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Fitting Conditions for Symmetric Algebras of Modules of Finite Projective Dimension (*).

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Sunto. – *Sia E un R-modulo finitamente generato, di dimensione proiettiva finita. Studiamo l'aciclicità del complesso di approssimazione $\mathcal{Z}(E)$ di E in termini di certe condizioni Fitting $F_k^{(i)}$ sugli ideali Fitting dell'i-esimo modulo di una risoluzione proiettiva di E. Ne deduciamo alcune buone proprietà dell'algebra simmetrica di E.*

Summary. – *Let E be a finitely generated R-module, having finite projective dimension. We study the acyclicity of the approximation complex $\mathcal{Z}(E)$ of E in terms of certain Fitting conditions $F_k^{(i)}$ on the Fitting ideals of the i-th module of a projective resolution of E. We deduce some good properties of the symmetric algebra of E.*

Introduction.

Let R be a commutative, Noetherian ring and let E be a finitely generated R-module.

An important algebraic object associated to E is its symmetric algebra $S_R(E)$ on R. A recurring theme in the study of the homological properties of the symmetric algebra $S_R(E)$ is the conversion of syzygetic properties of the module E (the fine details of a projective resolution of E) into ideal theoretic properties of $S_R(E)$. The $\mathcal{Z}(E)$-complex of approximation of E showed itself useful in order to study theoretic properties of $S(E)$ from syzygetic properties of the module. Complete results have been obtained in the case that the projective dimension of E, $\operatorname{pd}(E)$, is one. As soon as one considers modules E with $\operatorname{pd} E \geq 2$, things become more and more difficult. On the other hand, in the case of projective dimension two, if the second Betti number is 1, the modules that appear in the modified complex $\mathcal{Z}(E)$ can be connected to the syzygies of the ideal generated by the greatest order minors of the matrix representing the left-hand side map of the resolution of E.

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Our goal is to introduce new conditions on the ideals of minors of the maps that appear in a free resolution of a module E. These conditions can be useful in the study of symmetric algebras of modules of higher projective dimension. They are related to the Fitting conditions, introduced in [7] only for the first map of the relations of E. Moreover they appear in [9] and in [17] to establish the acyclicity of certain complexes of free modules that only if the dimension of E is one can coincide with the $\mathcal{Z}(E)$-complex of E. The idea is to reformulate these conditions in order to study the $\mathcal{Z}(E)$-complex, that is not a complex of free modules. We study modules E whose $S_R(E)$ has a nice free resolution and we notice the close link with some good properties of the symmetric algebra. More precisely in Section 1 we introduce the conditions $F_k^{(i)}$ on the Fitting ideals of the i-th module of a projective resolution of E and we give some links with the Fitting conditions, introduced in [7].

In Section 2 we study consequences of the previous conditions on the symmetric algebras of modules E of projective dimension 2 with a ZW-free resolution, on the $\mathcal{Z}(E)$-complex and we give some examples. For modules with the second Betti number one and such that $S_R(E)$ is a ZW-resolution, we deduce that grade $I_1(f_2)$ is m or $m-1$, if

$$0 \to R \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \to E \to 0$$

is a minimal free resolution of E. In some cases this fact implies $S_R(E)$ is Cohen-Macaulay.

We conclude with some approaches to the case of modules of projective dimension 3.

1. – Preliminaries.

Let R be a commutative Noetherian ring and let E be a finitely generated R-module. Denote by $Q(R)$ the total ring of quotients of R. Assume that E has finite rank e, that is $E \otimes Q(R)$ is a free $Q(R)$-module of rank e. Suppose also that E has a free presentation

$$R^m \xrightarrow{\varphi} R^n \to E \to 0$$

By the assumptions we have rank $\varphi = n - e$.

Very useful conditions are the following sliding requirements on the sizes of the ideals $I_t(\varphi)$, ideals generated by the $t \times t$ minors of φ, i.e. $I_{n-t}(\varphi) = F_t(E)$ is the t-th Fitting ideal of E.

If $k \geq 0$ is a fixed integer, in [7] the authors defined the condition F_k :

(1) $$F_k : \ \operatorname{ht}(I_t(\varphi)) \geq \operatorname{rank}(\varphi) - t + 1 + k, \qquad 1 \leq t \leq \operatorname{rank}(\varphi).$$

In terms of the Fitting ideals this can be written:

$$\text{(2)} \qquad \operatorname{ht}(F_s(E)) \geq s - \operatorname{rank}(E) + 1 + k, \qquad \operatorname{rank}(E) \leq s.$$

In [7] the authors gave the conditions F_k in terms of the local number of generators of the module M:

$F_k :=$ For each prime ideal $\wp$ of $R,$ such that $E_\wp$ is not a free $R_\wp$ – module,

we have

$$\text{(3)} \qquad \upsilon(E_\wp) \leq \operatorname{ht}(\wp) + \operatorname{rank}(E) - k$$

where $\upsilon(E_\wp)$ is the minimum number of generators of $E_\wp$.

Suppose now that E has finite projective dimension. Then condition F_k can be rephrased as follows: for any prime ideal $\wp$ such that $E_\wp$ is not a free module,

$$\operatorname{rank} L \leq \operatorname{ht}(\wp) - k,$$

where L is the first module of syzygies of $E_\wp$.

The condition F_k on E, together with some other conditions, leads to the acyclicity of the approximation complex $\mathcal{Z}(E)$ of E if $\operatorname{pd}(E) = 1$:

$$E \quad \text{is} \quad (F_1) \Longleftrightarrow \mathcal{Z}(E) \quad \text{is acyclic.}$$

If $\operatorname{pd}(E) \geq 2$ one has to consider other conditions.

Another use of the conditions (F_k) lies in the following characterization of complete intersections.

If R is a Cohen-Macaulay local ring, then $S(E)$ is a complete intersection if and only if the projective dimension of E is at most one and E has (F_0).

In addition, if R is a domain (resp. R is U.F.D.), then $S(E)$ is a domain (resp. $S(E)$ is U.F.D.) if and only if (F_1) (resp. (F_2)) is satisfied ([1], [15]*).*

In this paper we study the acyclicity of $\mathcal{Z}(E)$, when E is a finitely generated module with projective dimension 2 in terms of certain conditions $F_k^{(i)}$ on the modules of syzygies of E.

REMARK 1.1. – We shall use the following notion of grade of an ideal $I \subset R$.

If I an ideal of R such that $IE \neq E$, then by $\operatorname{grade}_I(E)$ we mean the common length of all maximal E-sequences contained in I.

If R is local with maximal ideal $\mathfrak{m}$, we set $\operatorname{depth} E = \operatorname{grade}_{\mathfrak{m}}(E)$.

We put also $\operatorname{grade}(I) := \operatorname{grade}_I(R)$. We have

$$\operatorname{grade}(I) = \operatorname{grade}_R(R/I) = \inf\{\operatorname{depth} R_\wp \mid \wp \in V(I)\}.$$

DEFINITION 1.2. – *Let E be a finitely generated R-module having a finite free resolution*

$$F.: \ 0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$$

with $f_j \neq 0$ for $j = 1, \ldots, n$. Denote by r_j the rank of f_j and by $I_s(f_j)$ the ideal of its $s \times s$ minors.

For integers $j, k \geq 1$ we say that E satisfies the condition $F_k^{(j)}$ if

$$\text{ht}\,(I_t(f_j)) \geq \text{rank}\,(f_j) - t + 1 + k, \qquad 1 \leq t \leq \text{rank}\,(f_j). \tag{4}$$

REMARK 1.3. – Let

$$\cdots \to R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0$$

be a free resolution of E and let L be the kernel of f_0. Suppose that E has a rank and that rank $(E) = e$. It is clear that:

i) E has $F_k^{(2)} \iff L$ has F_k.
ii) E has $F_k^{(1)} \iff E$ has F_k.
iii) E has $F_k^{(2)} \iff$ for each prime ideal $\wp$, such that pd $E_\wp > 1$,

$$\upsilon(L_\wp) \leq \text{ht}\,\wp + n - k - e.$$

EXAMPLE 1.4. – Let E be a finitely generated R-module such that pd $E = 2$ and rk $E = e$, with a minimal free resolution

$$0 \to R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0$$

If E has $F_k^{(1)}, k \neq 2$ and $m \geq 2s$, then E has $F_k^{(2)}$.

Indeed, let $L := \ker(f_0)$ and consider a prime ideal $\wp$ such that pd $(E_\wp) > 1$. From the condition $F_k^{(1)}$ it follows that

$$\upsilon(E_\wp) \leq \text{ht}\,(\wp) + e - k$$

or in other words

$$\text{ht}\,(\wp) \geq m - s + k \geq s + k.$$

But one can easily check that this is exactly the condition $F_k^{(2)}$.

REMARK 1.5. – Let E be a module, pd $E = 2$ having a resolution

$$0 \to R \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \to E \to 0.$$

Let $I = I_1(f_2)$ and suppose that E has $F_1^{(2)}$. Then ht $\wp \geq 2$, where $\wp \in$ Min (I).

In fact, E has $F_1^{(2)}$ implies that rank $(f_2) \leq$ ht $\wp - 1$, that is ht $\wp \geq 2$.

REMARK 1.6. – Let E be a module, pd $E = 3$, having a resolution:

$$0 \to R \xrightarrow{f_3} R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \to E \to 0.$$

Suppose that E has $F_1^{(2)}$. Then $I := I_1(f_3)$ has F_1.

In fact, since E has $F_1^{(2)}$, it follows that $\operatorname{rank} f_2 \leq \operatorname{ht} \wp - 1$. But

$$\operatorname{rank} f_2 = \operatorname{rank}(\operatorname{Im} f_2) = s - 1$$

and it follows that $\operatorname{ht} \wp \geq s \geq \upsilon(I)$.

REMARK 1.7. – In [17] Tchernev constructed canonical complexes of free modules denoted by $\mathcal{S}^i F.$ and $\mathcal{L}^i F.$, whose zero-th homology is $S_i(E)$ and $\overset{i}{\Lambda}(E)$ respectively. These complexes are combinations of symmetric and exterior powers of the free modules in $F.$, with naturally induced maps between them.

If $\lambda(F.) := \sup \{i \mid F_i \neq 0\} = m < \infty$ and $F.$ has no gaps (i.e. $F_i \neq 0$ for $0 \leq i < \lambda(F.)$), then $\mathcal{S}^k F.$ and $\mathcal{L}^k F.$ are complexes of finite free modules such that for each $k \geq 0$ one has

$$\lambda(\mathcal{S}^k F.) = \begin{cases} mk & \text{for even } m \\ (m-1)k + \min(k, r_m) & \text{for odd } n \end{cases}$$

$$\lambda(\mathcal{L}^k F.) = \begin{cases} mk & \text{for odd } m \\ (m-1)k + \min(k, r_m) & \text{for even } n \end{cases}$$

where $r_m = \operatorname{rank} F_m$. Using the divided power algebra, Tchernev constructed also the complexes $\mathcal{D}^i F.$ and $\mathcal{G}^i F.$ having similar properties with $\mathcal{L}^i F.$ and $\mathcal{S}^i F.$ respectively(see [17]).

REMARK 1.8. – Let E be as in 1.7 and let

$$F. = 0 \to F_m \to \cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} E \to 0$$

be a free resolution of E with $F_i \neq 0$, $\forall i = 0, \ldots, n$. Let $b_j = \operatorname{rank} F_j$ and let $r_j = \sum_{k=1}^{n} (-1)^{k-j} b_k$ be the expected rank of f_j. We denote by $I_t(f_j)$ the ideal of $t \times t$ minors of f_j, where

$$I_t(f_j) = \begin{cases} 0 & \text{for } t > \min(b_j, b_{j-1}) \\ R & \text{for } t \leq 0. \end{cases}$$

We shall also consider the following *grade conditions* on the ideals $I_t(f_j)$:

i) The *grade condition* $\mathbf{GC}^k(j)$. For integers $j, k \geq 1$ we say that the condition $\mathbf{GC}^k(j)$ is fulfilled if

$$\operatorname{grade} I_{r_j}(f_j) \geq kj.$$

ii) The *sliding grade condition* $\mathbf{SGC}^k(j)$. For integers $j, k \geq 1$ we say that the condition $\mathbf{SGC}^k(j)$ is fulfilled if

$$\operatorname{grade} I_{r_j - t}(f_j) \geq k(j-1) + 1 + t, \ \forall\, t = 0, \ldots, k-1$$

REMARK 1.9. – It is easy to show that the condition $F_k^{(2)}$ coincides with the condition $\mathbf{SGC}^k(2)$.

REMARK 1.10. – In [7] the authors introduced the following condition, that we call $F_1^{(1)*}$:

$$F_1^{(1)*} : \upsilon(E_\wp) \leq \frac{1}{2}(\mathrm{ht}(\wp) - 1) + \mathrm{rank}\,(E), \forall\ \wp \in \mathrm{Spec}\,(R).$$

By this condition it is possible to have the reflexivity of the exterior powers of the module $L = Z_1(E)$ (see [7]).

2. – Symmetric algebra of modules of projective dimension 2 with a ZW-free resolution.

Let A be a noetherian *-local graded ring with unit, such that the *-maximal ideal is also a maximal ideal in the usual sense(see [4]). If we consider a graded finitely generated A-module G, it has a minimal graded resolution by finitely generated free A-modules,

$$F. := \ldots \to \underset{j}{\oplus} A[-a_{ij}]^{\beta_{ij}} \to \underset{j}{\oplus} A[-a_{i-1,j}]^{\beta_{i-1,j}} \to \ldots$$

$$\ldots \to \underset{j}{\oplus} A[-a_{0j}]^{\beta_{0j}} \to G \to 0$$

where $A[-t]$ is the graded A-module $A[-t] = \overset{\infty}{\underset{i=t}{\oplus}} A_{i-t}$.

Since the resolution is graded, its differentials are homogeneous homomorphisms of degree 0. By our assumption on A it follows that β_{ij} are the graded Betti numbers of G and that for every i, $\beta_i = \sum_j \beta_{ij}$ is the usual i-th Betti number of G. Moreover, since the resolution is minimal, we have $\min_j \beta_{ij} < \min_j \beta_{i-1,j}$. Consequently, for a given j, there exist only finitely many i with $\beta_{ij} \neq 0$.

Now suppose that R is a local ring and let E be a finitely generated R-module with a free presentation

$$R^m \to R^n \to E \to 0.$$

Then $S(E)$ is naturally an S-module, where $S = R[X_1, \ldots, X_n]$. We can consider a minimal graded S-resolution of $S(E)$, $\{\mathcal{G}, \partial\}$. This resolution in general is infinite.

The fact that $\{\mathcal{G}, \partial\}$ is minimal means that $\partial(\mathcal{G}) \subset (\mathfrak{m}S + S_+)\mathcal{G}$, where S_+ is the maximal irrelevant ideal of S, $S_+ = (X_1, \ldots, X_n)S$ and m is the maximal ideal of R. We repeat here the argument in [7].

For all i we consider the filtration $\{\mathcal{F}_{-i}\mathcal{G}\}$ on $\mathcal{G}$, given by

$$(\mathcal{F}_{-i}\mathcal{G})_j = \underset{a_{jk} \leq i}{\oplus} S[-a_{jk}]$$

where a_{jk} are the shifts in the resolution

$$\ldots \to \underset{k}{\oplus} S[-a_{jk}]^{\beta_{jk}} \to \ldots \to \underset{k}{\oplus} S[-a_{1k}]^{\beta_{1k}} \to \underset{k}{\oplus} S[-a_{0k}]^{\beta_{0k}} \to S(E) \to 0.$$

It is clear that $\mathcal{F}_{-i+1}\mathcal{G}$ is a subcomplex of $\mathcal{F}_{-i}\mathcal{G}$ and that

$$\mathcal{F}_{-i}\mathcal{G}/\mathcal{F}_{-i+1}\mathcal{G} = \underset{a_{jk}=i}{\oplus} S[-a_{jk}] = \underset{a_{jk}=i}{\oplus} S[-i]^{\beta_{jk}} = \underset{a_{jk}=i}{\oplus} S[-i] \otimes R^{\beta_{jk}}.$$

Now, for a graded S-module M, put $M^* = M/S_+M$. Then we have:

$$(S[-i] \otimes R^{\beta_{jk}})/S_+ \cong R[-i]^{\beta_{jk}}.$$

Finally

$$\mathcal{G}^* = \underset{i\geq 0}{\oplus} \left(S[-i] \otimes R^{\beta_{jk}}\right)^* = \underset{i\geq 0}{\oplus} \mathcal{C}_i,$$

where $\mathcal{C}_i$ are complexes of R-modules, $\mathcal{C}_i = \underset{k}{\oplus} \mathcal{C}_{ik}$ which, considered as S-modules, are concentrated in degree i.

Now, for the Koszul homology we have

$$H_i(S_+; S(E)) = \mathrm{Tor}_i^S(R, S(E)) = H_i(\mathcal{G}^*)$$

and

$$H_i(S_+; S(E))_j = H_i(\mathcal{C}_j).$$

It follows:

THEOREM 2.1. – *The following conditions are equivalent:*

1) $\mathcal{Z}(E)$ *is acyclic;*
2) *all the complexes* $\mathcal{C}_i$ *are acyclic.*

If the equivalent conditions hold, then $\mathcal{C}_i$ *is a minimal R-free resolution of* $\mathcal{Z}_i(E)$ *shifted i steps to the left,* $\mathcal{C}_i \otimes S[-i]$, *and one has:*

$$\beta_i^S(S(E)) = \sum_j \beta_{i-j}^R(\mathcal{Z}_j(E)) = \sum_j \beta_{i-j}^R(\mathcal{C}_j).$$

PROOF. – See [7], Theorem 5.8. □

PROPOSITION 2.2. – *Suppose* $\mathcal{Z}(E)$ *is acyclic and* $\mathcal{C}'_i$ *is any free minimal resolution of* $Z_i(E)$ *as an R-module, for every* $i < \operatorname{rank} E$. *Then*

$$\beta_i^S(S(E)) = \sum_j \beta_{i-j}^R(\mathcal{C}'_j).$$

PROOF. – Since $\mathcal{Z}(E)$ is acyclic, all complexes $\mathcal{C}_i$ of Theorem 2.1 are acyclic and $\mathcal{C}_i$ is a minimal free resolution of $\mathcal{Z}_i(E)$ over R. Since $\mathcal{C}'_i$ is a minimal R-free resolution of $\mathcal{Z}_i(E)$, we have $\beta_j^R(\mathcal{C}'_i) = \beta_j(\mathcal{C}_i)$ for every j. The assertion follows. □

If $\mathcal{Z}(E)$ is acyclic, then the S-resolution of $S(E)$ looks like

$$\begin{aligned} \dots \quad \to \quad & \bigoplus_{j=1}^{i} S[-j]^{\beta_{j,i-j}} \to \dots \to S[-2]^{\beta_{2,0}} \oplus S[-1]^{\beta_{1,1}} \\ \to \quad & S[-1]^{\beta_{1,0}} \to S \to S(E) \to 0. \end{aligned}$$

In [11] the conditions (EW_i) and (SW_i) for a module E with a free resolution $F.$ have been introduced.

DEFINITION 2.3. – *Let $i \geq 1$ be an integer. We say that E satisfies (EW_i) (respectively (SW_i)) if the complex $\mathcal{L}_i(F.)$ (resp. $\mathcal{S}_i(F.)$) is a finite free resolution of $\overset{i}{\Lambda} E$ (resp. $S_i(E)$).*

REMARK 2.4. – i) $\mathcal{L}_i(F.)$ and $\mathcal{S}_i(F.)$ are the Weyman-Tchernev complexes (see 1.7).

ii) If we consider a torsion free module E, with rank e, free at the primes $\wp$ with depth $R_\wp \leq 1$, then the complex $\mathcal{Z}(E)$ can be identified with the complex $\mathcal{K}(E)^{**}$, where $\mathcal{K}(E)$ is the Koszul complex associated to a presentation of E:

$$0 \to Z_1(E) = L \to R^n \to E \to 0.$$

Now suppose that the $\mathcal{Z}(E)$-complex of the module E is acyclic. Then $S(E)$ has graded S-free minimal resolution given by 2.2.

DEFINITION 2.5. – *Let E be a torsion free R-module of finite projective dimension, with* $\operatorname{rk} E = e$. *We say that $S(E)$ has the property ZW if:*

1) *the R-module $L = Z_1(E)$ has the property EW_i, for all i, $i \leq n - e$;*
2) *$\overset{i}{\Lambda} L$ are reflexive, for $0 \leq i \leq n - e$.*

REMARK 2.6. – It follows that for an R-module E as in 2.5, if $S(E)$ has the property ZW, then $Z_i(E) \cong \overset{i}{\Lambda} L$ admits a finite free resolution given by the Weyman-Tchernev complex, $\forall\ 1 \leq i \leq n - e$.

REMARK 2.7. – If $S(E)$ has ZW, then we can compute the projective dimension of $S(E)$(see [17]).

PROPOSITION 2.8. – *Suppose R is a Cohen-Macaulay ring, E is a finitely generated R-module,* $\operatorname{pd} E = 2$ *and consider a minimal free resolution of E*

$$0 \to R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0$$

Let $L = \ker f_0$. *Suppose that:*

1) *E has* $F^{(2)}_{m-s}$.
2) $s \geq 2$.

Then we have:

(i) $\operatorname{pd} \overset{i}{\Lambda} L \leq i, \forall\, i \leq m-s$;

(ii) $\overset{i}{\Lambda} L$ *is reflexive, for all* $i \leq m-s$.

PROOF. – (i) By Eagon and Northcott ([6]), $\operatorname{ht} I_s(f_2) \leq m-s+1$.

The fact that E has $F^{(2)}_{m-s}$ implies

$$\begin{aligned}\operatorname{ht}(I_s(f_2)) &\geq \operatorname{rank} f_2 - s + 1 + m - s\\ &= s - s + 1 + m - s = m - s + 1.\end{aligned}$$

It follows that $\operatorname{ht} I_s(f_2) = m-s+1$.

Then, since $\operatorname{pd} L = 1$, from ([17], Theorem 4.1) it follows that $\mathcal{D}^i F.$ is a free resolution of $\overset{i}{\Lambda} L$, $\forall\, i = 1, \dots, m-s$.

(ii) From $F^{(2)}_{m-s}$ we have that for any prime ideal $\wp$ of R such that $L_\wp$ is not free,

$$\upsilon(L_\wp) \leq \operatorname{ht} \wp + \operatorname{rank} L - (m-s),$$

hence

$$m \leq \operatorname{ht} \wp + m - s - (m-s),$$

so $\operatorname{ht} \wp \geq m \geq s \geq 2$. If $\operatorname{ht} \wp \geq 2$ we have:

$$\operatorname{depth}(\overset{i}{\Lambda} L)_\wp = \operatorname{ht} \wp - \operatorname{pd}(\overset{i}{\Lambda} L_\wp) \geq \operatorname{ht} \wp - i \geq m - m + s = s \geq 2.$$

It follows that $\overset{i}{\Lambda} L$ is reflexive, for $i \leq m-s$. □

COROLLARY 2.9. – *Suppose* R *is a Cohen-Macaulay ring,* E *is a finitely generated* R*-module,* $\operatorname{pd} E = 2$ *and consider a minimal free resolution of* E

$$0 \to R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0$$

Let $L = \ker f_0$. *Suppose that:*

1) *E has* $F^{(2)}_{m-s}$.
2) $s \geq 2$.

Then $S(E)$ *has a ZW-resolution.*

We can prove a variant of 2.8 if moreover R is regular.

PROPOSITION 2.10. – *Let* R *be a regular local ring,* E *a finitely generated* R*-module,* $\operatorname{pd} E = 2$ *and consider a minimal free resolution of* E

$$0 \to R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0$$

Let $L = \ker f_0$. Suppose that E has $F^{(2)}_{m-1}$. Then we have:

(i) $\operatorname{pd} \overset{i}{\Lambda} L \leq i, \forall\, i \leq m$;

(ii) $\overset{i}{\Lambda} L$ *is reflexive,* $\forall\, i \leq \min(m, m+s-3)$.

PROOF. – We use ([5], Theorem A) instead of [6]. Then we obtain that $\operatorname{ht}(I_s(f_2)) \leq m$. Using the property $F^{(2)}_{m-1}$ it follows that

$$\operatorname{ht}(I_s(f_2)) \geq s - s + 1 + m - 1 = m$$

and consequently $\operatorname{ht}(I_s(f_2)) = m$. Then i) follows as in 2.8.

As regarding ii), from the property $F^{(2)}_{m-1}$ it follows that for any prime ideal $\wp$ such that $L_\wp$ is not free

$$m \leq \operatorname{ht}(\wp) + m - s - (m-1)$$

that is

$$\operatorname{ht}(\wp) \geq m - s + 1.$$

Now

$$\operatorname{depth}(\overset{i}{\Lambda} L)_\wp = \operatorname{ht} \wp - \operatorname{pd}(\overset{i}{\Lambda} L_\wp) \geq m + s - 1 - i \geq 2.$$

COROLLARY 2.11. – *Let R be a regular local ring, E a finitely generated R-module,* $\operatorname{pd} E = 2$. *Suppose that E has $F^{(2)}_{m-1}$ and that $s \geq 2$. Then $S(E)$ has a ZW-resolution.*

THEOREM 2.12. – *Let R be a Cohen-Macaulay local ring and E a finitely generated torsion-free R-module, with* $\operatorname{pd} E = 2$, *with a minimal free resolution*

$$0 \longrightarrow R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \longrightarrow 0$$

Suppose that

1) *E has $F^{(1)*}_1$;*
2) *E has $F^{(2)}_{m-s}$, $s \geq 2$.*

Then $\mathcal{Z}(E)$ is acyclic and $S(E)$ is torsion free as an R-module.

PROOF. – We can suppose that $m - s \geq 1$. By 1), for any prime ideal $\wp$ such that $E_\wp$ is not free, we have $\operatorname{ht}(\wp) \geq 2(m-s)+1$, that is $\operatorname{ht}(\wp) \geq (m-s)+2$ and by 2.8 we obtain that $\overset{i}{\Lambda} L$ is reflexive for all $i < (m-s)+1$. Now we have to prove that $\operatorname{depth}(\overset{i}{\Lambda} L_\wp)^{**} \geq i+1$. By ii) of Proposition 2.8

$$\operatorname{depth}(\overset{i}{\Lambda} L_\wp)^{**} \geq \operatorname{ht}(\wp) - i \geq 2(m-s) + 1 - (m-s) - 1 = m - s,$$

so that $\mathcal{Z}(E)$ is acyclic and $\operatorname{depth} S(E) > 0$.

PROPOSITION 2.13. – *Let R be a Cohen-Macaulay local domain, containing a field of characteristic zero, E a finitely generated torsion free R-module with a minimal free resolution*

$$0 \to R \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0. \tag{5}$$

Suppose that $S_R(E)$ has a ZW-resolution. Then:

$$\operatorname{grade} I_1(f_2) = \rho \in \{m-1, m\}.$$

PROOF. – Let us denote $I_1 := I_1(f_2)$. Since $S_R(E)$ has a ZW-resolution, $\overset{i}{\Lambda} L$ has a resolution of length i, for all $i \leq m-1$. We have

$$\overset{i}{\Lambda} L \cong (\overset{i}{\Lambda} L)^{**} \cong Z_i(E), \forall i \leq m-1$$

and $\operatorname{pd} \overset{i}{\Lambda} L = i$. By [17], Theorem 1 we have

$$\operatorname{grade} I_1(f_2) \geq m-1 \geq i$$

because of the exactness of $\mathcal{L}_{m-2}F.$, which is a resolution of $\overset{m-2}{\Lambda} L$. Moreover,

$$L \cong Z_1(I_1)^*$$

and by duality,

$$Z_t(E) \cong Z_{(m-1)-t}(I_1).$$

For $t = m-2$ we have

$$Z_1(I_1) \cong Z_{m-2}(E),$$
$$\operatorname{depth} Z_1(I_1) \geq d-(m-2).$$

Consider the exact sequences:

$$0 \to Z_1(I_1) \to R^{m-1} \to I_1 \to 0 \tag{6}$$

$$0 \to I_1 \to R \to R/I_1 \to 0 \tag{7}$$

By (6) we have

$$\operatorname{depth} Z_1(I_1) = \operatorname{grade} I_1 + 1,$$

$$\operatorname{grade} I_1 = \operatorname{depth}(Z_1(I_1)) - 1 \geq d-(m-2)-1 = d-(m-1).$$

By (7), since $\operatorname{depth} R > \operatorname{depth} R/I_1$ it follows that

$$\operatorname{depth} I_1 = \operatorname{depth} R/I_1 + 1$$

and consequently

$$\operatorname{depth} R/I_1 = \operatorname{grade} I_1 - 1 = \operatorname{depth} Z_1(I_1) - 2 \geq d-(m-2)-2 = d-m,$$

hence $\operatorname{grade} I_1 \leq m$. □

In the following we examine some examples where grade $I_1 = m$.

EXAMPLE 2.14. – Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local domain, $\dim R = d > 3$. Let $x_1, \dots, x_d$ be a system of parameters of R and consider the ideal $I = (x_1, x_2, x_3)$. Let $\mathcal{K}(x_1, x_2, x_3; R)$ be the Koszul complex associated to the system of elements x_1, x_2, x_3. Then we have the resolution

$$0 \to R \xrightarrow{f_2} \overset{2}{\Lambda} R^3 \xrightarrow{f_1} R^3 \xrightarrow{f_0} I \to 0. \tag{8}$$

Consider the R-module $E = I$. We have $\operatorname{pd} I = 2$ and $I_1(f_2) = (x_1, x_2, x_3)$, so that $\operatorname{ht} I_1(f_2) = 3$. Moreover

$$0 \to R \xrightarrow{f_2} \overset{2}{\Lambda} R^3 \to L \to 0$$

is a resolution of $L := \ker(f_0)$ and $\operatorname{rank} L = 2$.

We have

$$(\overset{2}{\Lambda} L)^{**} \cong R,$$
$$S = S(R^3) = R[T_1, T_2, T_3]$$

and the $\mathcal{Z}(E)$-complex is:

$$\begin{array}{ccccccccccc} & & & & 0 & & & & & & \\ & & & & \downarrow & & & & & & \\ & & & & S[-1] & & & & & & \\ & & & & \downarrow & & & & & & \\ & & & & S[-1]^3 & & & & & & \\ & & & & \downarrow & & & & & & \\ 0 & \to & R \otimes S[-2] & \to & L \otimes S[-1] & \to & S & \to & S(E) & \to & 0 \\ & & & & \downarrow & & & & & & \\ & & & & 0 & & & & & & \end{array}$$

The ZW-resolution is

$$0 \to S[-2] \otimes S[-1] \to S[-1]^3 \to S \to S(E) \to 0. \tag{9}$$

Since $d > \operatorname{rank} L + 1 = 3$, E satisfies $F_1^{(1)}$, hence $S(E)$ has the expected Krull dimension,

$$\dim S(E) = d + \operatorname{rank} E = d + 1.$$

Now we compute $\operatorname{depth} S(E)$. We have $\operatorname{pd}_S(S(E)) \leq 2$, and

$$\operatorname{depth}_{(\mathfrak{m}, S^+)} S(E) \geq \dim S - 2 = d + 3 - 2 = d + 1.$$

Then $S(E)$ is a Cohen-Macaulay ring.

EXAMPLE 2.15. – Under the same hypotheses as in Example 2.14, let R be a Cohen-Macaulay local domain, $\dim R = d > 4$, $I = (x_1, x_2, x_3, x_4)$ an ideal generated by a subset of a system of parameters of R. The resolution of I is:

$$0 \to R \to R^4 \xrightarrow{f_2} \overset{2}{\Lambda} R^4 \to R^4 \to I \to 0. \tag{10}$$

Consider $E = \operatorname{Coker}(\psi)$ and the tail of the Koszul complex:

$$0 \to R \xrightarrow{f_2} R^4 \xrightarrow{f_1} \overset{2}{\Lambda} R^4 \xrightarrow{f_0} E \to 0. \tag{11}$$

If we put $L := \ker(f_0)$ we have

$$\operatorname{ht} I(f_2) = \operatorname{ht}(x_1, x_2, x_3, x_4) = 4, \ \operatorname{rank} L = 3.$$

Now the $\mathcal{Z}(E)$-complex is

$$0 \to R \otimes S[-3] \to (\overset{2}{\Lambda} L)^{**} \otimes S[-2] \to L \otimes S[-1] \to S \to S(E) \to 0$$

where $S = \operatorname{Sym}(R^4) = R[T_1, T_2, T_3, T_4]$.

Since $\operatorname{ht} I(f_2) = 4$ and $d > 3 + 1 = 4$, it follows that $\overset{2}{\Lambda} L$ is reflexive and has a free resolution given by the Weyman-Tchernev complex.

$$0 \to S_2(R) \otimes R \to S_1(R) \otimes R^4 \to \overset{2}{\Lambda} R^4 \to \overset{2}{\Lambda} L \to 0$$
$$0 \to R \otimes R \to R \otimes R^4 \to R^6 \to \overset{2}{\Lambda} L \to 0$$
$$0 \to R \to R^4 \to R^6 \to \overset{2}{\Lambda} L \to 0.$$

Then we have:

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & S[-2] & & 0 & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & & & \\
 & & & & S[-2]^4 & & S[-1] & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & & & \\
 & & & & S[-2]^6 & & S[-1]^4 & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & & & \\
0 & \to & R \otimes S[-3] & \to & (\overset{2}{\Lambda} L)^{**} \otimes S[-2] & \to & L \otimes S[-1] & \to & S & \to & S(E) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & & & \\
 & & & & 0 & & 0 & & & & & &
\end{array}$$

and the ZW-resolution is:

$$0 \to S[-2] \to S[-3] \otimes S[-2]^4 \to S[-2]^6 \otimes S[-1] \to S[-1]^4 \to S \to S(E) \to 0$$

We have

$$\operatorname{pd}_S S(E) = 4, \operatorname{rank} E = 3$$
$$\dim S(E) = d + \operatorname{rank} E = d + 3,$$
$$\operatorname{depth} S(E) = d + n - 4 = d + 4 - 4 = d,$$

hence $S(E)$ is not Cohen-Macaulay.

Finally we indicate an application of the previously introduced Fitting conditions for modules of projective dimension 3.

PROPOSITION 2.16. – *Let R be a Cohen-Macaulay local ring containing a field of characteristic zero. Let E be a finitely generated R-module, $\operatorname{pd}(E) = 3, \operatorname{rank}(E) = e$ and let $k \geq 1$ be an integer. Consider a free resolution of E*

$$0 \to R^t \xrightarrow{f_3} R^s \xrightarrow{f_2} R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0.$$

Let $L = \ker(f_0)$. Suppose that:

i) E has the property $F^{(2)}_{k-1}$;

ii) E has the property $F^{(3)}_{k}$.

Then:

a) $\overset{i}{\Lambda} L$ has a free resolution, for all $i \leq k$;

b) If $k \leq t$, then $\operatorname{pd} \overset{i}{\Lambda} L \leq 2k, \forall i \leq k$.

c) If $k > t$, then $\operatorname{pd} \overset{i}{\Lambda} L \leq k + t, \forall i \leq k$.

PROOF. – a) We apply ([17], 2.2) for the module L. In order to do that, we need to show that the conditions $\mathbf{GC}^k(1)$ and $\mathbf{SGC}^k(2)$ corresponding to the module L hold.

1) *Condition* $\mathbf{GC}^k(1)$.

We have to show that $\operatorname{ht}(I_{s-t}(f_2)) \geq k$. From the assumption i) it follows that

$$\operatorname{ht}(I_h(f_2)) \geq \operatorname{rk}(f_2) - h + 1 + k - 1, \ 0 \leq h \leq \operatorname{rk}(f_2) = s - t.$$

For $h = s - t$ we obtain

$$\operatorname{ht}(I_{s-t}(f_2)) \geq s - t - s + t + 1 + k - 1 = k.$$

3) *Condition* $\mathbf{SGC}^k(2)$.

From the condition $F^{(3)}_k$ it follows that

$$\operatorname{ht}(I_h(f_3)) \geq t - h + 1 + k, \ h = 1, \ldots, t.$$

From this one gets

$$\begin{aligned} \mathrm{ht}\,(I_t(f_3)) &\geq k+1, \\ \mathrm{ht}\,(I_{t-1}(f_3)) &\geq k+2, \\ &\vdots \\ \mathrm{ht}\,(I_{t-k+1}(f_3)) &\geq 2k. \end{aligned}$$

Now we can easily check that these are exactly the conditions $\mathbf{SGC}^k(2)$ for the module L(see also 1.9).

b), c) Follows imediately from a) and from ([17],1.9). □

COROLLARY 2.17. – *In the situation of* 2.16, *suppose moreover that* $k \geq \max(t+1, n-e)$ *and* $s \geq 2t+1$. *Then* $S(E)$ *has a ZW-resolution.*

PROOF. – Since $k > t$, from 2.16 it follows that $\mathrm{pd} \overset{i}{\Lambda} L \leq k+t, \forall i \leq k$. Let $\wp$ be a prime ideal such that $L_\wp$ is not free. From the condition $F^{(2)}_{k-1}$ it follows that

$$\mathrm{ht}\,(\wp) \geq s-t+k+1.$$

Then for all $i \leq k$ we have

$$\mathrm{depth} \overset{i}{\Lambda} L_\wp = \mathrm{ht}\,(\wp) - \mathrm{pd}\,(\overset{i}{\Lambda} L_\wp) \geq s-t+k+1-k-t = s+1-2t \geq 2.$$

It follows that $\overset{i}{\Lambda} L$ is reflexive for all $i \leq n-e$. □

REFERENCES

[1] L. L. AVRAMOV, *Complete intersections and symmetric algebras*, J. Algebra, **73** (1981), 248-263.

[2] M. AUSLANDER - M. BRIDGER, *Stable module theory*, Mem. Amer. Math. Soc. **94** (1969).

[3] D. BUCHSBAUM - D. EISENBUD, *What makes a complex exact?*, J. Algebra, **25** (1973), 259-268.

[4] W. BRUNS - J. HERZOG, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1998.

[5] D. EISENBUD - C. HUNEKE - B. ULRICH, *Heights of ideals of minors*, Amer. J. Math., **126** (2004), 417-438.

[6] J. EAGON - D. G. NORTHCOTT, *Ideals defined by matrices and a certain complex associated with them*, Proc. Royal Soc. London Ser. A, **269** (1962), 188-204.

[7] J. HERZOG - A. SIMIS - W. V. VASCONCELOS, *On the arithmetic and homology of algebras of linear type*, Trans. Amer. Math. Soc., **283** (1984), 661-683.

[8] J. HERZOG - A. SIMIS - W. V. VASCONCELOS, *Koszul homology and blowing-up rings*, Commutative algebra (Trento, 1981), pp. 79-169, Lecture Notes in Pure and Appl. Math., 84, Dekker, New York, 1983.

[9] M. KÜHL, *On the symmetric algebra of an ideal*, Manuscripta Math., **37** (1982), 49-60.

[10] K. LEBELT, *Freie Auflösungen äuerer Potenzen*, Manuscripta Math., **21** (1977), 341-355.
[11] G. MOLICA - G. RESTUCCIA, *Torsion free exterior powers of a module and their resolutions*, An. Şt. Univ. Ovidius Constanta, **10** (2002), 101-108.
[12] G. RESTUCCIA, *On the ideal of relations of a symmetric algebra*, Rend. Sem. Mat. Univers. Politecn. Torino, **49** (1991), 281-298.
[13] G. RESTUCCIA, *On the symmetric algebra for a module of projective dimension two*, An. St. Univ. Bucarest Mat., **40** (1991), 83-91.
[14] G. RESTUCCIA - C. IONESCU, *q-torsion freeness of symmetric powers*, Rend. Circ. Mat. Palermo, **XLVI** (1997), 329-346.
[15] A. SIMIS - W. V. VASCONCELOS, *The syzygies of the conormal module*, Amer. J. Math. **103** (1981), 203-224.
[16] R. UTANO - G. RESTUCCIA, *Integrity of the symmetric algebra of modules of projective dimension two*, Riv. Mat. Univ. Parma, **4** (1995), 193-204.
[17] A. B. TCHERNEV, *Acyclicity of symmetric and exterior powers of complexes*, J. Algebra, **184** (1996), 1113-1135.
[18] W. V. VASCONCELOS, *On linear complete intersections*, J. Algebra, **111** (1987), 306-315.
[19] W. V. VASCONCELOS, *Symmetric algebras*, Lect. Notes in Math., **1430** (1988), 115-160.
[20] J. WEYMAN, *Resolutions of the exterior and symmetric powers of a module*, J. Algebra, **58** (1979), 333-341.

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