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Sunto. – *Il concetto di convergenza statistica di una successione fu introdotto per la prima volta da H. Fast* [7] *nel* 1951. *Recentemente, nella letteratura è stato studiato il concetto di convergenza statistica di successioni doppie. Il risultato principale di questo lavoro è un teorema che dà significato all'affermazione:* $s = \{s_{ij}\}$ *converge A statisticamente a L se e solo se "la maggior parte" delle "sottosuccessioni" di s convergono a L nel senso ordinario. I risultati presentati qui sono l'analogo dei teoremi di* [12], [13] *e* [6] *e riguardano la convergenza A statistica introdotta per la prima volta da Freedman e Sember* [8]. *Vengono anche presi in considerazione altri problemi correlati.*

Summary. – *The concept of statistical convergence of a sequence was first introduced by H. Fast* [7] *in* 1951. *Recently, in the literature, the concept of statistical convergence of double sequences has been studied. The main result in this paper is a theorem that gives meaning to the statement:* $s = \{s_{ij}\}$ *converges statistically A to L if and only if "most" of the "subsequences" of s converge to L in the ordinary sense. The results presented here are analogue of theorems in* [12], [13] *and* [6] *and are concerned with A statistical convergence, first introduced by Freedman and Sember* [8]. *Other related problems are considered.*

1. – Introduction and preliminaries.

The concept of the statistical convergence of a sequence of reals $s = (s_n)_{n=1}^{\infty}$ was introduced by H. Fast [7]. $s = (s_n)$ is said to converge statistically to L and we write

$$\lim_{n\to\infty} s_n = L \text{ (stat) if for every } \varepsilon > 0,$$

$$\lim_{n\to\infty} n^{-1}\big|\{k \le n : |s_k - L| \ge \varepsilon\}\big| = 0.$$

Here $|A|$ denotes the number of elements in the finite set A. A double sequence $s = (s_{ij})_{i=1,j=1}^{\infty,\infty}$ of reals is said to converge statistically to L and we write

$$\lim_{\substack{i\to\infty\\ j\to\infty}} s_{ij} = L \text{ (stat) if for every } \ \varepsilon > 0$$

$$\lim_{\substack{p\to\infty\\ q\to\infty}} (pq)^{-1}\big|\{(i,j)\colon |s_{ij} - L| \ge \varepsilon,\ i \le p, j \le q\}\big| = 0,$$

i.e. if for every $\delta > 0$, there exists an N such that $p, q \geq N$ implies

$$(pq)^{-1}\big|\{(i,j)\colon |s_{ij} - L| \geq \varepsilon,\ i \leq p, j \leq q\}\big| < \delta.$$

The results in this paper are analogues of the "single" sequence results presented in [12], [13] and extensions of results in [6]. For earlier related results see [10] and [11].

Statistical convergence of single sequences, $s = (s_n)_{n=1}^{\infty}$, can be generalized using a regular nonnegative summability method A in place of C, the Cesàro matrix. This idea was first mentioned by R.C. Buck [2] in 1953 and further studied by Sember and Freedman [8] and Connor ([3] and [5]).

We will now introduce the concept of A statistical convergence of double sequences $s = (s_{i,j})_{i=1,j=1}^{\infty,\infty}$.

DEFINITION 1. – *a) A double sequence $A = (A^{pq}) = (A^{pq})_{p=1,q=1}^{\infty,\infty}$ of summability matrices is called a mean if*

(1) $A_{ij}^{pq} = 0$ *if* $i > p$ *or* $j > q$, $A_{ij}^{pq} > 0$ *if* $i \leq p$ *and* $j \leq q$,

(2) $\sum_{i,j} (A^{pq})_{ij} = 1$ *for each* $p, q \in \mathbb{N}$; *and*

(3) $\lim_{\substack{p\to\infty\\ q\to\infty}} \sum_{(i,j)\in M} (A^{pq})_{ij} = 0$ *for each* M *of the form* $M = (\mathbb{N}\times\mathbb{N}) \setminus T_{p_0,q_0}$ *where* $T_{p_0,q_0} = \big\{(i,j) \in \mathbb{N}\times\mathbb{N} :\ i \geq p_0$ *and* $j \geq q_0\big\}$.

Here $\lim_{\substack{p\to\infty\\ q\to\infty}} a_{pq}$ *denotes that usual definition of limit, i.e.* $\lim_{\substack{p\to\infty\\ q\to\infty}} a_{pq} = L$ *if for every* $\varepsilon > 0$, *there exists an* N *such that* $p, q \geq N$ *implies* $|a_{pq} - L| < \varepsilon$.

b) If A is a mean and $B \subseteq \mathbb{N}\times\mathbb{N}$, then we say B has A-density zero and we write $\delta_A(B) = 0$, *if* $\lim_{\substack{p\to\infty\\ q\to\infty}} \sum_{(i,j)\in B} (A^{pq})_{ij} = 0$.

We are now in a position to define A-statistical convergence of a double sequence.

DEFINITION 2. – *If $A = (A^{pq})$ is a mean and $s = (s_{ij})$ a double sequence then we say s converges (stat) A to L and we write $s_{ij} \to L$ (stat) A if for each* $\varepsilon > 0$, $\delta_A(B_\varepsilon) = 0$ *where*

$$B_\varepsilon = \{(i,j)\colon |s_{ij} - L| \geq \varepsilon\}.$$

We now define subsequences of double sequences, as in [6].

DEFINITION 3. – *a) Let* $X\colon = \big\{(x_{ij})_{i=1,j=1}^{\infty,\infty} \colon x_{ij} \in \{0,1\}$ *for every* $(i,j) \in \mathbb{N}\times\mathbb{N}\big\}$.

b) If $s = (s_{ij})$ *and* $x = (x_{ij}) \in X$ *then* $s(x)$ *denotes the double sequence* $\{s_{ij}(x)\}$ *where*

$$s_{ij}(x) = s_{ij} \quad \text{if} \quad x_{ij} = 1 \quad \text{and}$$
$$s_{ij}(x) = * \quad \text{if} \quad x_{ij} = 0$$

For example, for

$$s = \begin{pmatrix} \cdot & \cdot & \cdot \\ s_{21} & s_{22} & \cdot \\ s_{11} & s_{12} & \cdot \end{pmatrix}, \ x = \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & 0 & \cdot \\ 1 & 1 & \cdot \end{pmatrix} \text{ we have } s(x) = \begin{pmatrix} \cdot & \cdot & \cdot \\ s_{21} & * & \cdot \\ s_{11} & s_{12} & \cdot \end{pmatrix}.$$

c) $X' =: \{x \in X \colon \text{for each } N \in \mathbb{N} \text{ there exists } p, q \geq N \text{ such that } x_{pq} = 1\}$

We will now define A statistical convergence of subsequences $s(x)$*, where* $x \in X'$*. We restrict ourselves to* X' *to avoid finite subsequences.*

DEFINITION 4. – *Suppose* $s = (s_{ij})$, $x = (x_{ij}) \in X'$ *and* $A = (A^{pq})$ *is a mean. We say that* $s(x)$ *converges statistically A to L, and write* $s(x) \to L$ *(stat) A, if for every* ε, $\varepsilon' > 0$ *there exists an* $N(\varepsilon, \varepsilon')$ *such that for every* $p, q \geq N(\varepsilon, \varepsilon')$ *we have*

$$\frac{\sum_{(i,j)} \left[(A^{pq})_{ij} \colon |s_{ij} - L| \geq \varepsilon, \ x_{ij} = 1 \right]}{\sum_{(i,j)} \left[(A^{pq})_{ij} \colon x_{ij} = 1 \right]} < \varepsilon'.$$

2. – Main results.

Our first theorem is an analogue of a theorem of Fridy [9] which states: $s = (s_n)$ converges statistically to L if and only if there exists a subset B of $\mathbb{N}$, having density zero (i.e. $\lim_{n \to \infty} n^{-1} |\{k \leq n \colon k \in B\}| = 0$, where $|C|$ denotes the number of elements in the finite set C) and the subsequence $(s_{n_j})_{j=1}^{\infty}$ converges to L, in the ordinary sense, where $B = \mathbb{N} \setminus \{n_j \colon j \in \mathbb{N}\}$. Móricz [14] has a similar result for double sequences.

THEOREM 1. – *If* $A = (A^{pq})$ *is a mean, then the double sequence* $s = (s_{ij})$ *is* (stat) *A convergent to L if and only if there exists an* $x \in X'$ *such that* $s(x)$ *converges to L in the ordinary sense and the set* $\{(i,j) \colon x_{ij} = 0\}$ *has A-density zero.*

PROOF. – *a)* Suppose $s_{ij} \to L$ (stat) A, where $A = (A^{pq})$ is a mean. For each $\varepsilon_n = \dfrac{1}{n}$, $n = 1, 2, 3, \ldots$ there exists a positive integer r_n (with the sequence $(r_n)_{n=1}^{\infty}$ strictly increasing) such that:

If $p,q \geq r_n$, then $\sum\left[A_{ij}^{pq}: |s_{ij}-L| \geq \frac{1}{n}\right] < \frac{1}{n^2}$, for $n=1,2,3,\ldots$. Set $B := \bigcup_{n=1}^{\infty}\left\{(i,j)\in\Im_n: |s_{ij}-L| \geq \frac{1}{n}\right\}$ where $\Im_n := T_{r_n,r_n}\setminus T_{r_{n+1},r_{n+1}}$.

(See (3) in Definition 1 for the definition of T_{p_0,q_0}.)

We will now show that B has A-density zero. To see this observe that if $\varepsilon>0$, there exists an $n(\varepsilon)$ such that $\sum_{n=n(\varepsilon)}^{\infty}\frac{1}{n^2}<\frac{\varepsilon}{2}$. Condition (2), in Definition 1, implies there exists an $R_n(\varepsilon)$, a term in the sequence $\{r_n\}$ with index larger than $n(\varepsilon)$, i.e. $R_{n(\varepsilon)}=r_{m(\varepsilon)}$, with $m(\varepsilon)>n(\varepsilon)$, such that

$$\sum\left[A_{ij}^{pq}:(i,j)\in\bigcup_{i=1}^{r_{n(\varepsilon)}-1}\Im_n\right]<\frac{\varepsilon}{2}\text{ for all } p,q\geq R_{n(\varepsilon)}\ (=r_{m(\varepsilon)}).$$

Now suppose that $p,q\geq r_m(\varepsilon)$. Set $\Im_0 := (\mathbb{N}\times\mathbb{N})\setminus\bigcup_{n=1}^{\infty}\Im_n$. then we have

$$\sum\left[A_{ij}^{pq}:(i,j)\in B\right]\leq\sum\left[A_{ij}^{pq}:(i,j)\in\bigcup_{n=0}^{r_{n(\varepsilon)}-1}\Im_n\right]$$
$$+\sum\left[A_{ij}^{pq}:(i,j)\in B\cap\left((\mathbb{N}\times\mathbb{N})\setminus\bigcup_{n=0}^{r_{n(\varepsilon)}-1}\Im_n\right)\right]$$

with $(p,q)\in\Im_k$ for some $k>n(\varepsilon)$, so that the first sum on the right side of the above inequality is less than $\varepsilon/2$ and second term on the right side (which we will denote by (†)), because of (1) in Definition 1, satisfies:

$$(\dagger)=\sum\left[A_{ij}^{pq}:(i,j)\in\Im_{r_{n(\varepsilon)}}\cap B\right]+\sum\left[A_{ij}^{pq}:(i,j)\in\Im_{n(\varepsilon)+1}\cap B\right]+\cdots$$
$$+\sum\left[A_{ij}^{pq}:(i,j)\in\Im_k\cap B\right]$$
$$<\frac{1}{n^2(\varepsilon)}+\frac{1}{(n(\varepsilon)+1)^2}+\ldots+\frac{1}{k^2}<\sum_{n=n(\varepsilon)}^{\infty}\frac{1}{n^2}<\frac{\varepsilon}{2}.$$

Therefore $\sum\left[A_{ij}^{pq}:(i,j)\in B\right]<\varepsilon$ if $p,q\geq r_m(\varepsilon)$, hence B has A-density zero and clearly $s(x)$ converges to L in the ordinary sense where

$$x_{ij}=\begin{cases}1 & \text{if } (i,j)\notin B\\ 0 & \text{if } (i,j)\in B.\end{cases}$$

and $x\in X'$.

b) Now consider the converse. Suppose $x\in X'$ and $B=\{(i,j): x_{ij}=0\}$ has A-density zero and $s(x)$ converges in the ordinary sense to L. Examine $\sum\left[A_{ij}^{pq}: |s_{ij}-L|\geq\varepsilon\right]$. This sum equals:

$$\sum\left[A_{ij}^{pq}: |s_{ij}-L|\geq\varepsilon\text{ and }x_{ij}=0\right]+\sum\left[A_{ij}^{pq}: |s_{ij}-L|\geq\varepsilon\text{ and }x_{ij}=1\right]$$

The first sum converges to zero as $p \to \infty$ and $q \to \infty$ since B has A-density zero. The fact that $s(x)$ converges to L in the ordinary sense and property (3) of (A^{pq}) implies that the second sum converges to zero as $p \to \infty$ and $q \to \infty$. Hence

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \sum \left[A^{pq}_{ij}: |s_{ij} - L| \geq \varepsilon\right] = 0$$

or $s = (s_{ij})$ is (stat) A convergent to L. □

Our main result, the measure theoretical subsequence characterization of statistical A convergence of a double sequence, alluded to in the Abstract, requires one final measure theoretical preliminary before we can proceed.

DEFINITION 5. – *If $B \subseteq \mathbb{N} \times \mathbb{N}$, P_B will denote the unique probability measure defined on $\mathfrak{B}$, the smallest σ-algebra of subsets of X containing all sets of the form $\{x = (x_{ij}) \in X: x_{i_1 j_1} = a_1, \ldots, x_{i_n, j_n} = a_n\}$ (the (i_k, j_k)'s distinct and $a_1, \ldots, a_n \in \{0, 1\}$), such that*

$$P_B\big(\{x \in X: x_{ij} = 1\}\big) = \begin{cases} \dfrac{1}{2} & \text{if} \quad (i,j) \notin B \\[2ex] \dfrac{1}{2^{i+j}} & \text{if} \quad (i,j) \in B \end{cases}$$

and such that

$$\begin{aligned} &P_B\big(\{x \in X: x_{i_1 j_1} = a_1, \ldots, x_{i_n j_n} = a_n\}\big) \\ &\quad = P_B\big(\{x \in X: x_{i_1 j_1} = a_1\}\big) \cdots P_B\big(\{x \in X: x_{i_n j_n} = a_n\}\big) \end{aligned}$$

for every pairwise distinct pairs, in $\mathbb{N} \times \mathbb{N}$, $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ and every choice of $a_1, \ldots, a_n$ in $\{0, 1\}$. Notice that $P_B(X) = 1$ for each B and that $P_B(X') = 1$ provided that compliment of B is infinite.

THEOREM 2. – *Suppose $s = (s_{ij})$ is a double sequence and $A = (A^{pq})$ is a mean. Then s converges* (stat) *A to L, if and only if there exists a subset B of $\mathbb{N} \times \mathbb{N}$ having A-density zero such that*

$$P_B\big(\{x \in X': s(x) \to L \textit{ in the ordinary sense}\}\big) = 1.$$

PROOF. – *a*) Suppose $A = (A^{pq})$ is a mean and $s = (s_{ij})$ converges (stat) A to L. Then, by Theorem 1, there exists an $x' \in X'$, $x' = (x'_{ij})$, such that $s(x')$ converges to L in the ordinary sense and $B = \{(i,j): x'_{ij} = 0\}$ has A-density zero. Notice that $s(x)$ converges to L in the ordinary sense if $x = (x_{ij}) \in X'$ and

$\{(i,j) \in B: x_{ij} = 1\}$ is a finite set. More over, by the first part of the Borel-Cantelli Lemma (see [1], pg. 48)

$$P_B\big(\{x \in X': \{(i,j) \in B \text{ such that } x_{ij} = 1\} \text{ is infinite}\}\big) = 0$$

since

$$\sum_{(i,j)\in B} P_B\big(\{x \in X': x_{ij} = 1\}\big) = \sum_{(i,j)\in B} \frac{1}{2^{i+j}} \leq 1.$$

Therefore

$$P_B\big(\{x \in X': s(x) \text{ converges to } L\}\big) = 1.$$

b) Suppose now that $A = (A^{pq})$ is a mean, $s = (s_{ij})$ is not (stat) A convergent and B is any subset of $\mathbb{N} \times \mathbb{N}$ having A-density zero. Then, by Theorem 1, $s(x)$, where $x = (x_{ij})$ satisfies

$$x_{ij} = 1 \quad \text{if} \quad (i,j) \notin B,$$
$$x_{ij} = 0 \quad \text{if} \quad (i,j) \in B$$

is not convergent. Notice that $x \in X'$.

Then either we have:

Case 1 $\lim_{k\to\infty} s_{i_k,j_k} = +\infty$, where $i_{k+1} > i_k$ and $j_{k+1} > j_k$ for every k and $(i_k,j_k) \notin B$ for every k,

Case 2 $\lim_{k\to\infty} s_{i_k,j_k} = -\infty$, where $i_{k+1} > i_k$ and $j_{k+1} > j_k$ for every k and $(i_k,j_k) \notin B$ for every k,

Case 3 There exists $\lambda < \mu$ and $\{(i_k,j_k)\}_{k=1}^{\infty}$ and $\{(\widehat{i}_k,\widehat{j}_k)\}_{k=1}^{\infty}$ satisfying $i_{k+1} > i_k$, $j_{k+1} > j_k$ and $\widehat{i}_{k+1} > \widehat{i}_k$, $\widehat{j}_{k+1} > \widehat{j}_k$ for every k and $(i_k,j_k) \notin B$, $(\widehat{i}_k,\widehat{j}_k) \notin B$, $s_{i_k j_k} > \mu$, $s_{\hat{i}_k \hat{j}_k} < \lambda$ for every k.

In Case 1

$$\sum_{k=1}^{\infty} P_B\big(\{x = (x_{ij}) \in X': x_{i_k j_k} = 1\}\big) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

In Case 2

$$\sum_{k=1}^{\infty} P_B\big(\{x = (x_{ij}) \in X': x_{i_k j_k} = 1\}\big) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

In Case 3

$$\sum_{k=1}^{\infty} P_B\big(\{x=(x_{ij})\in X'\colon x_{i_k j_k}=1\}\big)$$
$$=\sum_{k=1}^{\infty} P_B\big(\{x=(x_{ij})\in X'\colon x_{\hat{i}_k \hat{j}_k}=1\}\big)=\sum_{k=1}^{\infty}\frac{1}{2}=\infty.$$

Therefore by the second part of the Borel-Cantelli Lemma ([1], pg. 48) we have:

In Case 1

$$P_B\big(\{x\in X'\colon x_{i_k j_k}=1 \text{ for infinitely many } k\}\big)=1.$$

In Case 2

$$P_B\big(\{x\in X'\colon x_{i_k j_k}=1 \text{ for infinitely many } k\}\big)=1.$$

In Case 3

$$P_B\big(\{x\in X'\colon x_{i_k j_k}=1 \text{ for infinitely many } k \text{ and } x_{\hat{i}_k \hat{j}_k}=1 \text{ for infinitely many } k\}\big)=1.$$

Therefore, in each of the three cases we have

$$P_B\big(\{x=(x_{ij})\in X'\colon x \text{ is convergent}\}\big)=0$$

□

We now will present another double sequence measure theoretical result. In [4] Connor proved that almost none, in the sense of Lebesque measure, of the sequences of 0's and 1's are almost convergent. Here we will give the analogous double sequence result. We first present the relevant definitions.

DEFINITION 6. – *A double sequence $s=(s_{ij})$ will be said to be almost convergent to L if*

$$\lim_{\substack{p\to\infty\\ q\to\infty}}\frac{1}{pq}\sum_{i=0,j=0}^{p-1,q-1} s_{n+i,m+j}=L,$$

uniformly in (n,m).

DEFINITION 7. – *Let P denote the unique probability measure defined on $\mathfrak{B}$, the Borel subsets of X satisfying:*

$$P\big(\{x=(x_{ij})\in X\colon x_{i_1,j_1}=a_1,\dots,x_{i_n,j_n}=a_n\}\big)=\frac{1}{2^n}$$

for every pairwise distinct elements in $\mathbb{N}\times\mathbb{N}$, $(i_1,j_1),(i_2,j_2),\dots,(i_n,j_n)$ and every choice of $a_1,a_2,\dots,a_n$ in $\{0,1\}$.

THEOREM 3. – *$P(A) = 0$, where $A = \{x = (x_{ij}) \in X : x$ is almost convergent$\}$.*

PROOF. – For each n, let

$$C_n := \{x = (x_{ij}) \in X : x_{ij} = 1 \text{ for all } (i,j) \text{ in one of the sets } B_k^{(n)},\ k = 1, 2, \dots\},$$

where $B_1^{(n)} = \{(i,j): 1 \le i \le n, 1 \le j \le n\}$, $B_2^{(n)} = \{(i,j): n+1 \le i \le 2n, n+1 \le j \le 2n\}, \dots$. Then we have

$$\sum_{k=1}^{\infty} P(B_k^{(n)}) = \sum_{k=1}^{\infty} \frac{1}{2^{n^2}} = \infty.$$

Furthermore, since the set $\{B_k^{(n)}\}_{k=1}^{\infty}$ are stochastically independent we have by the second part of the Borel-Cantelli Lemma (see [1], pg. 48),

$$P\left(\limsup_{k \to \infty} B_k^{(n)}\right) = 1$$

for each $n \in \mathbb{N}$. This implies $P(C_n) = 1$ for each n which in turn implies $P\left(\bigcap_{n=1}^{\infty} C_n\right) = 1$. Since

$$\lim_{\substack{p \to \infty \\ q \to \infty}} \frac{1}{pq} \sum_{i=0, j=0}^{p-1, q-1} x_{ij} = \frac{1}{2}$$

for P almost all sequence $x = (x_{ij})$ in X (see [6]), it follows that

$$P(A) = 0.$$

□

We complete this paper with a Baire category theorem that doubly extends Theorem 3.1 in [13]. This result deals with subsets of X, the collection of all double sequences of zeroes and ones. We need to introduce a topology on X.

DEFINITION 8. – *If $x = (x_{ij})$, $y = (y_{ij})$ are two double sequence in X we define*

$$d(x, y) := \sum_{i=1, j=1}^{\infty, \infty} \frac{|x_{ij} - y_{ij}|}{2^{i+j}}.$$

(X, d) is a compact metric space (in fact it can be viewed as the product space of $\mathbb{N} \times \mathbb{N}$ copies of the discrete space consisting of the elements 0 and 1). Therefore (X, d) is of the second category.

While (X', d) is not complete it is easy to show that it is of the second category.

From now on till the end of paper, we will assume that $A = (A^{pq})$ is a mean.

We have defined (see Definition 4) what it means for $s(x)$ to converge (stat) A to a number L. It is convenient to now define when $s(x)$ is a (stat) A Cauchy sequence. For an earlier related definition (see Fridy [9]).

DEFINITION 9. – *For a given $s = (s_{ij})$ and $x = (x_{ij}) \in X'$, $s(x)$ is said to be a* (stat) *A Cauchy sequence if for every $\varepsilon > 0$, there exists an $m, n \in \mathbb{N} \times \mathbb{N}$ such that $x_{mn} = 1$ and*

$$\lim_{\substack{p\to\infty\\ q\to\infty}} \frac{\sum_{(i,j)} \left[A_{ij}^{pq} : |s_{ij} - s_{mn}| \geq \varepsilon, \text{ and } x_{ij} = 1\right]}{\sum_{(i,j)} \left[A_{ij}^{pq} : x_{ij} = 1\right]} = 0.$$

We now characterize (stat) A convergence.

THEOREM 4. – *Suppose $x \in X'$ and $s = (s_{ij})$. Then $s(x)$ is* (stat) *A convergent if and only if $s(x)$ is a* (stat) *A Cauchy sequence.*

PROOF. – *a*) Suppose $s(x)$ is (stat) A convergent to L. Then, by Definition 4, given $\varepsilon, \delta > 0$, there exists a natural number N such that $p, q \geq N$ implies

$$\frac{\sum_{(i,j)} \left[A_{ij}^{pq} : |s_{ij} - L| < \frac{\varepsilon}{2},\ x_{ij} = 1\right]}{\sum_{(i,j)} \left[A_{ij}^{pq} : x_{ij} = 1\right]} > 1 - \delta.$$

Select $i(\varepsilon), j(\varepsilon) \in \mathbb{N}$ such that $x_{i(\varepsilon),j(\varepsilon)} = 1$ and $|s_{i(\varepsilon),j(\varepsilon)} - L| < \varepsilon/2$. Then if $p, q \geq N$ we have

$$\frac{\sum_{(i,j)} \left[A_{ij}^{pq} : |s_{ij} - s_{i(\varepsilon)j(\varepsilon)}| < \varepsilon,\ x_{ij} = 1\right]}{\sum_{(i,j)} \left[A_{ij}^{pq} : x_{ij} = 1\right]} > 1 - \delta.$$

and hence $s(x)$ is a (stat) A Cauchy sequence.

b) We now prove the reverse implication. Suppose $s(x)$ is a (stat) A Cauchy sequence. By Definition 9, for every $n \in \mathbb{N}$, there exists $i(n), j(n) \in \mathbb{N}$ such that $x_{i(n),j(n)} = 1$ and

$$(*) \qquad \lim_{\substack{p\to\infty\\ q\to\infty}} \frac{\sum_{(i,j)} \left[A_{ij}^{pq} : |s_{ij} - s_{i(n)j(n)}| < \frac{1}{n},\ x_{ij} = 1\right]}{\sum_{(i,j)} \left[A_{ij}^{pq} : x_{ij} = 1\right]} = 1.$$

We will complete the proof by showing that $(s_{i(n),j(n)})_{n=1}^{\infty}$ is a Cauchy sequence in the ordinary sense and that $L = \lim_{n\to\infty} s_{i(n),j(n)}$ is the (stat) A limit of $s(x)$, i.e. $s(x) \to L$ (stat A).

To see that $(s_{i(n),j(n)})_{n=1}^{\infty}$ is a Cauchy sequence observe that since $A = (A^{pq})$ is a mean $(*)$ implies that given $m, n \in \mathbb{N}$ there exists $\widehat{i}, \widehat{j} \in \mathbb{N}$ such that $x_{\hat{i},\hat{j}} = 1$ and

$$|s_{\hat{i}\hat{j}} - s_{i(m)j(m)}| < \frac{1}{m} \quad \text{and} \quad |s_{\hat{i}\hat{j}} - s_{i(n)j(n)}| < \frac{1}{n}$$

so that

$$|s_{i(m)j(m)} - s_{i(n)j(n)}| < \frac{1}{m} + \frac{1}{n}.$$

Now, let $L = \lim_{n\to\infty} s_{i(n)j(n)}$. It is now clear, by $(*)$, that

$$s(x) \to L \ \ (\text{stat}\, A).$$

□

The proof of our last theorem will make use of a fourth condition on (A^{pq}), i.e. one in addition to the three conditions required in the definition of a mean.

DEFINITION 10. – $A = (A^{pq})$ *is said to be a mean plus if it is a mean and satisfies the following:*

(4) *There exists a* δ, $0 < \delta < 1/2$ *and* $v_0 \in \mathbb{N}$ *such that if* $t \in \mathbb{N}$ *and* C, D *are finite non-empty sets satisfying* $C \subseteq [1,t] \times [1,t]$, $D \subseteq [t+1,\infty) \times [t+1,\infty]$ *and* $\frac{|D|}{|C|+|D|} > 1 - \delta$, *then for each* $k \in \mathbb{N}$, *there exists* $p, q \geq k$ *with*

$$\frac{\sum_{(i,j)\in D} A_{ij}^{pq}}{\sum_{(i,j)\in C\cup D} A_{ij}^{pq}} > \frac{1}{v_0}.$$

We now present our last theorem.

THEOREM 5. – *Suppose* $s = (s_{ij})$ *is a divergent sequence and* $A = (A^{pq})$ *is a mean plus. Then*

$$S_c := \{x \in X' : s(x) \text{ is a (stat) } A \text{ Cauchy sequence}\}$$

is of the first Baire category in (X', d).

PROOF. – Since A is a mean plus, there exists a δ, $0 < \delta < 1/2$ and a $v_0 \in \mathbb{N}$ satisfying the condition (4) on A in Definition 10. The set S_c can be represented as follows.

$$S_c = \bigcap_{u=1}^{\infty}\left(\bigcap_{v=1}^{\infty}\left(\bigcup_{\substack{(m,n)\in\\ \mathbb{N}\times\mathbb{N}}}\left(\bigcup_{k=1}^{\infty}\left(\bigcap_{\substack{(p,q),\,p\geq k\\ q\geq k}}\left(\left\{x \in X' : x_{mn} = 1,\ Y < \frac{1}{v}\right\}\right)\right)\right)\right)\right)$$

where

$$Y = \frac{\sum_{(i,j)} \left[A_{ij}^{pq} : |s(x)_{ij} - s(x)_{mn}| \geq \frac{1}{u},\ x_{ij} = 1 \right]}{\sum_{(i,j)} \left[A_{ij}^{pq} : x_{ij} = 1 \right]}$$

We will show now that for each fixed $(m, n) \in \mathbb{N} \times \mathbb{N}$ and each fixed $k \in \mathbb{N}$ the set

$$(**) \qquad \bigcap_{\substack{(p,q) \\ p \geq k \\ q \geq k}} \left(\left\{ x \in X' : x_{mn} = 1 \text{ and } Y < \frac{1}{v} \right\} \right)$$

is nowhere dense, for a properly selected pair (u, v) in $\mathbb{N} \times \mathbb{N}$.

To see this, note that s divergent implies that there exists a $u_0 \in \mathbb{N}$ and terms $(s_{a_i\beta_i})_{i=1}^{\infty}$, with $a_{i+1} > a_i$, $\beta_{i+1} > \beta_i$ for every i, such that $|s_{a_i\beta_i} - s_{(a_i+1)(\beta_i+1)}| > \dfrac{2}{u_0}$ for every i. Next notice that the set $T := \{x = (x_{ij}) \in X$: there exists an $N \in \mathbb{N}$ such that $x_{ij} = 0$ for every $(i,j) \notin [1, N] \times [1, N]\}$ is dense in (X, d).

Let $x = (x_{ij}) \in T$ be arbitrary, but fixed, with $x_{ij} = 0$ for every $(i,j) \notin [1, N] \times [1, N]$. Let $W = \sum x_{ij}$.

If $x_{mn} = 0$, then $x = (x_{ij})$ is an interior point of the complement of the set in $(**)$.

Now suppose $x_{mn} = 1$. Given $\gamma > 0$, there exists $L \in \mathbb{N}$, $L > N$ such that $d(x, z) < \gamma$ for every $z \in X$ satisfying $z_{ij} = x_{ij}$ for all $(i,j) \in [1, L] \times [1 \times L]$.

Now pick $M > L$ and define $x' = (x'_{ij})$ as follows:

$x'_{ij} = x_{ij}$ if $(i,j) \in [1, L] \times [1 \times L]$.
$x'_{ij} = 1$ for an even number (say $2Q$) of consecutive elements (i,j) in $\{(a_k, \beta_k)\}_{k=1}^{\infty}$ lying in $[L+1, M] \times [L+1, M]$.
$x'_{ij} = 0$ if x'_{ij} has not been defined till now.

Clearly, $x' \in T$ and if M is taken large enough, we can certainly insure that

$$\frac{Q}{Q + W} > 1 - \delta$$

Therefore, by condition (4), there exist $p, q > k$ such that

$$\frac{\sum_{(i,j) \in D} A_{ij}^{pq}}{\sum_{(i,j) \in C \cup D} A_{ij}^{pq}} > \frac{1}{v_0}$$

where

$$C = \{(i,j): x_{ij} = 1\} \quad \text{and}$$

$$D = \left\{(i,j) \in [L+1, M] \times [L+1, M]: |s(x')_{ij} - s(x')_{mn}| \geq \frac{1}{u_0},\ x'_{ij} = 1\right\}.$$

The above implies that if $z \in X'$ and $z_{ij} = x'_{ij}$ for every $(i,j) \in [1,M] \times [1,M]$, but otherwise arbitrary that z is not in the set given in $(**)$ for $u = u_0$ and $v = v_0$. Therefore $(**)$ is nowhere dense for $u = u_0$ and $v = v_0$. □

We finish the paper by remarking that:

(a) The statement in Theorem 5 is false if it is only assumed that $A = (A^{pq})$ is a mean,

(b) if $A = (A^{pq})$ is $(C,1,1)$ then A is a mean plus; i.e. if

$$A^{pq}_{ij} = \begin{cases} \dfrac{1}{pq} & \text{if } i \leq p \text{ and } j \leq q \\ 0 & \text{otherwise} \end{cases}$$

for each $p, q \in \mathbb{N}$, and

(c) statement (b) implies that Theorem 5 holds if A is $(C,1,1)$.

(d) Theorems 4 and 5 taken together yield: If $s = (s_{ij})$ is divergent and $A = A^{pq}$ is a mean plus then $\{x \in X': s(x) \text{ is (stat) } A \text{ convergent}\}$ is of the first category.

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