^4 \rangle$. The elements of $T_1(n)$ are $z, az_1, abz_2, bz_3$ where $z, z_i \in Z(T_1(n))$. Since $\langle abz_2 \rangle$ and $\langle b^2z_3 \rangle$ have order $2^n$, we have to prove that $\langle az_1 \rangle$ permutes with both $\langle b^2z_2 \rangle$ and $\langle abz_3 \rangle$.

$(az_1)(b^2z_3) = abz_1z_3 = a^2b^2a^3z_1z_3 = ba^2a_1z_3 = bz_3(az_1)^3z_1^{-2}$. Setting $z_3^2 = b^{4i}$ and $z_1^2 = b^{4i}$, we get: $\langle b^2z_3 \rangle = \langle b^{2i} \rangle$ and there exists an integer $r$ such that $az_1bz_3 = (b^2z_3)^r(az_1)^3$.

The same if we consider $abz_2$ instead of $b^2z_3$. □

**Proposition 4.2.** – $T_2(n)$ with $n \geq 3$ is in $T(2^n)$.

**Proof.** – One see easily that: $Z(T_2(n)) = \langle b^2 \rangle$, $[a^2, T_2(n)] = \langle a^4 \rangle$, $|T_2(n)| = 2^{n+2}$, and $T_2(n)/\langle a^4 \rangle \cong T_1(n - 1)$. Moreover, for each $g \in T_2(n) \setminus \langle a, b^2 \rangle$ we have $\langle b^2 \rangle = \langle g \rangle$, $|\langle g \rangle| = 2^n$. It follows that non-permutable subgroups of $T_2(n)$ containing $\langle a^4 \rangle$ have order $2^n$ by prop. 4.1.

A subgroup not containing $\langle a^4 \rangle$ is cyclic; the possibilities are: $\langle a^2b^{2^{n-2}} \rangle$ of order 2 and (if $n > 3$) $\langle ab^{2^{n-3}} \rangle$ of order 4. Now $a^2b^{2^{n-2}}$ normalizes every subgroup of $T_2(n)$, $\langle ab^{2^{n-3}} \rangle$ centralizes $\langle a, b^2 \rangle$ and, if $g \notin \langle a, b^2 \rangle$, we have $|g| = 2^n$, $\langle g \rangle \cap \langle ab^{2^{n-3}} \rangle = 1$, $\lvert \langle g, ab^{2^{n-3}} \rangle \rvert = |T_2(n)| = 2^{n+2}$, so that $\langle g \rangle$ and $\langle ab^{2^{n-3}} \rangle$ permute. □

**Proposition 4.3.** – Let $G \in T(2^n)$ with $n \geq 2$. Two non-permutable subgroups of order $2^n$ generate a group isomorphic to one of $T_1(n)$ ($n \geq 2$), $T_2(n)$ ($n > 2$).

**Proof.** – Let $A_1$ and $A_2$ be subgroups of $G$ of order $2^n$ such that $A_1A_2 \neq A_2A_1$. By prop. 3.2, $A_1 = \langle a_1 \rangle$, $A_2 = \langle a_2 \rangle$ and $\langle A_1, A_2 \rangle = A_1 \langle t \rangle A_2$ where $t \in \Omega_1(G) \setminus \Omega_1(A_1)$. Moreover we can suppose $a_1a^2 = a_2^2$.

$\langle a_1, t \rangle$ ($t = 1, 2$) has a maximal cyclic subgroup and then it is either abelian or (if $n \geq 3$) isomorphic to $M(2^{n+1})$. Then $a_1^t = a_1$ or (if $n \geq 3$) $a_1^t = a_1a_i^{2^{n-1}}$ for $i = 1, 2$. Moreover $\langle a_1, t \rangle \leq \langle a_1, a_2 \rangle$ and then $a_1a_2 = a_1t$ with $j$ odd.
Hence the possibilities are:

1. \( a_1 t = ta_1, a_2 t = ta_2 \).
   
   From \( a_1^2 = (a_1^2)^a_2 \), we have \( 2j \equiv 2 \mod(2^n) \) which gives \( a_1^{a_2} = a_1^{1+2^{n-1}}t \).
   
   Then we may choose \( t \) such that \( a_1^{a_2} = a_1 t, a_1 t = ta_1 \) and \( a_2 t = ta_2 \).
   
   Setting \( a = a_1 a_2^{-1} \) and \( b = a_2 \), we get the group \( T_1(n) \).

2. (if \( n \geq 3 \)) \( a_1 t = a_1, a_2 t = a_2^{1+2^{n-1}} \).
   
   As seen above, we may choose \( t \) such that \( a_1^{a_2} = a_1 t, a_1^t = a_1, a_2^t = a_2^{1+2^{n-1}} \). Now \( (a_2^2)^{a_1} = a_2 t a_2 t = a_2 a_2^{1+2^{n-1}} \neq a_2^2 \), a contradiction.

3. (if \( n \geq 3 \)) \( a_1 t = a_1, a_2 t = a_2^{1+2^{n-1}} \).
   
   As seen above, we may choose \( t \) such that \( a_1^{a_2} = a_2 t, a_1^t = a_1, a_2^t = a_2^{1+2^{n-1}} \). Now \( (a_1^2)^{a_2} = a_1 t a_1 t = a_1 a_1^{1+2^{n-1}} \neq a_1^2 \), a contradiction.

4. (if \( n \geq 3 \)) \( a_1 t = a_1^{1+2^{n-1}}, a_2 t = a_2^{1+2^{n-1}} \).
   
   From \( a_1^2 = (a_1^2)^{a_2} \), we have \( 2 = 2j + 2^{n-1} \mod(2^n) \) which gives \( a_1^{a_2} = a_1^{1+2^{n-2}}t \) where \( j = 1, 3 \). Hence we may choose \( t \) such that \( a_1^{a_2} = a_1^{1+2^{n-2}}t \). Setting \( a = a_1 a_2^{-1} \) and \( b = a_2 \), we get the group \( T_2(n) \).

5. – The groups in \( T(4) \).

**Theorem 5.1.** – There is no group \( G \in T(4) \) having exponent \( > 4 \).

**Proof.** – By prop. 4.3, a group \( G \in T(4) \) contains a subgroup isomorphic to \( T = \langle a, b : a^4 = b^4 = 1, a^b = a^3 \rangle \). Since \( \exp(G) > 4 \), we can suppose \( G = T \langle z \rangle \) where \( o(z) = 8 \). We note that \( \Omega_1(G) = \Omega_1(T) \).

We first prove that \( \langle z \rangle \trianglelefteq G \).

Since \( \langle z \rangle \) permG, every element of order 2 normalizes \( \langle z \rangle \).

Let \( t \) be an element of \( T \) of order 4. If \( t^2 = z^4 \) then \( t \in N_G(\langle z \rangle) \). Suppose that \( t^2 \neq z^4 \). Then \( \langle z, t^2 \rangle \trianglelefteq \langle z, t \rangle \) but if \( z^i = z^j t^k \), we have a contradiction:

\[
(tz)^2 = (z^i)^2 z^j \in \langle z \rangle, \quad |\langle tz, z \rangle | = 16 \quad \text{whereas} \quad |\langle t, z \rangle | = 32.
\]

It follows that \( T \leq N_G(\langle z \rangle) \).

Suppose now \( z^i \in T \). Since \( \langle ba^{2i}b^2 \rangle \) and \( \langle ba^{2i}b^2 \rangle \) are not normal in \( T \), we have \( z^i \in a^i(a^2, b^2) \). From \( (z^i)^a = z^b \), we get \( z^b = z^{3+4k} \) and \( (bz)^2 = y^2 z^{3+4k} z = y^2 z^{4+k} \).

Then \( bz \) has order 4 and \( |\langle bz, b \rangle| \leq 16 \), a contradiction because \( |\langle b, z \rangle| = 32 \). It follows that \( z^i \notin T \).

If \( z^i = b^2 \) then \( G/\langle z \rangle \cong D_8 \) and \( \langle z, b \rangle \) is not permutable in \( G \).

Suppose \( z^i = a^2 \). Since \( |\langle z, a \rangle| = 16 \) and \( G = \langle z, a \rangle \langle b \rangle \), we get \( \langle z, a \rangle \cap \langle b \rangle = 1 \), and \( |\Omega_1(\langle z, a \rangle)| = 2 \). Then \( \langle a, z \rangle \) is either cyclic or a generalized quaternion subgroup of order 16. Suppose that \( \langle a, z \rangle \) is cyclic. Then \( z^2 \) is in \( \langle a \rangle \) and we have a contradiction. Suppose now that \( \langle z, a \rangle \cong Q_{16} \). In this case the element \( az \) has order 4 and \( \langle az \rangle \cap \langle b \rangle = 1 \). Then \( \langle az, b \rangle \) has order 16 but it does not permute with
\( \langle a \rangle \), a contradiction. Assume now \( z^4 = a^2b^2 \). If \( t \) is an element of \( T \) of order 4, then 
\[ [z, t] \in \langle z^2 \rangle. \] 
If \([z, t] = z^{2k}\) with \( k \) odd then \( o(tz) \leq 4 \) and \( |\langle tz, t \rangle| \leq 16 \) a contradiction because \( |\langle t, z \rangle| = 32 \). Hence, \([z, b] \in \langle z^4 \rangle\), \([z, a] \in z^4\). Now \( az \) has order 8, 
\( (az)^4 = \langle z^4 \rangle \) but \([az, b] \not\in \langle az \rangle\), a contradiction. \( \square \)

**Observation 1.** – A finite 2-group of exponent 4 has derived subgroup contained in \( \Omega_1(G) \). In particular the derived subgroup of \( G \in T(4) \) has order 2 or 4.

**Proposition 5.2.** – Let \( G \) be a group of exponent 4 with \( |G| = 32\), \( |G'| = 2 \).
Then \( G \) is in \( T(4) \) if and only if 
\[ G \cong \langle a, b, c : c^4 = a^4 = 1, a^2 = b^2, ca = ac, bc = cb, b^a = b^3 \rangle = M \cong Q_8 \times C_4. \]

**Proof.** – Since \( |G| = 32 \), \( |G'| = 2 \) and \( T = \langle a, b : a^4 = b^4 = 1, b^a = b^3 \rangle \leq G \), \( G' = \langle b^2 \rangle \). Every element in \( G \setminus T \) has order 4. Let \( c \in G \setminus T \). Now \( c^2 \in \Omega_1(G) = \langle a^2, b^2 \rangle \leq Z(G) \), \([c, a] \in \langle b^2 \rangle \), \([c, b] \in \langle b^2 \rangle \) so that \([c, a] = b^{2k}, [c, b] = b^{2k}. c \) acts on \( T \) as \( a^k b^h \) and then, replacing \( c \) with \( c(a^k b^h) \), we can suppose \( c \in Z(G) \).
We can have neither \( c^2 = a^2 (ac)^2 = 1 \) nor \( c^2 = b^2 ((bc)^2 = 1) \).
Then, \( G = \langle a, b, c : c^4 = a^4 = 1, c^2 = a^2 b^2, b^a = b^3, ac = ca, bc = cb \rangle \).
Repeating \( a \) with \( ac \), we get the presentation of the proposition.
Conversely, in the group 
\[ \langle a, b, c : c^4 = a^4 = 1, a^2 = b^2, ca = ac, bc = cb, b^a = b^3 \rangle \]
the subgroups \( \langle ac \rangle \) and \( \langle bc \rangle \) are non-permutable subgroups of order 4. Theorem 2 of [6] proves that subgroups of \( M \) of order different from 4 are normal, hence permutable.

**Proposition 5.3.** – Let \( G \) be a group of exponent 4 with \( |G| = 32\), \( |G'| = 4 \).
Then \( G \) is in \( T(4) \) if and only if 
\[ G \cong \langle a, b, c : a^4 = b^4 = 1, b^2 = c^2, ca = ac, c^h = ca^2, b^a = b^3 \rangle = R. \]

**Proof.** – Since \( |G| = 32 \), \( |G'| = 4 \), \( T = \langle a, b : a^4 = b^4 = 1, b^a = b^3 \rangle \) is contained in \( G \) and \( G' \leq \Omega_1(G) = \Omega_1(T) \), we have \( G' = \langle b^2, a^2 \rangle \leq Z(G) \). Let \( c \in G \setminus T \).
Then, \( c^2 \in \Omega_1(G) = \langle a^2, b^2 \rangle \) and \( c^2 \neq 1 \). Since \([c, a] \in \langle a^2, b^2 \rangle, [c, b] \in \langle a^2, b^2 \rangle \) we have \([c, a] = a^2b^2, [c, b] = a^{2k}b^{2k} \) and then \([cb^ia^k, a] = a^{2i}, [cb^ia^k, b] = a^{2k}. \)
Replacing \( c \) with \( cb^ia^k \), we get \( c^i = a^{1+2i} \) and \( b^c = a^{2h}b. \) Since \( a^2 \in G' \), either \( i \) or \( h \) has to be odd. If they are both odd, \( (ab)^c = ab \) and we replace \( a \) with \( ab \). So, the possibilities are:

- \( a^c = a, b^c = a^2b \). It can be neither \( c^2 = a^2 ((ac)^2 = 1) \) nor \( c^2 = a^2 b^2 ((cb)^2 = 1) \). It follows that \( c^2 = b^2 \) and \( G \cong R. \)
- \( a^c = a^{-1}, b^c = b \). It can be neither \( c^2 = b^2 ((bc)^2 = 1) \) nor \( c^2 = a^2 ((c)^2 \leq G, G/\langle c \rangle \cong D_8 \) and then \( \langle a, c \rangle \) is non permutable in \( G \). It follows that \( c^2 = a^2 b^2 \). Replacing \( a' = c, b' = a, c' = bc \), we get \( G \cong R. \)
The subgroups $\langle ab \rangle$ and $\langle a \rangle$ of order 4 of $R$ are non-permutable subgroups. Theorem 2 of [6] proves that subgroups of $R$ of order different from 4 are normal, hence permutable.

**Observation 2.** $- M$ and $R$ are the only groups in $T(4)$ of order 32. Moreover in $R$ there are neither central elements of order 4 nor subgroups of order 8 isomorphic to the quaternion group.

**Proposition 5.4.** $- M$ is not contained in a group $G \in T(2^2)$ of order $\geq 64$. In particular a group $G$ in $T(2^2)$ with $|G'| = 2$ has order $\leq 32$.

**Proof.** $- $ Suppose that there exists a group $G \in T(2^2)$ of order 64 containing $M$, $G = \langle a, b, c, d \rangle$ where $d \notin M$. Since $G' \leq \Omega_1(G) = \langle b^2, c^2 \rangle$ the possibilities are:

1. $|G'| = 2$. Then $G' = \langle b^2 \rangle$ and $[a, d] = b^{2h}$, $[b, d] = b^{2k}$, $[c, d] = b^{2r}$. Replacing $d$ with $db^ka^k$, we get $[a, d] = 1$, $[b, d] = 1$, $[c, d] = b^{2r}$. Moreover $d \notin Z(G)$; for each $w \in \Omega_1(G)$ there is $t \in M$ such that $t^2 = w$ and so if $d^2 = w$ then $(db)^2 = 1$. Hence, we get: $[a, d] = 1$, $[b, d] = 1$, $[c, d] = b^{2}$. It can be neither $d^2 = a^2$ ($\langle da \rangle^2 = 1$) nor $d^2 = a^2c^2$ ($\langle dc \rangle^2 = d^2ca^2c = 1$), and if $d^2 = c^2$ then $(dac)^2 = d^2aca^2c = 1$.

2. $|G'| = 4$. Then $G' = \langle b^2, c^2 \rangle$ and $[a, d] = b^{2h}c^{2k}$, $[b, d] = b^{2i}c^{2j}$, $[c, d] = b^{2r}c^{2s}$ where $h, k, i, j, r, s \in 0, 1$. Since $[a, db^ka^i] = c^{2k}$, $[b, db^ka^i] = c^{2j}$ and $[c, db^ka^i] = b^{2r}c^{2s}$, replacing $d$ with $db^ka^i$, we get $[a, d] = c^{2k}$, $[b, d] = c^{2j}$, $[c, d] = b^{2r}c^{2s}$. If $[a, d] = c^2$, we have $d \notin N_G(\langle a, b \rangle)$ and so $d^2 = a^2c^2$. Now $(da)^2 = d^2ac^2a \in \langle a^2 \rangle$, hence $da \in N_G(\langle a, b \rangle)$, a contradiction. It follows that $[d, a] = 1$. Likewise we prove that $[b, d] = 1$. We can not have $d^2 = a^2$ because in this case $(da)^2 = 1$, a contradiction. Moreover, since $c^2 \in G'$, we get $[d, c] = c^2a^{2i}$. Hence, we have the following cases:

- (a) $d^2 = c^2$. If $[c, d] = c^2b^2$, we have that the groups $\langle dc, ac \rangle \cong Q_8$ and $\langle bd \rangle$ do not permute. If $[c, d] = c^2$, we have that the groups $\langle ac, db \rangle \cong Q_8$ and $\langle ad \rangle$ do not permute;

- (b) $d^2 = a^2c^2$. If $[c, d] = c^2$, $\langle dab, c \rangle \cong Q_8$ does not permute with $\langle db \rangle$. If $[c, d] = a^2c^2$ then $(dc)^2 = d^2ca^2c^2 = c^2$ and, replacing $d$ with $dc$, we are in the previous case.

In each case we reached a contradiction and then $M$ is not contained in a group $G \in T(2^2)$ of order $\geq 32$.

**Observation 3.** $- $ Let $G \in T(4)$ of order $\geq 64$ and let $K$ be a subgroup of order 32 of $G$. By prop. 5.4, if $K$ contains non-permutable subgroups then $K \cong R$. If $K$ is quasi-hamiltonian then it should be either abelian or isomorphic to $Q_8 \times E$ where $E$ is elementary abelian, but in both cases we should have $|\Omega_1(K)| > 4$.

Hence, a subgroup of order 32 of $G \in T(4)$ of order $\geq 64$ is isomorphic to $R$.
At this point, we note that we are in a situation already considered by Zappa in [5]. The argument of lemma 7 and prop. 3 of [5] allow to prove the following propositions:

**Proposition 5.5.** - Let $G \in T(2^2)$ with $|G| = 64$. Then:

$$G \cong V = \langle a, b, c, d : a^4 = b^4 = 1, b^2 = c^2, d^2 = a^2, ca = ac, cb = ca^2, b^n = b^3, db = bd, a^d = aa^2b^2, c^d = cb^2 \rangle.$$  

**Proposition 5.6.** - If $G \in T(4)$ then $|G| \leq 64$.

5.1 - The groups in $T(2^n)$, $n > 2$.

**Observation 4.** - Let $G \in T(2^n)$, $T_1(n) \leq G$. By prop.3.2, $|\Omega_1(G)| = 4$ and $\Omega_1(G) = \Omega_1(T_1(n)) = \langle a^2, b^{2^{n-1}} \rangle$. Let $K$ be a normal subgroup of $G$ containing $\Omega_1(G)$, $G/K$ is quasi-hamiltonian, and so if $u, v \in G$, $o(uK) = 2$ and $o(vK) \leq 4$, we get $[v, u] \in K$.

We always take $K = Z(T_1(n))$.

**Theorem 5.7.** - If $n > 2$ there is no group $G$ in $T(2^n)$ such that $|G| > 2^{n+2}$ and $T_1(n) \leq G$.

**Proof.** - Suppose $G \in T(2^n)$, $T_1(n) \leq G$. We may assume that $[G : T_1(n)] = 2$.

The subgroups generated by elements of order 2 or 4 are permutable and so $\Omega_2(G)$ is abelian or isomorphic to $Q_8 \times E$ where $E$ is elementary abelian. Since $\Omega_2(T_1(n))$ is the direct product of two cyclic groups of order 4, we get that $\Omega_2(G)$ is abelian.

Let $z \in G, z \not\in T_1(n)$ of order 4. Since every element of $\Omega_1(T_1(n))$ is a square in $T_1(n)$, we have $z^2 = t^2$ and $(zt)^2 = 1$, a contradiction. It follows that $\Omega_2(G) = \Omega_2(T_1(n))$.

Let $z \in G \setminus T_1(n)$ of order $2^m$ where $m \leq n$ and $z^2 \in T_1(n)$. The elements of order $\leq 2^{n-1}$ are $ab^{2^i}, a^3b^{2^i}, a^2b^{2^i}$ and $b^{2^i}$ ($i$ an integer).

If $z^2 = b^{2^i}$ then $\langle z \rangle$ and $\langle b \rangle$ permute. Otherwise, since $ba^2 = a^2b$ we would get $\langle b, z \rangle \cong T_1(n), \Omega_2(\langle z, b \rangle) = \Omega_2(G) = \Omega_2(T_1(n))$, $a \in \langle z, b \rangle$ and then $T_1(n) = \langle z, b \rangle$, a contradiction. $\langle z, b \rangle$ is either abelian or isomorphic to $M(2^n)$. In both cases $o(zb^{i-1}) \leq 4$, a contradiction.

If $z^2 = ab^{2^i}$ then, by obs.4, we get $[b, z] \in \langle z^4, b^2 \rangle$ so that $b^2 = b^{i+2}z^{4i}$. Now $ba^2 = b^2 = (b^{i+2}z^{4i})^{i+2}z^{4i} = b^{i+2}z^{8i(1+2)} \in \langle b \rangle$, a contradiction.

If $z^2 = a^3b^{2^i}$, replacing $z$ with $z^{-1}$, we are in the previous case.
If $z^2 = a^2 b^2$ then, by obs.4, we get $[a, z] = a^{2r} b^{2s}$ and $[b, z] = a^{2k} b^{2k}$. If $[a, z] = b^{2s}$ then $(za)^2 = z^2 a b^2 s a = z^2 a^2 b^{2s} \in \langle b^2 \rangle$ and, replacing $z$ with $za$, we are in a previous case. If $[b, z] = a^{2b} b^{2k}$ then $(zb)^2 = z^2 a^2 b^{1+2k} b = a^2 b^{2i} a^2 b^{1+2k} b \in \langle b^2 \rangle$ and, replacing $z$ with $zb$, we are in a previous case.

Finally if $a^z = a^{3b} z b$ and $b^z = b^{1+2k}$, $(zab)^2 \in \langle b^2 \rangle$ and, replacing $z$ with $z ab$, we are again in a previous case.

Suppose now $z \in G \setminus T_1(n)$ of order $2n+1$. Since $G \in T(2^n)$, $\langle z \rangle$ perm $G$ and it cannot contain a non-permutable subgroup. Now all the elements of order $2^n$ in $T_1(n)$ generate non-permutable subgroups and so this case is not possible.

**Theorem 5.8.** There is no group $G$ in $T(2^n)$ such that $T_2(n) \leq G$.

**Proof.** Suppose first $n = 3$, and let $G \in T(8)$, $T_2(3) \leq G$, $[G : T_2(3)] = 2$. $G/\Omega_1(G)$ is quasi-hamiltonian, $T_2(3)/\Omega_1(T_2(3)) \cong Q_8$ and then $G/\Omega_1(G) \cong Q_8 \times C_2$. This proves that elements outside $T_2(3)$ have order $\leq 4$.

Let $z \in G \setminus T_2(3)$ of order 4.

If $z^2 = a^4$ then $\langle z, a \rangle$ is either abelian or isomorphic to $M(2^4)$. In both cases $(a^2 z)^2 = 1$, a contradiction.

Suppose $z^2 = a^2 b^2$. Since $(z^2)^2 = z^4$ and $\langle b, z^2 \rangle \cong \langle b, z \rangle$, it can be neither $b^z = y^{1+2i}$ nor $b^z = b^{1+4k} z^2$. Hence we get $b^z = b^{3+4k} z^2$. $(zb)^2 = z^2 b^{3+4k} z^2 b = a^6 b^{3+4k} a^2 b = b^{7+4k} b = b^{4k}$ and so $|\langle b, z \rangle| = 16$, a contradiction because $|\langle b, z \rangle| = 32$.

Finally if $z^2 = a^2 b^6$, replacing $b$ with $b^{-1}$, we are in the previous case.

Assume now $n > 3$ and let $G \in T(2^n)$, $T_2(n) \leq G$. Since $\Omega_1(T_2(n)) \cap Z(T_2(n)) = \langle a^4 \rangle$, $\Omega_1(T_2(n)) \cap Z(G) \neq 1$, we have $\Omega_1(T_2(n)) \cap Z(G) = \langle a^4 \rangle$. The subgroups of $G/\langle a^4 \rangle$ are $H/\langle a^4 \rangle$ where $\langle a^4 \rangle < H \leq G$ and $b \langle a^4 \rangle$ is non-permutable in $G/\langle a^4 \rangle$. Let $H/\langle a^4 \rangle$ and $K/\langle a^4 \rangle$ be subgroups such that $H/\langle a^4 \rangle K/\langle a^4 \rangle \neq K/\langle a^4 \rangle H/\langle a^4 \rangle$ and $H$ and $K$ are non-permutable cyclic subgroups of $G$ of order $2^n$ and $\langle a^4 \rangle < H$. It follows that $|H/\langle a^4 \rangle| = 2^{n-1} = |b \langle a^4 \rangle|$, $G/\langle a^4 \rangle \in T(2^{n-1})$ and $T_1(n-1) \cong T_2(n)/\langle a^4 \rangle \leq G/\langle a^4 \rangle$. By theorem 5.7, $T_2(n)/\langle a^4 \rangle = G/\langle a^4 \rangle$ and then we have $T_2(n) = G$. □

6. Conclusions.

The task of classifying finite $p$-groups in $T(p^n)$ is now completed. Our results are collected in the following theorem:

**Theorem 6.1.** The following conditions are equivalent:

- The group $G$ is in $T(p^n)$;
- $G$ is isomorphic to one of the following groups:
1. \( \langle a, b : a^4 = 1, b^4 = a^2, b^a = b^{-1} \rangle \) where \( p = 2 \) and \( n = 2; \)

2. \( \langle a, b, c : a^p = b^p = c^p = 1, [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle \)
\[ \text{where } p \geq 3 \text{ and } n = 1; \]

3. \( \langle a, b, d : a^p = b^p = d^m = 1, [a, b] = d^{p-1}, [a, d] = 1, [b, d] = 1 \rangle \)
\[ \text{where } p \geq 3, n = 1 \text{ and } m > 1; \]

4. \( \langle a, c, b : a^9 = c^3 = 1, a^3 = b^3, ac = ca, a^b = ac, c^b = ca^{-3} \rangle \)
\[ \text{where } p = 3 \text{ and } n = 2; \]

5. \( \langle a, b : b^4 = 1 = a^2, b^a = b^{-1} \rangle \) where \( p = 2 \) and \( n = 1; \)

6. \( \langle a, b, c : a^4 = b^4 = a^2 = c^2 = b^2 = b^a = b^{-1}, bc = cb, ac = ca \rangle \)
\[ \text{where } p = 2, n = 1 \text{ and } m > 1; \]

7. \( \langle a, b, c, d : b^4 = a^2, b^2 = c^2 = d^2, b^a = b^{-1}, c^d = c^{-1}, bc = cb, ac = ca, bd = db, ad = da \rangle \)
\[ \text{where } p = 2 \text{ and } n = 1; \]

8. \( \langle a, b, c : a^4 = c^4 = 1, a^2 = b^2, b^a = b^3, a^c = a, b^c = b \rangle \)
\[ \text{where } p = 2 \text{ and } n = 2; \]

9. \( \langle a, b, c : a^4 = b^4 = 1, c^2 = b^2, b^a = b^3, ac = ca, b^c = ba^2 \rangle \)
\[ \text{where } p = 2 \text{ and } n = 2; \]

10. \( \langle a, b, c, d : b^4 = a^4 = 1, b^2 = c^2, d^2 = a^2, ca = ac, c^b = ca^2, d^a = ba^2 b^a, c^d = cb^2 \rangle \)
\[ \text{where } p = 2 \text{ and } n = 2; \]

11. \( \langle a, b : a^4 = b^2 = 1, a^b = a^3 \rangle \) where \( p = 2 \) and \( n \geq 2; \]

12. \( \langle a, b : a^8 = 1, a^4 = b^{2^{m-1}}, a^b = a^{-7} \rangle \) where \( p = 2 \) and \( n \geq 3. \)

Brandl [1] classified the finite groups in which the non normal subgroups are in a single conjugacy class. We can use the list given above to solve the analogous problem for non-permutable subgroups.

**Proposition 6.2.** – The group \( G = N \times P \) in prop. 1.1 has only a conjugacy class of non-permutable subgroup.

The groups listed in theorem 6.1 have at least two conjugacy classes of non-permutable subgroups.

**Proof.** – Let \( G = N \times P \) be a split extension where \( N \trianglelefteq G \) is of prime order \( q \), \( P \) is a cyclic \( p \)-group with \( p \neq q \) and a generator of \( P \) acts on \( N \) as a nontrivial automorphism of order \( p \). Then \( G \) has only a conjugacy class of non-permutable subgroup, whose representative \( P \) has \( q \) conjugates.

In groups 1, 5, 6, 7, 9 and 10, listed in theorem 6.1, the non-permutable subgroups \( \langle a \rangle \) and \( \langle ab \rangle \) are not conjugated. In fact \( N_G(\langle a \rangle) \) is maximal in \( G \) and
\langle ab \rangle \not\leq N_G(\langle a \rangle). \text{ In groups 2, 3, 4, 11 and 12, listed in theorem 6.1, the non-permutable subgroups \langle a \rangle and \langle b \rangle are not conjugated. } N_G(\langle a \rangle) \text{ is a maximal subgroup of } G \text{ and } \langle b \rangle \not\leq N_G(\langle a \rangle). \text{ In group 8 of theorem 6.1, non-permutable subgroups } \langle ac \rangle \text{ and } \langle bc \rangle \text{ are not conjugated. In fact } N_G(\langle a \rangle) = \langle ac, c \rangle \text{ is maximal in } G \text{ and } \langle bc \rangle \not\leq \langle ac, c \rangle.

\square

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