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Sunto. – *Consideriamo innanzitutto il nucleo biarmonico di Poisson per il disco unitario e studiamo il comportamento al bordo dei potenziali associati a questa funzione nucleo. Useremo poi alcune proprietà del nucleo biarmonico di Poisson per il disco unitario per calcolare l'analogo nucleo biarmonico di Poisson per il semipiano superiore.*

Summary. – *We first consider the biharmonic Poisson kernel for the unit disk, and study the boundary behavior of potentials associated to this kernel function. We shall then use some properties of the biharmonic Poisson kernel for the unit disk to compute the analogous biharmonic Poisson kernel for the upper half plane.*

1. – Introduction.

Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ denote the open unit disk and $\mathbb{T} = \partial\mathbb{D}$ denote its boundary in the complex plane. The upper half plane will be denoted by

$$\mathbb{C}_+ = \{x + iy \in \mathbb{C} : \quad y > 0\}.$$

Let Δ stand for the Laplace operator

$$\Delta = \Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad z = x + iy,$$

in the complex plane. It is well-known that the Green function for the Laplacian is the function

$$G(z, \zeta) = \log\left|\frac{z - \zeta}{1 - \bar{\zeta}z}\right|^2, \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

which is a fundamental solution to the Dirichlet problem for the unit disk; $\Delta_z G(z, \zeta) = \delta_\zeta(z)$, here δ_ζ denotes the Dirac measure at ζ. For details on the Green function see [7].

In this note we are concerned with the partial differential operator Δ^2. The Green function for the operator Δ^2 in the unit disk is the function

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 + \left(1-|z|^2\right)\left(1-|\zeta|^2\right), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.$$

This function solves, for fixed $\zeta\in\mathbb{D}$, the boundary value problem

$$\begin{cases} \Delta_z^2\Gamma(z,\zeta) = \delta_\zeta(z), & z\in\mathbb{D}, \\ \Gamma(z,\zeta) = 0, & z\in\mathbb{T}, \\ \partial_{n(z)}\Gamma(z,\zeta) = 0, & z\in\mathbb{T}, \end{cases}$$

where $\partial_{n(z)}$ stands for the inward normal derivative with respect to the variable $z\in\mathbb{T}$, and δ_ζ denotes the Dirac distribution concentrated at the point $\zeta\in\mathbb{D}$. For a physical interpretation suppose that we are given a thin elastic plate spread over all of the domain $\mathbb{D}$ and clamped at the boundary $\mathbb{T}$. If we apply a force on this plate at the point ζ, the biharmonic Green function $z\mapsto\Gamma(z,\zeta)$ describes the shape of the clamped plate at another point z (see [3]).

A real-valued function u defined on an open subset of the complex plane is said to be *biharmonic* if $\Delta^2 u = 0$ (either in the usual or in the sense of distributions). The main objective here is the computation of biharmonic Poisson kernel for the upper half plane. For this purpose we scrutinize the biharmonic Poisson kernel for the unit disk in more detail (see § 2). This suggests a direct method of calculating the desired kernel function for the upper half plane (this is done in section 3).

2. – The biharmonic Poisson kernel for the unit disk.

Let u be a C^∞-smooth function in a neighborhood of the closed unit disk. Using Green's formula twice we obtain

$$\begin{aligned} (2.1)\quad u(z) = &\int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 u(\zeta)\,dA(\zeta) \\ &-\frac{1}{2}\int_{\mathbb{T}} \partial_{n(\zeta)}(\Delta_\zeta\Gamma(z,\zeta))u(\zeta)\,d\sigma(\zeta) + \frac{1}{2}\int_{\mathbb{T}} \Delta_\zeta\Gamma(z,\zeta)\partial_{n(\zeta)}u(\zeta)\,d\sigma(\zeta), \end{aligned}$$

where $dA(\zeta)$ denotes the normalized area measure on the unit disk, and $d\sigma(\zeta)$ stands for the normalized arc-lengh measure on the unit circle. A computation shows that

$$\Delta_\zeta\Gamma(z,\zeta) = G(z,\zeta) + H(z,\zeta), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

where the second term is given by

$$H(z,\zeta) = \left(1 - |z|^2\right) \frac{1 - |\zeta z|^2}{|1 - \bar{\zeta} z|^2}, \qquad (z,\zeta) \in \mathbb{D} \times \overline{\mathbb{D}}.$$

Moreover, another computation shows that for every $(z,\zeta) \in \mathbb{D} \times \mathbb{T}$we have

$$F(z,\zeta) = -\frac{1}{2}\partial_{n(\zeta)} \varDelta_\zeta \varGamma(z,\zeta) = \frac{1}{2}\left\{ \frac{(1-|z|^2)^2}{|z-\zeta|^2} + \frac{(1-|z|^2)^3}{|z-\zeta|^4} \right\}. \tag{2.2}$$

For a possibly non-smooth function u satisfying some growth conditions, the author and Hedenmalm [2] succeeded to find a Riesz-type representation formula in terms of the functions $H(z,\zeta)$ and $F(z,\zeta)$. More precisely, the formula (2.1) generalizes to

$$u(z) = \int_{\mathbb{T}} \varGamma(z,\zeta) d\mu(\zeta) + \int_{\mathbb{T}} H(z,\zeta) d\lambda(\zeta) + \int_{\mathbb{T}} > F(z,\zeta) d\upsilon(\zeta), \quad z \in \mathbb{D},$$

where μ is a positive Borel measure on the unit disk, and υ and λ are two real-valued Borel measures on the unit circle.

For fixed $\zeta \in \mathbb{T}$, the function $F(z,\zeta)$ defined by (2.2) is biharmonic; in the sense that it satisfies the equation $\varDelta_z^2 F(z,\zeta) = 0$, $z \in \mathbb{D}$. The function $F(z,\zeta)$ is known as the *biharmonic Poisson kernel* for the unit disk. This kernel function has lots of interesting properties. Among other things, we shall see that Fatou's theorem concerning the almost everywhere existence of nontangential limits is valid, so that the biharmonic Poisson kernel resembles the usual one. This generalizes the classical Fatou's theorem valid for the (harmonic) functions defined by the usual Poisson kernel (see for instance [4]) to biharmonic functions defined by the biharmonic Poisson kernel. To see another property of this kernel function, we refer the reader to [1].

We now state a proposition which collects some intrinsic properties of the biharmonic Poisson kernel for the unit disk.

PROPOSITION 2.1. – *Let $F(z,\zeta)$ denote the biharmonic Poisson kernel for the unit disk. Then*

(a) $F(z,\zeta) > 0$ *for* $(z,\zeta) \in \mathbb{D} \times \mathbb{T}$,

(b) $\int_{\mathbb{T}} F(z,\zeta)\, d\sigma(\zeta) = 1$ *for* $z \in \mathbb{D}$,

(c) $F(r\zeta, z) = F(rz,\zeta)$, *for* $(z,\zeta) \in \mathbb{T} \times \mathbb{T}$ *and* $0 \le r < 1$,

(d) $F(z,\zeta) \to 0$ *uniformly as* $|z| \to 1$ *and* $z^* \in \mathbb{T} \setminus I_\zeta$, *where* $z^* = z/|z|$ *for* $z \neq 0$, *and* I_ζ *is an arc centered at* ζ.

PROOF. – The proof follows easily from the definition of $F(z,\zeta)$. $\square$

For $f \in L^1(\mathbb{T})$ the function

$$u(z) = F[f](z) = \int_{\mathbb{T}} F(z,\zeta) f(\zeta)\, d\sigma(\zeta)$$

is called the *F-integral* of f. We shall see that the F-integral of a function $f \in C(\mathbb{T})$ behaves very well in the closure of $\mathbb{D}$. For $f \in L^1(\mathbb{T})$ and $z \in \mathbb{T}$ we define $\widetilde{f}$ on $\overline{\mathbb{D}}$ by

$$\widetilde{f}(rz) = \begin{cases} f(z) & \text{if} \quad r = 1, \\ u(rz), & \text{if} \quad 0 \le r < 1. \end{cases}$$

As a consequence of the biharmonicity of $F(z,\zeta)$ in the z variable, we see that $\widetilde{f}$ is a biharmonic function inside $\mathbb{D}$.

LEMMA 2.2. – *Let* $f(z) = \sum_{n=-k}^{k} c_n z^n$, $z \in \mathbb{T}$, *be a trigonometric polynomial on* $\mathbb{T}$. *Then*

$$\widetilde{f}(rz) = \sum_{n=-k}^{k} c_n r^{|n|} z^n \Big(1 + \frac{n}{2}(1-r^2)\Big), \qquad 0 \le r \le 1, \quad z \in \mathbb{T}.$$

PROOF. – The case $r = 1$ follows from the definition. Assume that $0 \le r < 1$ and $z \in \mathbb{T}$. By definition

$$\widetilde{f}(rz) = \int_{\mathbb{T}} F(rz,\zeta) f(\zeta)\, d\sigma(\zeta) = \sum_{n=-k}^{k} c_n \int_{\mathbb{T}} F(rz,\zeta)\zeta^n\, d\sigma(\zeta).$$

We have to compute the following integrals:

$$\int_{\mathbb{T}} F(rz,\zeta)\zeta^n\, d\sigma(\zeta) = \frac{1}{2}\left\{ \int_{\mathbb{T}} \frac{(1-|rz|^2)^2}{|\zeta - rz|^2}\zeta^n\, d\sigma(\zeta) + \int_{\mathbb{T}} \frac{(1-|rz|^2)^3}{|\zeta - rz|^4}\zeta^n\, d\sigma(\zeta) \right\}.$$

We first assume that n is nonnegative. As for the first integral above, we see from the definition of the usual (harmonic) Poisson kernel $P(w,\zeta)$, for $w \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, that

$$\int_{\mathbb{T}} \frac{(1-|rz|^2)^2}{|\zeta - rz|^2}\zeta^n d\sigma(\zeta) = (1-r^2)\int_{\mathbb{T}} \frac{1-|rz|^2}{|\zeta - rz|^2}\zeta^n d\sigma(\zeta)$$

$$= (1-r^2)\int_{\mathbb{T}} P(rz,\zeta)\zeta^n d\sigma(\zeta) = (1-r^2)(rz)^n.$$

On the other hand,

$$\int_{\mathbb{T}} \frac{\zeta^n}{|1-rz\bar{\zeta}|^4}\,d\sigma(\zeta) = \sum_{p,q=0}^{\infty}(p+1)(q+1)r^{p+q}\int_{\mathbb{T}}(z\bar{\zeta})^p(\bar{z}\zeta)^q\zeta^n d\sigma(\zeta)$$

$$= \sum_{p=0}^{\infty}(p+n+1)(p+1)r^{2p+n}z^{p+n}\bar{z}^p = (rz)^n\left\{\sum_{p=0}^{\infty}(p+1)^2r^{2p} + n\sum_{p=0}^{\infty}(p+1)r^{2p}\right\}$$

$$= (rz)^n\left\{\frac{1+r^2}{(1-r^2)^3}+\frac{n}{(1-r^2)^2}\right\},$$

and consequently,

$$\int_{\mathbb{T}} \frac{(1-r^2)^3}{|\zeta - rz|^4}\zeta^n\,d\sigma(\zeta) = (rz)^n\{1+r^2+n(1-r^2)\}.$$

Hence

$$\int_{\mathbb{T}} F(rz,\zeta)\zeta^n\,d\sigma(\zeta) = \frac{1}{2}(rz)^n\{(1-r^2)+(1+r^2)+n(1-r^2)\}$$
$$= (rz)^n\Big(1+\frac{n}{2}(1-r^2)\Big).$$

Now assume that $n=-p$ is a negative integer. Similar argument, using the fact that for $\zeta \in \mathbb{T}$ we have $\zeta^n = (\bar{\zeta})^p$, we see that

$$\int_{\mathbb{T}} F(rz,\zeta)\zeta^n\,d\sigma(\zeta) = r^{|n|}z^n\Big(1+\frac{n}{2}(1-r^2)\Big),$$

from which it follows that

$$\widetilde{f}(rz) = \int_{\mathbb{T}} F(rz,\zeta)f(\zeta)\,d\sigma(\zeta) = \sum_{n=-k}^{k} c_n\int_{\mathbb{T}} F(rz,\zeta)\zeta^n\,d\sigma(\zeta)$$
$$= \sum_{n=-k}^{k} c_n r^{|n|}z^n\Big(1+\frac{n}{2}(1-r^2)\Big).$$

□

PROPOSITION 2.3. – *Let* $f \in C(\mathbb{T})$ *and* $u = F[f]$. *Then* $\widetilde{f}$ *is uniformly continuous on* $\overline{\mathbb{D}}$. *In particular, the functions* $u_r(z) = u(rz)$, $(z \in \mathbb{T}, 0 \le r < 1)$ *converge uniformly to* f *as* $r \to 1$.

PROOF. – It follows from Proposition 2.1(b) that for $0 \le r < 1$ we have

$$|u(rz)| = \left|\int_{\mathbb{T}} F(rz,\zeta)f(\zeta)\,d\sigma(\zeta)\right| \le \sup_{\zeta\in\mathbb{T}}|f(\zeta)| = \|f\|_{\mathbb{T}}.$$

Hence

$$\|\widetilde{f}\|_{\overline{\mathbb{D}}} = \|f\|_{\mathbb{T}}, \qquad f \in C(\mathbb{T}).$$

Let $p(z) = \sum_{n=-k}^{k} c_n z^n$ be a trigonometric polynomial on $\mathbb{T}$. By the previous lemma

$$\widetilde{p}(rz) = \sum_{-k}^{k} c_n r^{|n|} z^n \Big(1 + \frac{n}{2}(1-r^2)\Big), \qquad 0 \le r \le 1, \quad z \in \mathbb{T}.$$

In particular, $\widetilde{p}$ is continuous on $\overline{\mathbb{D}}$. Since the trigonometric polynomials are dense in $C(\mathbb{T})$, we can find a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ on $\mathbb{T}$ such that $\|p_n - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$. It now follows that

$$\big\|\widetilde{p}_n - \widetilde{f}\,\big\|_{\overline{\mathbb{D}}} = \|p_n - f\|_{\mathbb{T}} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence the sequence $\widetilde{p}_n$ converges uniformly to $\widetilde{f}$ on $\overline{\mathbb{D}}$. As we already observed, each $\widetilde{p}_n$ is continuous on $\overline{\mathbb{D}}$, so that $\widetilde{f}$ is uniformly continuous on $\overline{\mathbb{D}}$. In particular, if $0 \le r < 1$, then continuity of $\widetilde{f}$ on $\overline{\mathbb{D}}$ yields

$$\lim_{r \to 1} \widetilde{f}(rz) = f(z), \qquad z \in \mathbb{T},$$

or equivalently,

$$\|u_r - f\|_{\mathbb{T}} \to 0, \qquad \text{as} \quad r \to 1.$$

□

Let $P(z, \zeta)$ denote the Poisson kernel for the unit disk, and consider the Poisson integral of $f \in L^1(\mathbb{T})$, that is,

$$P[f](z) = \int_{\mathbb{T}} P(z, \zeta) f(\zeta) d\sigma(\zeta), \qquad z \in \mathbb{D}.$$

According to a theorem of Fatou, $P[f]$ has nontangential limits, almost everywhere on the boundary. We now consider the F-integral of f given by $u(z) = F[f](z)$. The main result of this section is an analog of Fatou's theorem: the function u has nontangential limit almost everywhere on the unit circle.

Let us fix a real number $a > 1$. For $\zeta \in \mathbb{T}$, we define

$$\Omega_a(\zeta) = \{z \in \mathbb{D} : \quad |z - \zeta| < a(1 - |z|)\,.$$

THEOREM 2.4. – *(Fatou's Theorem) Let $f \in L^1(\mathbb{T})$ and let $u = F[f]$. Then u has nontangential limit for almost every $\zeta \in \mathbb{T}$; that is*

$$\lim_{\Omega_a(\zeta) \ni z \to \zeta} u(z) = f(\zeta), \qquad \textit{for almost every} \quad \zeta \in \mathbb{T}.$$

Before we prove the theorem, we need a lemma which is key to the proof of Fatou's theorem. Exploiting the notations of Theorem, we define the nontangential maximal function of u at ζ by

$$u_a^*(\zeta) = \sup_{z \in \Omega_a(\zeta)} |u(z)|.$$

For a subset E of the unit circle, the notation $|E|$ stands for the one-dimensional Lebesgue measure of E.

LEMMA 2.5. – *Let $f \in L^1(\mathbb{T})$ and $u = F[f]$. Let u_a^* be the nontangential maximal function of u at $\zeta \in \mathbb{T}$. Then for every positive number λ we have*

$$\left|\left\{\zeta \in \mathbb{T} :\ u_a^*(\zeta) > \lambda\right\}\right| \le \frac{9+12a}{\lambda} \|f\|_{L^1(\mathbb{T})}.$$

Let us postpone the proof of the lemma and manage to deduce Fatou's theorem form this lemma.

PROOF OF THE THEOREM. – We can assume that f is real-valued (the same argument can be applied to real and imaginary parts of f). Put

$$\omega_f(\zeta) = \limsup_{\Omega_a(\zeta) \ni z \to \zeta} |u(z) - f(\zeta)|.$$

It is clear that ω_f is nonnegative, moreover,

$$\omega_f(\zeta) \le \limsup_{\Omega_a(\zeta) \ni z \to \zeta} |u(z)| + |f(\zeta)| \le u_a^*(\zeta) + |f(\zeta)|.$$

This implies that for every $\varepsilon > 0$ we have

$$\left|\left\{\zeta \in \mathbb{T} : \quad \omega_f(\zeta) > \varepsilon\right\}\right| \le \left|\left\{\zeta \in \mathbb{T} : \quad u_a^*(\zeta) > \varepsilon/2\right\}\right| + \left|\left\{\zeta \in \mathbb{T} : \quad |f(\zeta)| > \varepsilon/2\right\}\right|.$$

According to the above lemma,

$$\left|\left\{\zeta \in \mathbb{T} : \quad u_a^*(\zeta) > \varepsilon/2\right\}\right| \le \frac{18+24a}{\varepsilon} \|f\|_{L^1(\mathbb{T})}.$$

On the other hand, by Chebyshev's inequality (see [4])

$$\left|\left\{\zeta \in \mathbb{T} :\ |f(\zeta)| > \varepsilon/2\right\}\right| \le \frac{2}{\varepsilon} \|f\|_{L^1(\mathbb{T})}.$$

Combining these relations, we obtain

$$\left|\left\{\zeta \in \mathbb{T} :\ \omega_f(\zeta) > \varepsilon\right\}\right| \le \frac{20+24a}{\varepsilon} \|f\|_{L^1(\mathbb{T})}.$$

We now assume that g is a continuous function which approximates the function f in the $L^1(\mathbb{T})$-norm; that is $\|f - g\|_{L^1(\mathbb{T})} < \varepsilon^2$. Since g is continuous, we conclude that $\omega_g(\zeta) = 0$, hence $\omega_f = \omega_{f-g}$. We now apply the above estimate to the function

$f - g$ to get

$$\left|\{\zeta \in \mathbb{T} : \quad \omega_f(\zeta) > \varepsilon\}\right| = \left|\{\zeta \in \mathbb{T} : \quad \omega_{f-g}(\zeta) > \varepsilon\}\right|$$
$$= \frac{(20+24a)\varepsilon^2}{\varepsilon} = (20+24a)\varepsilon,$$

from which it follows that $\omega_f = 0$ almost everywhere on the unit circle. In other words,

$$\lim_{\Omega_a(\zeta)\ni z\to\zeta} u(z) = f(\zeta), \qquad \text{for almost every} \quad \zeta \in \mathbb{T}.$$

The proof is complete.

PROOF OF LEMMA 2.5. – Recall the Hardy-Littlewood maximal function of $f \in L^1(\mathbb{T})$ defined by

$$Mf(\zeta) = \sup_{I_\zeta} \frac{1}{|I_\zeta|} \int_{I_\zeta} |f|,$$

where I_ζ is an arc centered at $\zeta \in \mathbb{T}$. Our first objective is to show that

$$u_a^*(\zeta) \leq (3+4a)Mf(\zeta), \qquad \zeta \in \mathbb{T}.$$

Assume this temporarily and use the well-known fact that the operator $f \mapsto Mf$ is weak-L^1 (see [8]), meaning that for every λ positive

$$|\{\zeta \in \mathbb{T} : \quad Mf(\zeta) > \lambda\}| \leq \frac{3}{\lambda}\|f\|_{L^1(\mathbb{T})},$$

we conclude that

$$\left|\{\zeta \in \mathbb{T} : \quad u_a^*(\zeta) > \lambda\}\right| \leq \left|\left\{\zeta \in \mathbb{T} : \quad Mf(\zeta) > \frac{\lambda}{3+4a}\right\}\right| \leq \frac{9+12a}{\lambda}\|f\|_{L^1(\mathbb{T})}.$$

Hence the lemma follows if we verify that the above inequality holds. To this end, we may assume that $\zeta = 1$. Fix a point $z_0 = r_0 e^{i\theta_0}$ with the condition that $|\theta_0| \leq \pi$. Recall the usual (harmonic) Poisson kernel

$$P_{z_0}(\theta) = P(z_0, e^{i\theta}) = \frac{1 - r_0^2}{1 + r_0^2 - 2r_0\cos(\theta - \theta_0)}, \qquad z_0 = r_0 e^{i\theta_0} \in \mathbb{D}.$$

Since $P_{z_0}(\theta)$ is a decreasing function of $\theta \in [0, \pi]$, it follows that for $|\theta_0| < |\theta| \leq \pi$ we have

$$\sup\{P_{z_0}(t)|\theta| \leq t \leq \pi\} = P_{z_0}(|\theta|).$$

On the other hand, for $|\theta| \leq |\theta_0|$, the above supremum is attained when $t = \theta_0$, and its value is

$$\frac{1-r_0^2}{1+r_0^2-2r_0}=\frac{1+r_0}{1-r_0}.$$

Let us look at the biharmonic Poisson kernel $F(e^{i\theta},z_0)$ as a function of θ for fixed $z_0=r_0e^{i\theta_0}$. For this, we write

$$F_{z_0}(\theta)=F(e^{i\theta},z_0),\qquad z_0=r_0e^{i\theta_0}.$$

As a matter of fact, there is the following interesting relation between the Poisson kernel and the biharmonic Poisson kernel:

$$F(z,\zeta)=\frac{1}{2}(1-|z|^2)\{P(z,\zeta)+P^2(z,\zeta)\},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{T}.$$

We now define

$$\varPhi_{z_0}(\theta)=\sup\{F_{z_0}(t)|\theta|\le t\le\pi\}.$$

The function $\varPhi_{z_0}(\theta)$ is an even function on the interval $-\pi\le\theta\le\pi$, it dominates $F_{z_0}(\theta)$, and it is a decreasing function of $0\le\theta\le\pi$. Indeed, $\varPhi_{z_0}$ is the least decreasing majorant of F_{z_0}. Since $Mf=M(|f|)$, we may assume that $f\ge 0$. Suppose that $\varPhi_{z_0}$ is an increasing limit of a finite combination of characteristic functions of the intervals $(-\theta_k,\theta_k)$. More precisely, there is a sequence of positive numbers c_k with $\sum c_k\le\|\varPhi_{z_0}\|_{L^1(\mathbb{T})}$ such that

$$h_n(\theta)=\sum_{k=1}^{n}c_k\left(\frac{1}{2\theta_k}\chi_{(-\theta_k,\theta_k)}(\theta)\right)\to\varPhi_{z_0}(\theta),\qquad \text{as}\quad n\to\infty.$$

It follows from the monotone convergence theorem that

$$\begin{aligned}\int_{\mathbb{T}} f(\theta)\varPhi_{z_0}(\theta)d\theta&=\lim_{n\to\infty}\int_{\mathbb{T}} f(\theta)h_n(\theta)d\theta\\&=\lim_{n\to\infty}\sum_{k=1}^{n}c_k\frac{1}{2\theta_k}\int_{-\theta_k}^{\theta_k}f(\theta)d\theta\le Mf(1)\sum_{k=1}^{n}c_k\\&\le Mf(1)\|\varPhi_{z_0}\|_{L^1(\mathbb{T})}.\end{aligned}$$

To have an upper bound for the $L^1(\mathbb{T})$-norm of $\varPhi_{z_0}$, we first note that for $|\theta_0|<|\theta|\le\pi$, we have $\varPhi_{z_0}(\theta)=F_{z_0}(|\theta|)$ and

$$\varPhi_{z_0}(\theta)=\frac{1}{2}(1-r_0^2)\left\{\frac{1+r_0}{1-r_0}+\frac{(1+r_0)^2}{(1-r_0)^2}\right\},\qquad |\theta|\le|\theta_0|.$$

The second thing we need to know is the following estimate (see [5]):

$$\frac{|\theta_0|}{1-|z_0|}\le\pi a,\qquad z_0\in\varOmega_a(1).$$

Since Φ_{z_0} is an even function, we can write

$$\|\Phi_{z_0}\|_{L^1(\mathbb{T})} = 2\int_{|\theta_0|}^{\pi} \Phi_{z_0}(\theta)\frac{d\theta}{2\pi} + 2\int_{0}^{|\theta_0|} \Phi_{z_0}(\theta)\frac{d\theta}{2\pi}.$$

It follows from the definition of Φ_{z_0} that

$$\begin{aligned} 2\int_{0}^{|\theta_0|} \Phi_{z_0}(\theta)\frac{d\theta}{2\pi} &\leq \frac{2|\theta_0|}{2\pi}\frac{1}{2}(1-r_0^2)\left\{\frac{1+r_0}{1-r_0} + \frac{(1+r_0)^2}{(1-r_0)^2}\right\} \\ &= \frac{|\theta_0|}{2\pi}\left\{(1+r_0)^2 + \frac{(1+r_0)^3}{1-r_0}\right\} \\ &\leq \frac{|\theta_0|}{2\pi}\left(4 + \frac{8}{1-|z_0|}\right) \\ &\leq \frac{2|\theta_0|}{\pi} + \frac{4|\theta_0|}{\pi(1-|z_0|)} \leq 2+4a. \end{aligned}$$

This yields

$$\|\Phi_{z_0}\|_{L^1(\mathbb{T})} \leq 1 + (2+4a) = 3+4a.$$

And finally it follows that

$$\begin{aligned} u(z_0) = \int_{\mathbb{T}} f(\theta)F_{z_0}(\theta)\frac{d\theta}{2\pi} &\leq \int_{\mathbb{T}} f(\theta)\Phi_{z_0}(\theta)\frac{d\theta}{2\pi} \\ &\leq Mf(1)\|\Phi_{z_0}\|_{L^1(\mathbb{T})} \\ &\leq (3+4a)Mf(1). \end{aligned}$$

Since z_0 was arbitrarily chosen in $\Omega_a(1)$, we conclude that

$$u_a^*(1) \leq (3+4a)Mf(1),$$

completing the proof of the lemma.

3. – The biharmonic Poisson kernel for the upper half plane.

In this section we intend to find the upper half plane analog of $F(z,\zeta)$ studied in the previous section. Given the usual Poisson kernel for the unit disk, it is easy to find the Poisson kernel for the upper half plane (or any other simply connected region). What we need is a Moebius transformation which maps the given region onto the unit disk, then a change of variables does the job. Unfortunately, the biharmonic functions are not preserved under Moebius transformations; therefore this method does not work. Instead, we have to appeal to a direct computation of the desired kernel function.

We first recall (see for instance [6]) the biharmonic Green function for the upper half plane; this is the function

$$U(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{z-\overline{\zeta}}\right|^2 + 4\,Im(z)\,Im(\zeta), \qquad (z,\zeta)\in\mathbb{C}_+\times\mathbb{C}_+,$$

which solves, for fixed $\zeta\in\mathbb{C}_+$, the boundary value problem

$$\begin{cases} \Delta_z^2 U(z,\zeta) = \delta_\zeta(z), & z\in\mathbb{C}_+,\\ U(z,\zeta)=0, & z\in\mathbb{R},\\ \partial_{n(z)}U(z,\zeta)=0, & z\in\mathbb{R}. \end{cases}$$

We start by the following lemma:

LEMMA 3.1. – *For fixed $z\in\mathbb{C}_+$ we have*

$$\Delta_\zeta U(z,\zeta) = \log\left|\frac{\zeta-z}{\zeta-\overline{z}}\right|^2 + 2\,Re\left(\frac{z-\overline{z}}{\zeta-\overline{z}}\right), \qquad \zeta\in\mathbb{C}_+\,.$$

PROOF. – Since $U(z,\zeta)$ is symmetric, we can write

$$U(z,\zeta) = |\zeta-z|^2 \log\left|\frac{\zeta-z}{\zeta-\overline{z}}\right|^2 + 4\,Im(\zeta)\,Im(z), \qquad (z,\zeta)\in\mathbb{C}_+\times\mathbb{C}_+\,.$$

It is enough to compute the bilaplacian of the first term, since

$$\Delta_\zeta^2\big((Im(z)\,Im(\zeta)\big) = 0.$$

Writing

$$|\zeta-z|^2\log\left|\frac{\zeta-z}{\zeta-\overline{z}}\right|^2 = (\zeta-z)(\overline{\zeta}-\overline{z})\log\frac{(\zeta-z)(\overline{\zeta}-\overline{z})}{(\zeta-\overline{z})(\overline{\zeta}-z)},$$

we see that

$$\frac{\partial}{\partial\zeta}\left(|\zeta-z|^2\log\left|\frac{\zeta-z}{\zeta-\overline{z}}\right|^2\right) = (\overline{\zeta}-\overline{z})\log\left|\frac{\zeta-z}{\zeta-\overline{z}}\right|^2 + (\overline{\zeta}-\overline{z})\frac{z-\overline{z}}{\zeta-\overline{z}}.$$

Applying the differential operator $\dfrac{\partial}{\partial\overline{\zeta}}$ to the expression above, we end up with the desired result. □

From now on, we adhere to the following convention. For z and ζ in the upper half plane we write

$$z = x+iy, \qquad y>0,$$
$$\zeta = t+is, \qquad s>0\,.$$

LEMMA 3.2. – *Let $z \in \mathbb{C}_+$ be fixed. Then we have*

$$\Delta_\zeta U(z,\zeta) = \log\left(\frac{(t-x)^2+(s-y)^2}{(t-x)^2+(s+y)^2}\right) + \frac{4y(s+y)}{(t-x)^2+(s+y)^2},$$

and

$$\frac{\partial}{\partial s}\Delta_\zeta U(z,\zeta) = \frac{2(s-y)}{(t-x)^2+(s-y)^2} - \frac{2(s+y)}{(t-x)^2+(s+y)^2} + 4y\frac{(t-x)^2-(s+y)^2}{\left((t-x)^2+(s+y)^2\right)^2}.$$

PROOF. – The first statement follows from Lemma (3.1). As for the second equality we can just differentiate the expression $\Delta_\zeta U(z,\zeta)$ with respect to the variable s. □

LEMMA 3.3. – *The outward normal derivative of $\Delta_\zeta U(z,\zeta)$ on the boundary of the upper half plane, $\mathbb{R}$, is*

$$\frac{-4y}{y^2+(x-t)^2} + 4y\frac{(x-t)^2-y^2}{\left((y^2+(x-t)^2\right)^2}.$$

PROOF. – It is enough to put $s=0$ in the second expression of the Lemma (3.2). □

Motivated by the biharmonic Poisson kernel for the upper half plane given by the equation (2.2), we can define the biharmonic Poisson kernel for the upper half plane.

DEFINITION 3.4. – *For $z \in \mathbb{C}_+$ and $t \in \mathbb{R}$ we define the biharmonic Poisson kernel for the upper halp plane as*

$$\begin{aligned} F(z,t) &= -\frac{1}{4\pi}\frac{\partial}{\partial s}\Delta_\zeta U(z,\zeta)\Big|_{s=0} \\ &= \frac{1}{\pi}\left\{\frac{y}{y^2+(x-t)^2} + y\frac{y^2-(x-t)^2}{\left((y^2+(x-t)^2\right)^2}\right\}. \end{aligned}$$

In the following lemma, we shall see that for fixed $t \in \mathbb{R}$, the function $F(z,t)$ is biharmonic in its first variable.

LEMMA 3.5. – *For fixed $t \in \mathbb{R}$, we have*

$$\Delta_z^2 F(z,t) = 0, \qquad z \in \mathbb{C}_+ .$$

PROOF. – It is well-known that

$$\frac{y}{(x-t)^2+y^2} = Im\left(\frac{1}{t-z}\right), \qquad z \in \mathbb{C}_+ ,$$

is the usual (harmonic) Poisson kernel for the upper half plane. In particular, it is biharmonic. What remains is to verify that

$$K(x,y) = \frac{y^3 - y(x-t)^2}{\left((x-t)^2+y^2)\right)^2}$$

is biharmonic. A direct computation shows that the expression

$$\Delta^2 K(x,y) = \frac{\partial^4}{\partial x^4}K(x,y) + 2\frac{\partial^4}{\partial x^2 \partial y^2}K(x,y) + \frac{\partial^4}{\partial y^4}K(x,y),$$

vanishes identically for $z = x + iy$ in the upper half plane. □

We shall at times refer to $F(z,t)$ defined in Definition (3.4) as the *biharmonic Poisson kernel for the upper half plane*. Note that $F(z,t)$ can be written in the form

$$F(z,t) = F(x+iy,t) = F_y(x-t), \qquad z \in \mathbb{C}_+, \quad t \in \mathbb{R},$$

where

$$F_y(t) = \frac{1}{\pi}\left\{\frac{y}{y^2+t^2} + y\frac{y^2-t^2}{(y^2+t^2)^2}\right\}, \qquad t \in \mathbb{R}.$$

We now proceed to study this new biharmonic kernel function in more details. Indeed, we should verify that $F(z,t)$ enjoys the intrinsic properties of a kernel function; it is an approximate identity.

THEOREM 3.6. – *Assume that*

$$F(z,t) = F(x+iy,t) = F_y(x-t), \qquad z \in \mathbb{C}_+, \quad t \in \mathbb{R},$$

where

$$F_y(t) = \frac{1}{\pi}\left\{\frac{y}{y^2+t^2} + y\frac{y^2-t^2}{(y^2+t^2)^2}\right\}, \qquad t \in \mathbb{R}$$

denotes the biharmonic Poisson kernel for the upper half plane. Then

(a) *the integral of the biharmonic Poisson kernel over the real line is* 1*:*

$$\int_{\mathbb{R}} F_y(t)\,dt = 1, \qquad y > 0,$$

(b) *for fixed* $y > 0$, F_y *is a positive and even function on* $\mathbb{R}$ *which is decreasing for* $0 \le t < \infty$,

(c) *for every* $\delta > 0$ *we have*

$$\sup_{|t|>\delta} F_y(t) \to 0, \quad \text{as} \quad y \to 0,$$

(d) *for every* $\delta > 0$ *we have*

$$\int\limits_{|t|>\delta} F_y(t)\,dt \to 0, \quad \text{as} \quad y \to 0.$$

PROOF. – To prove (a), it is easy to see that for $y > 0$ we have

$$\int\limits_{-\infty}^{\infty} \frac{y^3 - yt^2}{(y^2+t^2)^2}\,dt = \left[\frac{yt}{y^2+t^2}\right]_{t=-\infty}^{t=\infty} = 0.$$

It now follows from the definition of $F_y(t)$ that

$$\int\limits_{\mathbb{R}} F_y(t)\,dt = \frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{y}{y^2+t^2}\,dt = \frac{1}{\pi}\left[\arctan\left(\frac{t}{y}\right)\right]_{t=-\infty}^{t=\infty} = 1.$$

As for part (b) we see that $F_y(t) = F_y(-t)$, that is F_y is an even function on $\mathbb{R}$. A computation shows that

$$\frac{d}{dt}F_y(t) = -\frac{8y^3t}{\pi(y^2+t^2)^3} \leq 0,$$

for $y > 0$ and $t \geq 0$. Therefore F_y is decreasing on the interval $0 \leq t < \infty$. Hence its maximum value is attained for $t = 0$, that is

$$F_y(t) \leq F_y(0) = \frac{2}{\pi y}, \qquad t > 0.$$

Since F_y is strictly decreasing on $0 < t < \infty$, and $F_y(t) \to 0$, as $t \to \infty$, we conclude that F_y is positive.

It follows from part (b) that

$$\sup_{|t|>\delta} F_y(t) = F_y(\delta) = \frac{y}{\delta^2+y^2} + \frac{y^3 - y\delta^2}{(\delta^2+y^2)^2} \to 0, \quad y \to 0.$$

This proves part (c). Finally, part (d) follows from the fact that

$$\int\limits_{\delta}^{\infty} F_y(t)\,dt \to 0, \quad y \to 0,$$

which in turn is a consequence of the monotonicity of F_y. □

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