Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 797–818.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_797_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 797-818

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Sunto. – *L'obiettivo di questo lavoro é quello di dimostrare una nuova regola di derivazione per funzioni composte e un teorema di semicontinuitá inferiore rispetto alla topologia L^1 per un funzionale integrale definito su $BV(\Omega)$. Si applica poi quest'ultimo risultato per ottenere nuovi risultati di rilassamento e Γ-convergenza in assenza di coercivitá e in assenza di continuitá della funzione integranda $f(x, s, p)$ nella variabile s.*

Summary. – *The aim of this work is to prove a chain rule and an L^1-lower semicontinuity theorems for integral functional defined on $BV(\Omega)$. Moreover we apply this result in order to obtain new relaxation and Γ-convergence result without any coerciveness and any continuity assumption of the integrand $f(x, s, p)$ with respect to the variable s.*

1. – Introduction

The L^1-lower semicontinuity of an integral functional of the type:

$$F(u) = \int_{\Omega} f(x, u, \nabla u)dx \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $u \in W^{1,1}(\Omega)$ has been extensively studied, in order to find the minimal assumptions on integrand f that ensure its L^1-lower semicontinuity on $W^{1,1}(\Omega)$, and to obtain an integral representation for its relaxed functional in the larger space $BV(\Omega)$.

The most of recent studies on this subject originate from celebrated result by Serrin. In [24] the author establishes the lower semicontinuity of (1.1) with respect to the L^1-topology by requiring that f is continuous in all its variables, convex in the last variable and by assuming on the integrand one of the following conditions: f is coercive; f is strictly convex in the gradient variable; the derivatives $f_x(x, s, p)$, $f_p(x, s, p)$, $f_{xp}(x, s, p)$ exist and are continuous (for further improvements see, among others, also [11, 13, 15, 21, 22]).

It is well known that a natural extension of (1.1) to the larger space $BV(\Omega)$ is given by the functional

$$(1.2)\quad \mathcal{F}(u)=\int_{\Omega} f(x,u,\nabla u)dx+\int_{\Omega} f^{\infty}\left(x,\tilde{u},\frac{D^c u}{|D^c(u)|}\right)d|D^c u| \\ +\int_{J_u}\left[\int_{u^-}^{u^+} f^{\infty}(x,s,\nu_u)ds\right]d\mathcal{H}^{N-1}.$$

Indeed, in [6] it is proved, by assuming continuity and coerciveness on integrand f, that $\mathcal{F}$ coincides with the relaxed functional of F on $BV(\Omega)$. Also for this functional, in the literature there exist several L^1-lower semicontinuity results. Among others we recall [10, 12, 18, 19] and the reference therein. In particular in ([12]) the authors prove an L^1-lower semicontinuity result for the functional (1.2), by assuming $W^{1,1}$ dependance of the integrand in the variable x. Moreover in that paper no continuity with respect to x is considered and this lack of regularity is compensated by assuming on the contrary the continuity with respect to s.

In this paper we investigate a similar problem. Roughly speaking in the same $W^{1,1}$ dependance on x as in ([12]), we prove an L^1-lower semicontinuity result for the functional (1.2), by weakening the regularity assumptions with respect to s, but assuming the continuity with respect to x.

In this direction, after Serrin's theorem, many authors extended his result by assuming weaker conditions. We recall, for instance, a classical result due to De Giorgi-Buttazzo-Dal Maso (see [15]), where they proved that for autonomous functionals the continuity of s is not necessary in order to obtain the L^1-lower semicontinuity of (1.1). A similar result for the functional (1.2) was proven by De Cicco in [10]. In this last paper the lower semicontinuity is stated with respect to the weak* convergence of $BV(\Omega)$, instead of the L^1-convergence and then extended to non autonomous functionals in [9].

Here, we generalize both the result of De Giorgi-Buttazzo-Dal Maso and of De Cicco, by proving the lower semicontinuity for the functional (1.2) with respect to the L^1-convergence on $BV(\Omega)$ (see Theorem 4.2). This theorem is stated by only requiring that the integrand f is $W^{1,1}$ in x with a uniform control of the weak gradient (see hypothesis (4.1)) and continuous in x (not uniformly with respect to the other variables). Our result improves also the lower semicontinuity theorem of Fonseca-Leoni (see [18]), since they assume a continuous dependance of integrand f in x uniformly with respect to the other variables (see [20] for the consequence of this assumption). Moreover, we generalize also the lower semicontinuity result of Fusco-Giannetti-Verde (see [19]), where they assume the continuity of f in all its variables. Finally our result is an extension of the lower semicontinuity theorem of De Cicco-Leoni (see [13]), since they deal only with the space $W^{1,1}$.

The main tools of the proof of the lower semicontinuity theorem are a new chain rule formula for the function $x \to \int_0^{u(x)} b(x,t)dt$ (see Theorem 3.1) and an approximation result for convex functions due to De Giorgi.

Moreover,we generalize a relaxation result stated in [18] (where no coerciveness assumption and continuity with respect to s are assumed), by removing also the uniform continuity of x with respect to the other variables.

Finally, we apply this result in order to obtain a Γ-convergence result for a sequence of functionals whose integrals pointwise converge. This last result generalizes an analogous theorem proven in [6], since here no continuity with respect to s and no coerciveness condition are required.

The paper is organized as follows: Section 2 is devoted to notations, defintions and preliminaries. In Section 3 we state the new chain rule formula. In Section 4 we establish the lower semicontinuity theorem and we give an improved version of this result. In the last section we state and prove the relaxation formula and Γ-convergence theorem.

Acknowledgments. I would like to thank Micol Amar and Virginia De Cicco, for suggesting me this research work and for many useful discussions.

2. – Definitions and Preliminary Results.

2.1 – *BV-functions*

In this section we give basic definitions and we collect some technical results on $BV(\Omega)$.
Let Ω be a bounded open subset of $\mathbb{R}^N$ and $u \in L^1_{loc}(\Omega)$.

We say that u has an approximate limit in x if there exists $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r \to 0} \fint_{B_r(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

The set C_u of all points where u has an approximate limit is a Borel set.
The function $\tilde{u} : C_u \to \mathbb{R}$, called precise representative of u, is a Borel function.
We say that u is approximately continuous at x if $x \in C_u$ and $\tilde{u}(x) = u(x)$.

We say that a point $x \notin C_u$ is an approximate jump point if there exist $u^+(x)$, $u^-(x) \in \mathbb{R}$ with $u^-(x) < u^+(x)$ and $\upsilon_u(x) \in \mathcal{S}^{N-1}$ such that

$$\lim_{r \to 0} \fint_{B_r^+(x;\upsilon_u(x))} |u(y) - u^+(x)| dy = 0, \quad \lim_{r \to 0} \fint_{B_r^-(x;\upsilon_u(x))} |u(y) - u^-(x)| dy = 0,$$

where $B_r^+(x;\upsilon_u(x)) = \{y \in B_r(x) : \langle x, \upsilon_u(x) \rangle > 0\}$ and $B_r^-(x;\upsilon_u(x))$ is defined analogously. Also the set $J_u = \{x \in \Omega : u^-(x) < u^+(x)\}$ of all approximate jump points of u is a Borel set and the function $(u^+(x), u^-(x), \upsilon_u(x)) : J_u \to \mathbb{R} \times \mathbb{R} \times \mathcal{S}^{N-1}$ is a Borel function.

Let $x \in C_u$. We say that u is approximately differentiable at x if there exists a vector $P \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \fint_{B_r} |u(y) - \tilde{u}(x) - \langle P, y - x\rangle| dy = 0.$$

The vector P is called the approximate differential at x and it is denoted $\nabla u(x)$. The set D_u of all points where u is approximately differentiable is a Borel set and the map $\nabla u(x) : D_u \to \mathbb{R}^N$ is a Borel map.

We recall that, if $u \in BV(\Omega)$, we have $H^{N-1}(\Omega \setminus (C_u \cup J_u)) = 0$ and we can split the measure Du in the following way

$$Du = D^a u + D^s u = D^a u + D^c u + D^j u$$

where

$$D^a u = \nabla u \mathcal{L}^N \lfloor D_u,\ D^c u = Du \lfloor (C_u \setminus D_u),\ D^j u = (u^+ - u^-) \otimes \upsilon_u \mathcal{H}^{N-1} \lfloor J_u, \tag{2.1}$$

where $\nabla u \in L^1(\Omega)$ and $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$ (see Proposition 3.92 pag. 184 of [3]).

For every $u \in BV(\Omega)$ we define the subgraph of u given by

$$S(u) = \{(x, s) \in \Omega \times \mathbb{R} : s < u^+(x)\}. \tag{2.2}$$

We recall (see Theorem 3.2.23 of [23]) that $S(u)$ is a set with locally finite perimeter in $\Omega \times \mathbb{R}$, i.e. $\chi_{S(u)} \in BV_{loc}(\Omega \times \mathbb{R})$. We indicate by $a(u) = (a_1(u), .., a_{n+1}(u))$ the distributional derivative of $\chi_{S(u)}$.

We recall also the following results

PROPOSITION 2.1 (see [3], Proposition 3.64 pag. 160). – *Let $u : \mathbb{R}^N \to \mathbb{R}$ and $\{\varphi_\varepsilon\}$ be a mollifying sequence. Then, if u is approximately continuous at $x \in \mathbb{R}^N$,*
$$(\varphi_\varepsilon * u)(x) \to u(x) \quad \text{for } \varepsilon \to 0.$$

PROPOSITION 2.2 (see Appendix of [8]). – *Let $u \in BV(\Omega)$. Let $M \subset \mathbb{R}$ such that $\mathcal{L}^1(M) = 0$ and let $E = C_u \cap (\tilde{u})^{-1}(M)$. Then $|Du|(E) = 0$.*

2.2 – *Preliminary Lemmas*

LEMMA 2.1 (see [13], Proposition 2.5). – *Let E be an open subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^d$. Let $g : E \times G \to \mathbb{R}$ be a Borel function in $L^\infty_{loc}(E \times G)$ such that for $\mathcal{L}^N$ almost every $x \in E$ the function $g(x, \cdot)$ is continuous in G. Then there exists a set $M \subset \mathbb{R}^N$ with $\mathcal{L}^N(M) = 0$ such that for every $t \in G$ the function $g(\cdot, t)$ is approximately continuous in $E \setminus M$.*

LEMMA 2.2 ([10], Lemma 7). – *Let μ be a positive Radon measure on $\Omega \times \mathbb{R}$ and let $\{f_\kappa\}$ be a sequence of nonnegative functions of $L^1(\Omega \times \mathbb{R}; d\mu)$. Set $f := \sup_{\kappa \in \mathbb{N}} f_\kappa \geq 0$. Then for every open subset A of $\Omega \times \mathbb{R}$ we have*

$$\int_A f d\mu = \sup_{\mathcal{D}} \sum_{i \in I} \int_A f_{\kappa_i}(x,s)\varphi_i(x)\psi_i(s)d\mu,$$

where $\mathcal{D}$ is the set of all families $(\kappa_i, \varphi_i, \psi_i)_{i \in I}$ with I finite, $\kappa_i \in \mathbb{N}$, $\varphi_i \in C_0^\infty(\Omega)$, $\psi_i \in C_0^\infty(\mathbb{R})$, $\varphi_i \geq 0$, $\psi_i \geq 0$, $\sum_{i \in I} \varphi_i \otimes \psi_i \leq 1$ and $supp(\varphi_i) \times supp(\psi_i) \subset A$.

2.3 – *Functionals and their properties*

If f is a Borel function such that the map $p \to f(x,s,p)$ is convex on $\mathbb{R}^N$ for every $(x,s) \in \Omega \times \mathbb{R}$, we consider the following functionals defined on the space $BV(\Omega)$:

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega); \end{cases} \tag{2.3}$$

$$\mathcal{F}(u) = \int_\Omega f(x,u,\nabla u)dx + \int_\Omega f^\infty\left(x, \tilde{u}, \frac{D^c u}{|D^c(u)|}\right) d|D^c u| + \int_{J_u} \left[\int_{u^-}^{u^+} f^\infty(x,s,\nu_u)ds \right] d\mathcal{H}^{N-1}, \tag{2.4}$$

where $f^\infty(x,s,p)$ is the recession function, defined by

$$f^\infty(x,s,p) = \lim_{t \to +\infty} \frac{f(x,s,pt)}{t}.$$

We define also

$$\hat{f}(x,s,p,t) = \begin{cases} f\left(x,s,\dfrac{p}{t}\right)t & t > 0, \\ f^\infty(x,s,p) & t = 0. \end{cases} \tag{2.5}$$

It is easy to verify that $\hat{f}$ is a Borel function and that for each $(x,s) \in \Omega \times \mathbb{R}$ the map $(p,t) \mapsto \hat{f}(x,s,p,t)$ is convex and positively homogeneous of degree 1.

We assume, for any Borel function, the following convention:

$$\fint_a^b h(t)dt = \begin{cases} \dfrac{1}{b-a} \displaystyle\int_a^b h(t)dt & a \neq b, \\ h(a) & a = b. \end{cases} \tag{2.6}$$

We notice also that, taking into account (2.6), the functional (2.4) can be rewritten as

$$\mathcal{F}(u) = \int_{\Omega} f(x, u, \nabla u)dx + \int_{\Omega} \left[\fint_{u^-}^{u^+} f^{\infty}\left(x, s, \frac{D^s u}{|D^s(u)|}\right) ds \right] d|D^s u|. \tag{2.7}$$

Let us recall the following result:

LEMMA 2.3 ([6], Lemma 2.2). – *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ *be a Borel function such that, for each* $(x, s) \in \Omega \times \mathbb{R}$, *the map* $p \mapsto f(x, s, p)$ *is convex on* $\mathbb{R}^N$. *Then*

$$\mathcal{F}(u) = \int_{\Omega\times\mathbb{R}} \hat{f}\left(x, s, \frac{a(u)}{|a(u)|}\right) d|a(u)|(x, s).$$

2.4 – *Approximation of convex functions*

One of the main tool in order to prove the lower semicontinuity of the functional (2.4) is an approximation result for convex functions due to De Giorgi. This result states that any convex function $f : \mathbb{R}^N \to \mathbb{R}$ can be approximated by mean of a sequence of affine functions $a_a + \langle b_a, p\rangle$, where

$$a_a := \int_{\mathbb{R}^n} f(p)\big((n+1)a(p) + \langle \nabla a(p), z\rangle\big)dp \tag{2.8}$$

$$b_a := -\int_{\mathbb{R}^n} f(p)\nabla a(p)dp, \tag{2.9}$$

with $a \in C^1_0(\mathbb{R}^N)$ a nonnegative function and $\int_{\mathbb{R}^N} a(p)dp = 1$. The main feature of De Giorgi's theorem is that the coefficients a_a and b_a explicitly depend on f. When f depends also on x, s the explicit formulas permit to deduce regularity properties of De Giorgi's coefficients from hypotheses satisfied by f. We recall therefore De Giorgi's theorem.

THEOREM 2.1 (see [14]). – *Let* $f : \mathbb{R}^N \to \mathbb{R}$ *be a convex function and* a_a, b_a *be defined as in* (2.8) *and* (2.9) *then:*

(*i*) $f(p) \geq a_a + \langle b_a, p\rangle$ *for any* $p \in \mathbb{R}^N$;

(*ii*) $f(p) = \sup_{\beta\in\mathcal{B}} [a_\beta + \langle b_\beta, p\rangle]$ *for* $p \in \mathbb{R}^N$,
where $\mathcal{B} := \{\beta : \beta(z) := \kappa^N a(\kappa(q - z)), \kappa \in \mathbb{N}, q \in \mathbb{Q}^N z \in \mathbb{R}^N\}$;

(*iii*) $f(p) = \lim_{j\to\infty} f_j(p)$ *for any* $p \in \mathbb{R}^N$, *where* $f_j(p) := \sup_{i\leq j}\{a_{\beta_i} + \langle b_{\beta_i}, p\rangle\}$ *for any* $p \in \mathbb{R}^N$, *where* β_i *is an ordering of the class* $\mathcal{B}$.

2.5 – *Relaxation and Γ-convergence*

Let F be the functional defined in (2.3). For every $u \in BV(\Omega)$, we can define the relaxed functional $\overline{F}$ of F, with respect the L^1-topology, given by

$$\overline{F}(u) = \inf\Big\{\liminf_{n\to\infty} F(u_h) : u_n \in W^{1,1}(\Omega), u_n \to u \text{ in } L^1(\Omega)\Big\}. \tag{2.10}$$

We recall that $\overline{F}$ is the greatest lower semicontinuous functional not greater than F.

We recall also the defintion of $\Gamma-$convergence for a sequence $\{F_h\}$ of functionals defined on $BV(\Omega)$ of the type (2.3), with respect to the $L^1(\Omega)$-topology. We recall that the functional $F^\Gamma := \Gamma - \lim F_h$, if it exists, is characterized by the following two inequalities:

for every $u \in BV(\Omega)$ and every $\{u_h\} \in W^{1,1}(\Omega)$, such that $u_h \to u$ strongly in $L^1(\Omega)$,

$$F^\Gamma(u) \leq \liminf_{h\to\infty} F_h(u_h), \tag{2.11}$$

for every $u \in BV(\Omega)$ there exists $\{\overline{u}_h\} \in W^{1,1}(\Omega)$, such that $\overline{u}_h \to u$ strongly in $L^1(\Omega)$,

$$F^\Gamma(u) = \lim_{h\to\infty} F_h(\overline{u}_h). \tag{2.12}$$

We recall that F^Γ is L^1-lower semicontinuous functional on $BV(\Omega)$. If $F_h \equiv F$ for every $h \in \mathbb{N}$, then $F^\Gamma = \overline{F}$ and from every sequence of functionals, it is possible to find a Γ-convergent subsequence. For further properties of the relaxation and Γ-convergence we refere to [4, 7, 16, 17].

3. – Chain rule.

In this section we improve both the chain rule of [13] and of [19]. Indeed with respect to [13] we deal with the space $BV(\Omega)$, and with respect to [19] we do not require continuity with respect to t.

THEOREM 3.1. – *Let $b : \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded Borel function with compact support in $\Omega \times \mathbb{R}$, satisfying the following properties*

(*i*) $b(\cdot, t) \in W^{1,1}(\Omega) \cap C(\Omega)$ *for almost every* $t \in \mathbb{R}$,
(*ii*) $\nabla_x b \in L^1(\Omega \times \mathbb{R})$.

Then for every $\varphi \in C_0^1(\Omega)$ and for every $u \in BV(\Omega)$ we have:

$$
\begin{aligned}
&-\int_{\Omega}\Big(\int_{0}^{u(x)} b(x,t)dt\Big)\nabla\varphi dx = \int_{\Omega} b(x,u)\varphi\nabla u dx \\
&+\int_{\Omega}\Big(\fint_{u^-}^{u^+} b(x,t)dt\Big)\varphi dD^s u + \int_{\Omega}\Big(\int_{0}^{u(x)} \nabla_x b(x,t)dt\Big)\varphi dx.
\end{aligned}
\tag{3.1}
$$

PROOF. – Let $\{\psi_\delta\}_{\delta>0}$ be a mollifying sequence in $\mathbb{R}$. Let us define $b_\delta(x,t) = \int_{\mathbb{R}} \psi_\delta(t-s)b(x,s)ds$. We claim that, for evey $\delta > 0$, $b_\delta(x,t)$ is a continuous function in $\Omega \times \mathbb{R}$. In order to prove this, we notice the following properties: for every $\delta \in \mathbb{R}$, the function $\psi_\delta(\,\cdot\, - s)b(\cdot, s)$ is continuous in Ω for almost every $s \in \mathbb{R}$ thanks to the hypothesis (i) and to the regularity properties of mollifiers. Furthermore, since b has compact support, there exist two compact sets $K \subset \Omega$ and $\Lambda \subset \mathbb{R}$ such that the support of b is contained in $K \times \Lambda$ and the support of the function $b(x,\cdot)$ is contained in Λ for every $x \in K$. Hence we have that, for almost every $s \in \mathbb{R}$, $|\psi_\delta(t-s)b(x,s)| \le \|b\|_\infty \|\psi_\delta\|_\infty \chi_\Lambda(s) \in L^1(\mathbb{R})$. It follows, by the dominated convergence theorem, that $b_\delta(x,t)$ is continuous in $\Omega \times \mathbb{R}$. Let us show that, for every $\delta > 0$, $b_\delta(\cdot, t) \in W^{1,1}(\Omega)$ for every $t \in \mathbb{R}$. Indeed using Tonelli's theorem we get:

$$
\begin{aligned}
\int_{\Omega} |b_\delta(x,t)|dx &\le \int_{\Omega} dx \int_{\mathbb{R}} |\psi_\delta(t-s)b(x,s)|ds = \int_{\Omega\times\mathbb{R}} |\psi_\delta(t-s)|\,|b(x,s)|dxds \\
&= \int_{K\times\Lambda} |\psi_\delta(t-s)|\,|b(x,s)|dxds \le \mathcal{L}^N(K)\|b\|_\infty \int_{\mathbb{R}} |\psi_\delta(t-s)|ds \le C,
\end{aligned}
$$

so that $b_\delta(\cdot,t) \in L^1(\Omega)$ for every $t \in \mathbb{R}$. Furthermore the following equality holds in the weak sense for almost every $x \in \Omega$ and for every $t \in \mathbb{R}$,

$$
\nabla_x\Big(\int_{\mathbb{R}} \psi_\delta(t-s)b(x,s)ds\Big) = \int_{\mathbb{R}} \psi_\delta(t-s)\nabla_x b(x,s)ds. \tag{3.2}
$$

In fact, let S be the set of $s \in \mathbb{R}$ such that $b(\cdot,s) \notin W^{1,1}(\Omega)$. By hypothesis (i), $\mathcal{L}^1(S) = 0$. Multiplying by $\varphi \in C_0^1(\Omega;\mathbb{R}^N)$ the righthand side of (3.2), integrating over Ω, and applying Fubini's theorem (taking into account hyphotesis (ii)), we get

$$
\begin{aligned}
\int_{\Omega} \varphi dx \int_{\mathbb{R}} \psi_\delta(t-s)\nabla_x b(x,s)ds &= -\int_{\mathbb{R}\setminus S} \psi_\delta(t-s)ds \int_{\Omega} b(x,s)\operatorname{div}_x\varphi \\
&= -\int_{\Omega} \operatorname{div}_x\varphi dx \int_{\mathbb{R}} \psi_\delta(t-s)b(x,s)ds
\end{aligned}
$$

and (3.2) is proved.

It remains to show that $\nabla_x b_\delta(\cdot,t) \in L^1(\Omega)$ uniformly with respect to $t \in \mathbb{R}$. From (3.2) and hypothesis (ii) we have:

$$\int_\Omega |\nabla_x b_\delta(x,t)|dx \leq \int_\Omega dx \int_{\mathbb{R}\setminus S} |\psi_\delta(t-s)|\,|\nabla_x b(x,s)|ds$$
$$= \int_{\mathbb{R}\setminus S} \psi_\delta(t-s)ds \int_\Omega |\nabla_x b(x,s)|dx \leq \|\psi_\delta\|_\infty \int_{\Omega\times\mathbb{R}} |\nabla_x b(x,s)|dxds \leq C.$$

This implies that $b_\delta(x,t)$ satisfies all the hypotheses of Lemma 2.4 of [19] and so (3.1) holds for $b_\delta(x,t)$, i.e.

$$\begin{aligned}-\int_\Omega \Big(\int_0^{u(x)} b_\delta(x,t)dt\Big)\nabla\varphi dx =& \int_\Omega b_\delta(x,u)\varphi\nabla u dx \\ +\int_\Omega \Big(\fint_{u^-}^{u^+} b_\delta(x,t)dt\Big)\varphi dD^s u +& \int_\Omega \Big(\int_0^{u(x)} \nabla_x b_\delta(x,t)dt\Big)\varphi dx,\end{aligned} \tag{3.3}$$

for evey $\varphi \in C_0^1$. Now we pass to the limit as $\delta \to 0$.

Let us consider the first term in (3.3). We remark that $b_\delta(x,t)$ is continuous in $\Omega \times \mathbb{R}$ and there exists $M \subset \mathbb{R}$ with $\mathcal{L}^1(M) = 0$, such that by Lemma 2.1, $b(x,\cdot)$ is approximately continuous in $\mathbb{R}\setminus M$ for every $x \in \Omega$. Then, by Proposition 2.1, $b_\delta(x,t) \to b(x,t)$ for every $x \in \Omega$ and every $t \in \mathbb{R}\setminus M$. It is not difficult to prove that

$$\Big|\int_\Omega \Big(\int_0^{u(x)} b_\delta(x,t)dt\Big)\nabla\varphi dx - \int_\Omega \Big(\int_0^{u(x)} b(x,t)dt\Big)\nabla\varphi dx\Big|$$
$$\leq \int_\Omega \Big(\int_{\mathbb{R}\setminus M} \chi_{[0,u(x)]}|b_\delta(x,t) - b(x,t)|dt\Big)|\nabla\varphi|dx \to 0,$$

since

$$\chi_{[0,u]}|b_\delta(x,t) - b(x,t)|\,|\nabla\varphi| \leq (\|b_\delta\|_\infty + \|b\|_\infty)|\nabla\varphi|\chi_H \leq 2\|b\|_\infty|\nabla\varphi|\chi_H \in L^1(\Omega\times\mathbb{R}), \tag{3.4}$$

for a proper compact set $H \subset \Omega \times \mathbb{R}$ and independent of δ.

Let us consider the second term of (3.3). As we have already remarked $b_\delta(x,t) \to b(x,t)$ for every $x \in \Omega$ and every $t \in \mathbb{R}\setminus M$. Moreover, reasoning as in (3.4), it follows

$$|b_\delta(x,t) - b(x,t)|\,|\varphi|\,|\nabla u| \leq 2\|b\|_\infty|\varphi|\,|\nabla u| \in L^1(\Omega), \tag{3.5}$$

for every $\delta > 0$.

Hence by Proposition 2.2, we get

$$\left|\int\limits_{\Omega} b_\delta(x,u)\varphi\nabla u dx - \int\limits_{\Omega} b(x,u)\varphi\nabla u dx\right| = \left|\int\limits_{\Omega\setminus(\tilde{u})^{-1}(M)} b_\delta(x,u)\varphi\nabla u dx - \int\limits_{\Omega\setminus(\tilde{u})^{-1}(M)} b(x,u)\varphi\nabla u dx\right|,$$

and, letting $\delta \to 0$,

$$\left|\int\limits_{\Omega\setminus(\tilde{u})^{-1}(M)} b_\delta(x,u)\varphi\nabla u dx - \int\limits_{\Omega\setminus(\tilde{u})^{-1}(M)} b(x,u)\varphi\nabla u dx\right| \to 0,$$

as a consequence of (3.5) and the dominated convergence theorem.

Let us consider the third term of (3.3). Thanks to (2.1) and (2.6), we can rewrite this term as

$$\int\limits_{\Omega\cap J_u} \Big(\fint_{u^-}^{u^+} b_\delta(x,t)dt\Big)\varphi dD^j(u) + \int\limits_{\Omega\cap C_u} b_\delta(x,\tilde{u}(x))\varphi dD^c u. \tag{3.6}$$

Clearly for every $x \in \Omega\cap J_u$ we have

$$\fint_{u^-}^{u^+} |b_\delta(x,t) - b(x,t)|dt \to 0, \text{ as } \delta \to 0.$$

Furthermore, the function $g_\delta(x) = |\varphi(x)| \fint_{u^-}^{u^+} |b_\delta(x,t) - b(x,t)|dt$ satisfies the following estimate

$$0 \le g_\delta(x) \le 2\|b\|_\infty\|\varphi\|_\infty \in L^1(\Omega\cap J_u, |D^j u|)$$

so that, letting $\delta \to 0$, we get

$$\left|\int\limits_{\Omega\cap J_u} \Big(\fint_{u^-}^{u^+} b_\delta(x,t)dt\Big)\varphi dD^j(u) - \int\limits_{\Omega\cap J_u} \Big(\fint_{u^-}^{u^+} b(x,t)dt\Big)\varphi dD^j(u)\right| \to 0.$$

As far as the second term of (3.6), for every $t \in \mathbb{R}$ and for every $x \in \Omega\cap C_u \setminus (\tilde{u})^{-1}(M)$, we have

$$|b_\delta(x,t) - b(x,t)|\,|\varphi| \le 2\|b\|_\infty|\varphi| \in L^1(\Omega\cap C_u, |D^c u|), \tag{3.7}$$

so that, by the dominated convergence theorem and Lemma 2.2, we obtain

$$\int\limits_{\Omega\cap C_u} b_\delta(x,\tilde{u}(x))\varphi dD^c u \to \int\limits_{\Omega\cap C_u} b(x,\tilde{u}(x))\varphi dD^c u,$$

so that

$$\int\limits_{\Omega} \Big(\fint_{u^-}^{u^+} b_\delta(x,t)dt\Big)\varphi dD^s u \to \int\limits_{\Omega} \Big(\fint_{u^-}^{u^+} b(x,t)dt\Big)\varphi dD^s u.$$

Let us consider the last term of (3.3). Thanks to the hypothesis (ii), we have that for $\mathcal{L}^N$-almost every $x \in \Omega$ the function $\nabla_x b(x,\cdot) \in L^1(\mathbb{R})$. Therefore, from (3.2), it follows that for $\mathcal{L}^N$-almost every $x \in \Omega$,

$$\nabla_x b_\delta(x,\cdot) = \psi_\delta * \nabla_x b(x,\cdot) \to \nabla_x b(x,\cdot) \text{ in } L^1(\mathbb{R}),$$

as $\delta \to 0$. This implies that, for $\mathcal{L}^N$-almost every $x \in \Omega$, we obtain

$$\lim_{\delta\to 0} \int_0^{u(x)} |\nabla_x b_\delta(x,t) - \nabla_x b(x,t)| dt = 0.$$

In order to conclude, we note that, thanks to the hypothesis (ii),

$$\begin{aligned} |\varphi(x)| \int_0^{u(x)} |\nabla_x b_\delta(x,t)| &\le \|\varphi\|_\infty \int_{\mathbb{R}} dt \int_{\mathbb{R}} \psi_\delta(t-s) |\nabla_x b(x,s)| ds \\ &= \int_{\mathbb{R}} |\nabla_x b(x,s)| ds \int_{\mathbb{R}} \psi_\delta(t-s) dt = \int_{\mathbb{R}} |\nabla_x b(x,s)| ds \in L^1(\Omega), \end{aligned}$$

for a.e. $x \in \Omega$ and hence

$$\int_\Omega \Big(\int_0^{u(x)} \nabla_x b_\delta(x,t) dt \Big) \varphi dx \to \int_\Omega \Big(\int_0^{u(x)} \nabla_x b(x,t) dt \Big) \varphi dx,$$

as $\delta \to 0$. The proof is now complete. □

We establish a refinement of previous result, which will be useful in the next section.

THEOREM 3.2. – *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a Borel function with a compact support in $\Omega \times \mathbb{R}$ satisfying the following properties*

- (i) *there exists $g \in L^1(\mathbb{R})$ such that $|b(x,t)| \le g(t)$ for every $x \in \Omega$ and for every $t \in \mathbb{R}$;*
- (ii) *$b(\cdot,t) \in W^{1,1}(\Omega;\mathbb{R}^N) \cap C(\Omega;\mathbb{R}^N)$ for almost every $t \in \mathbb{R}$,*
- (iii) *$\nabla_x b \in L^1(\Omega \times \mathbb{R})$.*

Then for every $u \in BV(\Omega)$ such that

$$\int_\Omega \langle b(x,u), \nabla u \rangle^+ dx < +\infty;$$

$$\int_\Omega \Big(\fint_{u^-}^{u^+} \Big\langle b(x,t), \frac{D^s u}{|D^s u|} \Big\rangle^+ dt \Big) d|D^s u| < +\infty.$$

and for every $\varphi \in C_0^1(\Omega)$ we have

$$\begin{aligned}(3.8)\qquad &\int_\Omega \langle b(x,u),\nabla u\rangle \varphi dx + \int_\Omega \Big(\fint_{u^-}^{u^+} \Big\langle b(x,t), \frac{D^s u}{|D^s u|}\Big\rangle dt\Big)\varphi d|D^s u| \\ &= -\int_\Omega \Big\langle \int_0^{u(x)} b(x,t)dt, \nabla\varphi\Big\rangle dx - \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b(x,t)dt\Big)\varphi dx.\end{aligned}$$

PROOF. – Let us define

$$b_h(x,t) = b(x,t)\chi_{A_h}(t) \quad \text{where } A_h = \{t \in \mathbb{R} : g(t) \le h\}.$$

Clearly $b_h \in L^\infty(\Omega \times \mathbb{R})$ for every $h \in \mathbb{N}$ and $b_h(x,t) \to b(x,t)$ for a.e. $x \in \Omega$ and for a.e. $t \in \mathbb{R}$. Therefore (3.8) holds for b_h, i.e.

$$\begin{aligned}(3.9)\qquad &\int_\Omega \langle b_h(x,u),\nabla u\rangle \varphi dx + \int_\Omega \Big(\fint_{u^-}^{u^+} \Big\langle b_h(x,t), \frac{D^s u}{|D^s u|}\Big\rangle dt\Big)\varphi d|D^s u| \\ &= -\int_\Omega \Big\langle \int_0^{u(x)} b_h(x,t)dt, \nabla\varphi\Big\rangle dx - \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b_h(x,t)dt\Big)\varphi dx,\end{aligned}$$

for every $\varphi \in C_0^1(\Omega)$ Moreover, $\mathrm{div}_x b_h(x,t) = \chi_{A_h}(t)\mathrm{div}_x b(x,t) \to \mathrm{div}_x b(x,t)$ for a.e. $(x,t) \in \Omega \times \mathbb{R}$. Since $|\mathrm{div}_x b_h(x,t)| \le |\nabla_x b(x,t)|$ for a.e. $(x,t) \in \Omega \times \mathbb{R}$, and, by (iii), $|\nabla_x b(x,\cdot)| \in L^1(\mathbb{R})$ for a.e. $x \in \Omega$, we get a.e.

$$\varphi(x) \int_0^{u(x)} \mathrm{div}_x b_h(x,t)dt \to \varphi(x) \int_0^{u(x)} \mathrm{div}_x b(x,t)dt.$$

Using again (iii), it follows

$$\Big|\varphi \int_0^{u(x)} \mathrm{div}_x b_h(x,t)dt\Big| \le |\varphi| \int_{\mathbb{R}} |\nabla_x b(x,t)|dt \in L^1(\Omega),$$

and hence

$$\int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b_h(x,t)dt\Big)\varphi dx \to \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x b(x,t)dt\Big)\varphi dx.$$

Let us consider the lefthand side of (3.9). Since $\langle b_h(x,s),p\rangle^+$ and $\langle b_h(x,s),p\rangle^-$ are increasing sequences which converge to $\langle b(x,s),p\rangle^+$ and $\langle b(x,s),p\rangle^-$ respectively, from Beppo Levi's theorem and hypothesis (ii), we obtain that

$$\lim_{h\to+\infty} \int_\Omega \langle b_h(x,u),\nabla u\rangle\varphi dx = \int_\Omega \langle b(x,u),\nabla u\rangle \varphi dx.$$

Analogously, using again hypothesis (ii), we get

$$\lim_{h\to+\infty}\int_\Omega \Big(\fint_{u^-}^{u^+} \Big\langle b_h(x,t), \frac{D^s u}{|D^s u|}\Big\rangle dt\Big)\varphi d|D^s u| = \int_\Omega \Big(\fint_{u^-}^{u^+} \Big\langle b(x,t), \frac{D^s u}{|D^s u|}\Big\rangle dt\Big)\varphi d|D^s u|.$$

Therefore passing to the limit, as $h \to +\infty$, in (3.9) we get (3.8). The thesis is achieved. □

4. – Lower semicontinuity.

In the same spirit of [13] and [19], but on the space $BV(\Omega)$ and without continuity with respect to the variable s, we obtain, by using Theorem 3.1, a lower semicontinuity result with respect L^1-topology for the functional (2.4).

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function such that:

(i) $f(x,s,\cdot)$ is convex on $\mathbb{R}^N$ for every $(x,s) \in \Omega \times \mathbb{R}$;

(ii) $f(\cdot,s,p) \in C(\Omega) \cap W^{1,1}_{loc}(\Omega)$ for almost every $s \in \mathbb{R}$ and for every $p \in \mathbb{R}^N$;

(iii) for every bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$, there exists a costant $L(B)$ such that

$$\int_\Omega |\nabla_x f(x,s,p)|dx \le L(B) \text{ for every } (s,p) \in B. \tag{4.1}$$

THEOREM 4.1. – *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a locally bounded Borel function, satisfying* (4.1), *such that*

$$f(x,s,0) = 0 \quad \forall (x,s) \in \Omega \times \mathbb{R}. \tag{4.2}$$

Then the functional (2.4) *is lower semicontinuous on $BV(\Omega)$ with respect to the L^1-topology.*

PROOF. – By Theorem 2.1 there exists a sequence $\{a_\kappa\} \subset C_0^\infty(\Omega)$ with $a_\kappa \ge 0$ and $\int_{\mathbb{R}^N} a_\kappa dx = 1$ such that for any $(x,s,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ we have

$$f(x,s,p) = \sup_{\kappa\in\mathbb{N}} (a_\kappa(x,s) + \langle b_\kappa(x,s), p\rangle)^+$$

and

$$f^\infty(x,s,p) = \sup_{\kappa\in\mathbb{N}} \langle b_\kappa(x,s), p\rangle^+,$$

where, recalling (2.8) and (2.9),

$$\begin{aligned} a_\kappa(x,s) &= \int_{\mathbb{R}^n} f(x,s,p)\Big((n+1)a_\kappa(p) + \langle \nabla a_\kappa(p), p\rangle\Big)dp \\ b_\kappa(x,s) &= -\int_{\mathbb{R}^n} f(x,s,p)\nabla a_\kappa(p)dp. \end{aligned} \tag{4.3}$$

Hence, if we set $f_\kappa(x,s,p)=(a_\kappa(x,s)+\langle b_\kappa(x,s),p\rangle)^+$, we obtain $\hat f(x,s,p,t)=\sup_\kappa \hat f_\kappa(x,s,p,t)$. Therefore, applying Lemma 2.2 with f, f_k and μ replaced by $\hat f, \hat f_k$ and $|a(u)|$ respectively, we obtain

$$
\begin{aligned}
\mathcal{F}(u) &= \int_{\Omega\times\mathbb{R}} \hat f\Big(x,s,\frac{a(u)}{|a(u)|}\Big)d|a(u)|(x,s)\\
&= \sup_{\mathcal{D}} \sum_{i\in I} \int_{\Omega\times\mathbb{R}} \hat f_{\kappa_i}\Big(x,s,\frac{a(u)}{|a(u)|}\Big)\varphi_i(x)\psi_i(u)d|a(u)|(x,s)\\
&= \sup_{\mathcal{D}} \sum_{i\in I} \Big\{\int_\Omega \psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx\\
&+\int_\Omega \Big(\fint_{u^-}^{u^+} \psi_i(s)\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big)\varphi_i(x)d|D^s u|\Big\},
\end{aligned}
\tag{4.4}
$$

where the first and the last equality are due to Lemma 2.3 and we used the notation in (2.7). Let us define

$$
\begin{aligned}
G_i(u) :=& \int_\Omega \psi_i(u)\Big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\Big)^+\varphi_i(x)dx\\
&+\int_\Omega \Big(\fint_{u^-}^{u^+} \psi_i(s)\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\rangle^+ ds\Big)\varphi_i(x)d|D^s u|.
\end{aligned}
\tag{4.5}
$$

We remark that, by (ii) of (4.1) and (4.3), $a_{\kappa_i}(\cdot,s)$ is continuous for almost every $s\in\mathbb{R}$. By Scorza-Dragoni theorem it is possible to find an increasing sequence K_h of compact subsets of $\mathbb{R}$ such that, if we set $E:=\bigcup_{h\in\mathbb{N}} K_h$, $\mathcal{L}^1(\mathbb{R}\setminus E)=0$, and for every $\kappa_i\in\mathbb{N}$ $a_{\kappa_i}\in C^0(\Omega\times K_h)$. We remark that, by hypothesis (4.2), we have $a_{\kappa_i}\le 0$, hence, by Lemma 2.2 it follows that

$$
\begin{aligned}
G_i(u) =& \int_\Omega \chi_E(u)\psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx\\
&+\int_\Omega \Big(\fint_{u^-}^{u^+} \chi_E(s)\psi_i(s)\Big\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\Big\rangle^+ ds\Big)\varphi_i(x)d|D^s u|\\
=& \sup_{h\in\mathbb{N}}\Big\{\int_\Omega \chi_{K_h}(u)\psi_i(u)\big(a_{\kappa_i}(x,u)+\langle b_{\kappa_i}(x,u),\nabla u\rangle\big)^+\varphi_i(x)dx\\
&+\int_\Omega \Big(\fint_{u^-}^{u^+} \chi_{K_h}(s)\psi_i(s)\Big\langle b_{\kappa_i}(x,s),\frac{D^s u}{|D^s u|}\Big\rangle^+ ds\Big)\varphi_i(x)d|D^s u|\Big\}.
\end{aligned}
$$

As $\mathcal{L}^n$ and $|D^s u|$ are mutually singular measures,

$$
\begin{aligned}
G_i(u) = \sup_{h\in\mathbb{N}} \sup_{0\le\eta\le 1} \Big\{ & \int_\Omega \chi_{K_h}(u)\psi_i(u)(a_{\kappa_i}(x,u)\eta(x)\varphi_i(x)dx \\
& + \int_\Omega \chi_{K_h}(u)\langle \psi_i(u) b_{\kappa_i}(x,u), \nabla u\rangle \eta(x)\varphi_i(x)dx \qquad (4.6)\\
& + \int_\Omega \Big(\fint_{u^-}^{u^+} \chi_{K_h}(t)\eta(x)\Big\langle \psi_i(t) b_{\kappa_i}(x,s), \frac{D^s u}{|D^s u|}\Big\rangle ds\Big)\varphi_i(x) d|D^s u| \Big\};
\end{aligned}
$$

where $\eta \in C_0^\infty(\Omega)$. Since $a_{\kappa_i} \in C^0(\Omega\times K_h)$ and $a_{\kappa_i}\le 0$, the function $\chi_{K_h}(s)\psi_i(s)a_{\kappa_i}(x,s)$ is lower semicontinuous with respect to $s\in\mathbb{R}$. Therefore, as a consequence of Fatou's lemma, the first term in (4.6) is lower semicontinuous with respect to the L^1-topology. Now we prove the lower semicontinuity with respect to the L^1-topology of the last two terms of (4.6). Since $u_n \to u$ strongly in $L^1(\Omega)$, without loss of generality, we may assume that $u_n\to u$ almost everywhere in Ω. Let us define

$$
\begin{aligned}
H(u_n) := & \int_\Omega \chi_{K_h}(u_n)\langle \psi_i(u) b_{\kappa_i}(x,u_n), \nabla u_n\rangle \eta(x)\varphi_i(x)dx \\
& + \int_\Omega \Big(\fint_{u_n^-}^{u_n^+} \chi_{K_h}(s)\eta(x)\Big\langle \psi_i(s) b_{\kappa_i}(x,s), \frac{D^s u_n}{|D^s u_n|}\Big\rangle ds\Big)\varphi_i(x) d|D^s u_n|.
\end{aligned}
$$

We claim that the scalar function $\eta(x)\psi(s)b^j_{\kappa_i}(x,s)$ satisfies for $1\le j\le n$ all the hypotheses of theorem 3.1. Indeed $\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)$ has compact support in $\Omega\times\mathbb{R}$ and it is bounded in $\Omega\times\mathbb{R}$, since $f\in L^\infty_{loc}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$. Moreover, by (ii) of (4.1) and the dominated convergence theorem, it follows that $\eta(\,\cdot\,)\psi_i(s)b^j_{\kappa_i}(\cdot,s)$ is continuous for almost every $s\in\mathbb{R}$. Finally by (ii) and (iii) of (4.1), we have that $\eta(\,\cdot\,)\psi(s)b^j_{\kappa_i}(\cdot,s)$ belongs to $W^{1,1}(\Omega)$ with $\nabla_x\Big(\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)\Big)\in L^1(\Omega\times\mathbb{R})$.

Therefore, by applying Theorem 3.1, we get

$$
\begin{aligned}
\liminf_{n\to+\infty} H(u_n) = \lim_{n\to+\infty} \Big\{ & -\int_\Omega \Big(\int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds\Big)\varphi_i dx \\
& -\int_\Omega \Big\langle \int_0^{u_n(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\Big\rangle dx\Big\}.
\end{aligned}
$$

From (iii) of (4.1) and from the absolute continuity of the integral it follows that as $n\to+\infty$

$$
\lim_{n\to+\infty} \int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds = \int_0^{u(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds.
$$

Moreover,

$$\left|\varphi_i(x)\int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds\right| \tag{4.7}$$
$$\leq \|\varphi_i\|_\infty \int_{\mathbb{R}} |\operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))|ds \in L^1(\Omega),$$

so that

$$\lim_{n\to\infty}\int_\Omega \varphi_i(x)\int_0^{u_n(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds$$
$$= \varphi_i(x)\int_0^{u_(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds.$$

Analogously we get

$$\lim_{n\to+\infty}\int_\Omega \left\langle \int_0^{u_n(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\right\rangle dx$$
$$= \int_\Omega \left\langle \int_0^{u(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\right\rangle dx.$$

Therefore letting $n \to +\infty$ in (4.7) we obtain

$$\liminf_{n\to+\infty} H(u_n) = -\int_\Omega \left(\int_0^{u(x)} \operatorname{div}_x(b_{k_i}(x,s)\chi_{K_h}(s)\psi_i(s)\eta(x))ds\right)\varphi_i dx \tag{4.8}$$
$$-\int_\Omega \left\langle \int_0^{u(x)} b_{\kappa_i}(x,s)\chi_{K_h}(s)\eta(x)\psi_i(s)ds, \nabla\varphi_i\right\rangle dx.$$

Hence, applying Theorem 3.1 to (4.8), we obtain the lower semicontinuity of the second and the third term of (4.6). This implies that G_i, being the supremum of lower semicontinuous functions is lower semicontinuous itself, so that, by (4.4) and (4.5), $\mathcal{F}$ is lower semicontinuous too.

The thesis is then achieved. □

REMARK 4.1. – It is not very difficult to verify that Theorem 4.1 continues to hold under a weaker assumption than (*iii*) of (4.1), which is the following

$$\nabla_x f \in L^1_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n). \tag{4.9}$$

Indeed in the proof of the previous theorem we only need to know that $\eta(\,\cdot\,)\psi_i(s)b^j_{\kappa_i}(\cdot, s)$ belongs to $W^{1,1}(\Omega)$ with $\nabla_x\big(\eta(x)\psi_i(s)b^j_{\kappa_i}(x,s)\big) \in L^1(\Omega\times\mathbb{R})$, and it is guaranteed by hypothesis (4.9).

In the same spirit of the papers of De Giorgi-Buttazzo-Dal Maso (see [15]) and Ambrosio (see [2]) we give a further lower semicontinuity result, where assumption (4.2) is replaced by a weaker one.

THEOREM 4.2. – *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a locally bounded Borel function satisfying* (4.1) *such that:*

(*a*) *$f(x, \cdot, 0)$ is lower semicontinuous on $\mathbb{R}$ for $\mathcal{L}^N$ a.e $x \in \Omega$*

(*b*) *there exists a Borel function*

$$\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^N,$$

with $\lambda(x, s) \in \partial_p f(x, s, 0)$ for every $(x, s) \in \Omega \times \mathbb{R}$, such that

(*i*) $g(s) = \sup_{x\in\Omega} |\lambda(x, s)| \in L^1_{loc}(\mathbb{R})$

(*ii*) $\lambda(\cdot, s) \in C(\Omega; \mathbb{R}^N)$ for $\mathcal{L}^1$ a.e $s \in \mathbb{R}$

(*iii*) $\lambda(\cdot, s) \in W^{1,1}_{loc}(\Omega; \mathbb{R}^N)$ for $\mathcal{L}^1$ a.e $s \in \mathbb{R}$ with $\nabla_x\lambda \in L^1_{loc}(\Omega \times \mathbb{R})$.

Then the functional (2.4) *is lower semicontinuous in $BV(\Omega)$ with respect to the strong L^1-topology.*

PROOF. – Without loss of generality, we may suppose that there exists a constant $C > 0$ such that $f(x, s, p) = 0$ for every $(x, s, p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}$, with $|s| \geq C$. Indeed, in the general case, we can write

$$f(x, s, p) = \sup_{k\in\mathbb{N}} f(x, s, p)\chi_{(-k,k)}(s).$$

Moreover since $\lambda(x, s) \in \partial_p f(x, s, 0)$ and that $f \geq 0$, it follows that $f(x, s, p) \geq \langle\lambda(x, s), p\rangle^+$ for every $(x, s, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Hence we may assume that $\lambda(x, s) = 0$ for every $x \in \Omega$ and $s \in \mathbb{R}$, with $|s| \geq C$. Besides, since f is locally bounded, λ is locally bounded, too. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ be defined by

$$g(x, s, p) = f(x, s, p) - f(x, s, 0) - \langle\lambda(x, s), p\rangle.$$

Then for every $\varphi \in C_0^\infty(\Omega)$ and for every open set $A \subset\subset \Omega$ we have

$$\mathcal{F}_A(f, u, \varphi) = \mathcal{F}_A(g, u, \varphi) + G_A(f, u, \varphi) + H_A(\lambda, u, \varphi), \tag{4.10}$$

where

$$\mathcal{F}_A(f, u, \varphi) = \int_A f(x, u, \nabla u)\varphi dx + \int_A \Big[\fint_{u^-}^{u^+} f^\infty(x, s, \frac{D^s u}{|D^s(u)|})ds\Big]\varphi d|D^s u|,$$

$$G_A(f, u, \varphi) = \int_A f(x, u, 0)\varphi dx$$

$$H_A(\lambda, u, \varphi) = \int_A \langle\lambda(x, u), \nabla u\rangle\varphi dx + \int_A \Big[\fint_{u^-}^{u^+} \Big\langle \lambda(x, s), \frac{D^s u}{|D^s(u)|} \Big\rangle ds\Big]\varphi d|D^s u|.$$

Let $u_n \to u \in BV(\Omega)$ strongly in $L^1(\Omega)$. Without loss of generality, we may suppose that $u_n \to u$ almost everywhere in Ω and that $F(u_n) \leq M$, for every $n \in \mathbb{N}$. Since the function $g\varphi$ satifies all the hypotheses of Theorem 4.1 we obtain that

$$\mathcal{F}_A(g, u, \varphi) \leq \liminf_{n\to+\infty} \mathcal{F}_A(g, u_n, \varphi). \tag{4.11}$$

Moreover, by hypothesis (a) and Fatou's lemma it follows that

$$G_A(f, u, \varphi) \leq \liminf_{n\to+\infty} G_A(f, u_n, \varphi). \tag{4.12}$$

Since $f(x,s,p) \geq \langle \lambda(x,s), p\rangle^+$ we have, for every $n \in \mathbb{N}$,

$$\int_A \langle \lambda(x, u_n), \nabla u_n\rangle^+ dx \leq F(u_n) \leq M \tag{4.13}$$

and

$$\int_A \Big[\fint_{u_n^-}^{u_n^+} \langle \lambda(x,s), \frac{D^s u_n}{|D^s u_n|}\rangle^+ ds\Big] d|D^s u_n| \leq F(u_n) \leq M. \tag{4.14}$$

We remark that, since λ is locally bounded, we have

$$\int_A \langle \lambda(x, u), \nabla u\rangle^+ dx \leq M \tag{4.15}$$

and

$$\int_A \Big[\fint_{u^-}^{u^+} \Big\langle \lambda(x,s), \frac{D^s u}{|D^s u|}\Big\rangle^+ ds\Big] d|D^s u| \leq M. \tag{4.16}$$

Furthermore, if we define

$$\tilde{\lambda}(x,s) = \begin{cases} \lambda(x,s) & (x,s) \in supp\, \varphi \times [-C, C], \\ 0 & (x,s) \notin supp\, \varphi \times [-C, C]. \end{cases} \tag{4.17}$$

The function $\tilde{\lambda}$ satifsfies all the hypotheses of Lemma 3.2. Then using, by (4.13) and (4.14), Lemma 3.2, we get

$$\liminf_{n\to+\infty} H_A(\lambda, u_n, \varphi)$$
$$= \lim_{n\to+\infty} \Big\{ -\int_\Omega \Big\langle \int_0^{u_n(x)} \lambda(x,s)ds, \nabla\varphi \Big\rangle dx - \int_\Omega \Big(\int_0^{u_n(x)} \mathrm{div}_x \lambda(x,s) ds\Big) \varphi dx \Big\};$$

so that, by (4.15) and (4.16), using again Lemma 3.2

$$\begin{aligned}\lim_{n\to+\infty} H_A(\lambda, u_n, \varphi) &= -\int_\Omega \Big\langle \int_0^{u(x)} \lambda(x,s)ds, \nabla\varphi\Big\rangle dx - \int_\Omega \Big(\int_0^{u(x)} \mathrm{div}_x \lambda(x,s)ds\Big)\varphi dx \\ &= H_A(\lambda, u, \varphi).\end{aligned} \tag{4.18}$$

Therefore from (4.10) (4.11), (4.12) and (4.18) we have

$$\mathcal{F}_A(f, u, \varphi) \le \liminf_{n\to+\infty} \mathcal{F}_A(f, u_n, \varphi) \le \liminf_{n\to+\infty} \mathcal{F}_\Omega(f, u_n, \varphi).$$

Then, since A is arbitrary, the functional $u \to \mathcal{F}_\Omega(f, u, \varphi)$ is lower semicontinuous. The conclusion follows by

$$\mathcal{F}(u) = \sup\{\mathcal{F}_\Omega(f, u, \varphi) : \varphi \in C_0^\infty(\Omega),\ 0 \le \varphi \le 1\}. \qquad \square$$

5. – Applications.

In this section, as a consequence of the result in §4, we give first an integral representation theorem for the relaxed functional (2.10), then we prove Γ -limit result for a sequence of functionals $\{F_h\}$ of the type (2.3).

5.1 – *Relaxation*

Here we will show that for every $u \in BV(\Omega)$ the following representation holds

$$\mathcal{F}(u) = \overline{F}(u), \tag{5.1}$$

where $\overline{F}(u)$ is defined in (2.10) and $\mathcal{F}(u)$ in (2.4). In order to get (5.1) we need a result due to Fonseca and Leoni (see [18], Theorem 1.6). We will assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ is a Borel function such that

$$0 \le f(x, s, p) \le C(1 + |p|) \quad \text{for all } (x, s, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{5.2}$$

PROPOSITION 5.1. – *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a Borel function convex with respect to p for every $(x, s) \in \Omega \times \mathbb{R}$, and continuous with respect to x for every $(s, p) \in \mathbb{R} \times \mathbb{R}^N$. Assume that f satisfies* (5.2), *and $f^\infty(\cdot, s, p)$ is upper semicontinuous in Ω for every $(s, p) \in \mathbb{R} \times \mathbb{R}^N$. Then*

$$\overline{F}(u) \le \mathcal{F}(u).$$

REMARK 5.1. – Following the proof in [18], it is not difficult to see that the Proposition 5.1 holds even if the hypothesis that f is continuous with respect to x

for every $(s,p) \in \mathbb{R} \times \mathbb{R}^N$, is replaced by

$$|f(x,s_1,0) - f(x,s_2,0)| \leq C\rho(s_1 - s_2),$$

for every $x \in \Omega$ and s_1, $s_2 \in \mathbb{R}$, where ρ is a modulus of continuity, i.e. a non-negative, increasing and continuous function ρ such that $\rho(0) = 0$, or by the assumption that for every $s \in \mathbb{R}$ exists $N \subset \Omega$ such that $\mathcal{H}^{N-1}(N) = 0$ and $f(\cdot, s, 0)$ is approximately continuous in $\Omega \setminus N$ (these conditions are in particular implied by $f(x,s,0) = 0$).

THEOREM 5.1. – *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function, which satisfies hypotheses* (4.1), (5.2) *and* (*a*), (*b*) *of Theorem* 4.2. *Assume that $f^\infty(\cdot, s, p)$ is upper semicontinuous in Ω for every $(s,p) \in \mathbb{R} \times \mathbb{R}^N$. Then $\mathcal{F}(u) = \overline{F}(u)$.*

PROOF. – Since $\overline{F}$ is the greatest lower semicontinuous functional not greater than F, $\mathcal{F} \leq F$ and, by Theorem 4.2, $\mathcal{F}$ is L^1-lower semicontinuous, it follows that

$$\mathcal{F}(u) \leq \overline{F}(u).$$

The opposite inequality is stated in Proposition 5.1. □

5.2 – *Γ-convergence*

In this subsection, in the same spirit of [1, 6], we state a Γ-convergence result for a sequence of integral functionals of the type (2.3), whose integrands pointwise converge to an integrand, which is not necessarily continuous with respect to s nor coercive.

THEOREM 5.2. – *Let $f_h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a sequence of Borel functions such that*

$$0 \leq f_h(x,s,p) \leq \Lambda(1 + |p|) \quad \textit{for every } (x,s,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{5.3}$$

where $0 < \Lambda < +\infty$ is a fixed costant. For every $u \in BV(\Omega)$ we define

$$F_h(u) = \begin{cases} \displaystyle\int_\Omega f_h(x, u(x), \nabla u(x))dx & \textit{if } u \in W^{1,1}(\Omega) \\ +\infty & \textit{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega). \end{cases} \tag{5.4}$$

Assume that $\{f_h\}$ converges pointwise to a locally bounded Borel function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ satsfying all the hypotheses of Theorem 5.1.

Finally, let $\{\varepsilon_h\}$ be an infinitesimal sequence, such that

$$(1 + \varepsilon_h) f_h(x,s,p) \geq f(x,s,p) - \varepsilon_h \quad \textit{for every } (x,s,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \forall h \in \mathbb{N}. \tag{5.5}$$

Then for every $u \in BV(\Omega)$, we have

$$F^\Gamma(u) := \Gamma - \lim F_h(u) = \mathcal{F}(u).$$

PROOF. – By the compactness of Γ- convergence, we may assume that, up to a subsequence, there exists $\Gamma - \lim F_h$. Firstly, we will prove that $\Gamma - \lim F_h \geq \mathcal{F}$. Given $u \in BV(\Omega)$, by (5.5), for every $h \in \mathbb{N}$ we obtain that $F_h(u) \geq F(u) - \varepsilon_h[\mathcal{L}^N(\Omega) + F_h(u)]$, where F is defined in (2.3). By (2.12), we have that for every $u \in BV(\Omega)$, there exists $\overline{u}_h \to u$ strongly in $L^1(\Omega)$, such that

$$\left(\Gamma - \lim_{h\to\infty} F_h\right)(u) = \lim_{h\to\infty} F_h(\overline{u}_h).$$

We may assume that the previous limit is finite (otherwise the conclusion is trivial). Therefore, taking into account Theorem 5.1, it follows

$$\begin{aligned}\left(\Gamma - \lim_{h\to\infty} F_h\right)(u)& \\ &= \lim_{h\to\infty} F_h(\overline{u}_h) \geq \liminf_{h\to\infty} F(\overline{u}_h) - \lim_{h\to\infty} \varepsilon_h[\mathcal{L}^N(\Omega) + F_h(\overline{u}_h)] \geq \overline{F}(u) = \mathcal{F}(u).\end{aligned}$$

In order to prove the opposite inequality, we note that, by dominated convergence theorem, we have

$$\lim_{h\to\infty} F_h(u) = F(u) \quad \text{for every } u \in W^{1,1}(\Omega).$$

Hence, by (2.11)

$$\left(\Gamma - \lim_{h\to\infty} F_h\right)(u) \leq F(u) \quad \text{for every } u \in BV(\Omega).$$

So that, by the lower semicontinuity of the $\Gamma - \lim$ and Theorem 5.1, it follows

$$\left(\Gamma - \lim_{h\to\infty} F_h\right)(u) \leq \overline{F}(u) = \mathcal{F}(u) \quad \text{for every } u \in BV(\Omega).$$

Since this in independent from the subsequence, we obtain that the whole sequence F_h Γ- converges to F. Then the thesis is achieved.

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Pervenuta in Redazione
il 22 marzo 2005