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L^p Regularity of Transmission Problems in Dihedral Domains.

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Sunto. – *Consideriamo il problema di trasmissione per l'operatore di Laplace per un cilindro retto con dati in L^p. Applicando la teoria delle somme di operatori negli spazi di Banach, dimostriamo che la soluzione ammette una decomposizione in una parte regolare in $W^{2,p}$ e una parte singolare esplicita.*

Summary. – *We consider the transmission problem for the Laplace operator in a straight cylinder with data in L^p. Applying the theory of the sums of operators in Banach spaces, we prove that the solution admits a decomposition into a regular part in $W^{2,p}$ and an explicit singular part.*

1. – Introduction.

Let $\mathcal{B} = G \times \mathbb{R}$ where G is the bounded plane sector of opening $\omega > 0$ defined by

$$G = \{(r\cos\theta, r\sin\theta); 0 < \theta < \omega, 0 < r < 1\}.$$

It is well known that the variational solution of the Dirichlet problem

$$\left\{\begin{array}{rcll} -\Delta u & = & f & \text{in } \mathcal{B}, \\ u & = & 0 & \text{on } \partial\mathcal{B}, \end{array}\right. \tag{1}$$

where $f \in L^p(\mathcal{B})$, does not have the optimal regularity $W^{2,p}(\mathcal{B})$ if $\dfrac{\pi}{\omega} < 2 - \dfrac{2}{p}$. In that case, this solution admits the decomposition

$$u = u_R + cS, \tag{2}$$

where $u_R \in W^{2,p}(\mathcal{B})$, S is the so-called singular function which can be written explicitly in this case, and the coefficient c depend continuously on the data f, see [12]. In the hilbertian case ($p = 2$), the technique used is essentially based on the partial Fourier transform and Plancherel's theorem. For $p \neq 2$, Clément-Grisvard [5] proved the decomposition (2) by using the theory of sums of operators in Banach spaces, as developped by Da Prato-Grisvard [8] and successfully improved by Dore-Venni [9]. These two theories complement to lead to optimal regularity and singularity results of the solution of (1). We also refer to [18, 10] for some applications of such theories.

In this paper, we suppose that G is constituted of two bounded plane sectors G_1, G_2 with respective opening ω_1 and ω_2, separated by an interface Σ.

$$G_1 = \{(r\cos\theta, r\sin\theta); -\omega_1 < \theta < 0, 0 < r < 1\},$$
$$G_2 = \{(r\cos\theta, r\sin\theta); 0 < \theta < \omega_2, 0 < r < 1\},$$
$$\Sigma = \{(r, 0); 0 < r < 1\}.$$

We consider the following transmission problem

$$\begin{cases} -\Delta u_i & = & f_i & \text{in } G_i \times \mathbb{R}, \\ u_i & = & 0 & \text{on } (\partial G_i \setminus \Sigma) \times \mathbb{R}, \\ u_1 & = & u_2 & \text{on } \Sigma \times \mathbb{R}, \\ \displaystyle\sum_{i=1}^{2} a_i \frac{\partial u_i}{\partial \upsilon_i} & = & 0 & \text{on } \Sigma \times \mathbb{R}, \end{cases} \tag{3}$$

where υ_i denotes the unit normal vector to $\Sigma \times \mathbb{R}$ directed outside $G_i \times \mathbb{R}$, u_i means the restriction of u to $G_i \times \mathbb{R}, f \in L^p(G \times \mathbb{R})$, and a_1, a_2 are two positive real numbers such that $a_1 \neq a_2$ (only this case is of interest).

Our aim is to establish a decomposition similar to (2) of the variational solution of (3) in the case $p \neq 2$. Here, due to the interface, the regular part u_R will belong to $PW^{2,p}(\mathcal{B})$, where

$$PW^{2,p}(\mathcal{B}) := \{u \in H^1(\mathcal{B});\ u_i \in W^{2,p}(\mathcal{B}_i),\ i = 1, 2\}$$

is the space of piecewise $W^{2,p}$ functions on $\mathcal{B}$.

A decomposition similar to (2) of the solution of (3) in the case $p = 2$ was obtained in [17].

The paper is organised as follows. In section 2, we recall the main results on the two strategies of sums of operators, the one of [8] which rely on the following explicit form of the inverse of the sum of two operators A and B

$$(A + B)^{-1} = -\frac{1}{2\pi i} \int_{\gamma} (A + \lambda)^{-1} (B + \lambda)^{-1} d\lambda, \tag{4}$$

under appropriate assumptions on the resolvents of A and B and on the path γ, and the one by [9] based on the formula

$$(A + B)^{-1} = \frac{1}{i\pi} \int_{\gamma} \frac{A^{-z} B^{z-1}}{\sin \pi z} dz \tag{5}$$

under appropriate assumptions on the imaginary powers of A and B.

In section 3, we consider the following transmission problem with complex parameter λ

$$(6) \qquad \begin{cases} -\Delta u_i + \lambda u_i & = & f_i & \text{in } G_i, \\ u_i & = & 0 & \text{on } \partial G_i \setminus \Sigma, \\ u_1 & = & u_2 & \text{on } \Sigma, \\ \displaystyle\sum_{i=1}^{2} a_i \frac{\partial u_i}{\partial \upsilon_i} & = & 0 & \text{on } \Sigma, \end{cases}$$

where $f \in L^p(G)$, $1 < p < +\infty$, υ_i here denotes the normal vector to Σ directed outside G_i and u_i is still the restriction of u to G_i.
By analogy with [1] who considered the Dirichlet problem (without interface), we shall prove the estimate

$$\|u\|_{0,p} \leq \frac{c(p)}{|\lambda|} \|f\|_{0,p} \qquad \forall \lambda : \Re\lambda \geq 0, \quad \lambda \neq 0,$$

for all $1 < p < \infty$, and we deduce that the operator A_p defined by

$$\begin{aligned} D_{A_p} &= \left\{ u \in H_0^1(G); \Delta u_i \in L^p(G_i), u_1 = u_2 \text{ and } \sum_{i=1}^{2} a_i \frac{\partial u_i}{\partial \upsilon_i} = 0 \text{ on } \Sigma \right\}, \\ A_p &: u \mapsto \{\Delta u_i\}_{i=1,2}, \end{aligned}$$

is the infinitesimal generator of a strongly continuous semigroup of contraction, which is also analytic. Such property will be useful to show that A_p verifies the assumptions concerning its resolvent in the application of the first strategy of the sum of operators, and the ones concerning its imaginary powers when we apply the second strategy.

In section 4 we study the behavior of the solution of the transmission problem (6) with parameter λ. We give a decomposition of the solution into a regular and a singular part, with a priori estimates of the regular part and the coefficients of singularities uniform with respect to the parameter λ. This extends to the case $p \neq 2$ $(1 < p < \infty)$ Theorem 4.7 of [17] proved for $p = 2$. In other terms this gives the analogue of Theorem 3.1 of [13] for the transmission case.

Finally in section 5, following the techniques of sections 5 and 6 of [16], we shall apply respectively the two strategies of sums of operators to study the boundary value problem (3). The first application allows us using (4) and the results of section 4, to show existence and uniqueness of a strong solution of problem (3) which admits a decomposition similar to (2). By applying the second strategy we obtain the optimal regularity of u_R.

The theory of sums of operators consists in writing the Laplace equation in 3d in the form of a sum of two operators with values in Banach spaces, one operating on the variable z, the other on the variables x and y. This replaces the effect of the partial Fourier transfom which reduces the space dimension.

2. – Sums of linear operators.

2.1 – *The first strategy.*

Let E be a complex Banach space and A, B two closed linear operators whith dense domains $D(A)$ and $D(B)$ respectively.
Their sum is defined by

$$Lx = Ax + Bx,$$

for every $x \in D(L) = D(A) \cap D(B)$.

We shall make the following assumptions on A and B:

H$_1$ There exist positive numbers M_A, M_B, R, θ_A, θ_B such that $\theta_A + \theta_B > \pi$ and the resolvent $\rho(-A)$ of $-A$ contains the truncated sector

$$S_A = \{\lambda; |\lambda| \geq R, |\arg \lambda| \leq \theta_A\},$$

while the resolvent $\rho(-B)$ of $-B$ contains the truncated sector

$$S_B = \{\lambda; |\lambda| \geq R, |\arg \lambda| \leq \theta_B\},$$

and

$$\|(A+\lambda)^{-1}\| \leq \frac{M_A}{|\lambda|}, \qquad \forall \lambda \in S_A,$$

$$\|(B+\lambda)^{-1}\| \leq \frac{M_B}{|\lambda|}, \qquad \forall \lambda \in S_B.$$

H$_2$ The spectra $\sigma(-A)$ of $-A$ and $\sigma(B)$ of B do not intersect.

H$_3$ The resolvents of A and B commute, i.e.

$$(A+\lambda)^{-1}(B+\mu)^{-1} = (B+\mu)^{-1}(A+\lambda)^{-1}$$

for every $\lambda \in \rho(-A)$ and every $\mu \in \rho(-B)$.

Now we can recall the (see [8])

THEOREM 2.1. – *Under the assumptions* $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, $\boldsymbol{H}_3$, *the closure* $\overline{L}$ *of* L *is invertible.*

The inverse of $\overline{L}$ is defined by the Dunford integral (4) where the path γ separates $\sigma(-A)$ and $\sigma(B)$ and joins $\infty e^{-i\theta_\gamma}$ to $\infty e^{i\theta_\gamma}$ where θ_γ is chosen so that $\pi - \theta_B < \theta_\gamma < \theta_A$.

The unique solution $v \in D(\overline{L})$ of the equation

$$\overline{L}v = (\overline{A + B})v = f$$

is called the strong solution of $Lv = f$.

2.2 – *The second strategy.*

We introduce some different assumptions:

$\mathbf{H}_4$ E is a U.M.D. space, i. e., for some $p \in]1, +\infty[$, or equivalently for all $p \in]1, +\infty[$, the Hilbert transform is continuous in the space $L^p(\mathbb{R}, E)$ (see [4]). In practice all L^p spaces are U.M.D.

$\mathbf{H}_5$ $\rho(A) \supset]-\infty, 0]$ and there exists $M_A > 0$:

$$\|(A + t)^{-1}\| \leq \frac{M_A}{t + 1}, \qquad \forall t \geq 0,$$

$\rho(B) \supset]-\infty, 0]$ and there exists $M_B > 0$:

$$\|(B + t)^{-1}\| \leq \frac{M_B}{t + 1}, \qquad \forall t \geq 0.$$

$\mathbf{H}_6$ $A^{is} \in \mathcal{L}(E)$, $B^{is} \in \mathcal{L}(E)$ for all $s \in \mathbb{R}$ and there exist $K > 0$, τ_A, τ_B such that $\tau_A + \tau_B < \pi$, and

$$\|A^{is}\| \leq Ke^{|s|\tau_A} \qquad \forall s \in \mathbb{R},$$

$$\|B^{is}\| \leq Ke^{|s|\tau_B} \qquad \forall s \in \mathbb{R},$$

where A^{is}, B^{is} are the complex powers of A and B respectively. The main result proved in [9] is summarized in the following theorem

THEOREM 2.2. – *Under the assumptions $\boldsymbol{H}_3$, $\boldsymbol{H}_4$, $\boldsymbol{H}_5$ and $\boldsymbol{H}_6$, the operator L is invertible.*

The explicit construction of the inverse of L is given by the integral (5) where γ is any vertical line within the strip $0 < \Re z < 1$.

3. – Transmission problem and generation of a semigroup.

Let λ be a complex parameter and $f \in L^p(G)$, $1 < p < \infty$. We consider the solution $u \in V$ of the variational problem

$$a_\lambda(u, v) = \int_G a\, f\overline{v}\, dx \qquad \forall v \in V, \tag{7}$$

with

$$a_\lambda(u,v) = \int_G a\left\{\sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial \overline{v}}{\partial x_i} + \lambda u\overline{v}\right\}dx,$$

where

$$V = H_0^1(G) = \{v \in H^1(G); v = 0 \text{ on } \partial G\},$$

equipped with the natural norm $\|\cdot\|_{1,2}$. The function $a(x)$ is piecewise constant, i.e.,

$$a(x) = a_i > 0 \text{ for } x \in G_i,\ i = 1,2,$$

with $a_1 \neq a_2$.

From now on, $L^p(G)$ will be equipped with the norm $\|\cdot\|_{0,p}$ defined by

$$\|u\|_{0,p} = \left(\int_G a|u(x)|^p\ dx\right)^{1/p}.$$

It is clear that (7) is the weak formulation of problem (6).

The aim of this section is to prove that the following estimate holds

$$\|u\|_{0,p} \leq \frac{c}{|\lambda|}\|f\|_{0,p}, \quad \Re\lambda \geq 0,\ \lambda \neq 0, \tag{8}$$

where the constant c depends only on p (and not on λ).

It is well know that the estimate (8) is linked with the problem of knowing wether the operator A_p defined in section 1 generates an analytic semigroup. When G is a smooth domain, the estimate (8) is well know for a general strongly elliptic operator [2, 3] (for estimates in the C^0-norm see [24] and in Hölder norms, see [7]). In the case when G is a polygonal domain, Adeyeye [1] proves (8) for the Laplace operator with different boundary conditions.

First we recall the results concerning existence, uniqueness and regularity of the variational solution u of (7). It is clear that the sesquilinear form a_λ is continuous on $V \times V$, it is also coercive for $\Re\lambda \geq 0$ thanks to Poincaré's inequality. On the other hand, since $H_0^1(G) \hookrightarrow L^{p'}(G)$ (p' being the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) and then $L^p(G) \hookrightarrow H^{-1}(G) = V'$, we deduce that the right-hand side of (7) belongs to V'. Consequently applying the Lax-Milgram lemma, we get

LEMMA 3.1. – *For each $f \in L^p(G)$, there exists a unique solution $u \in H_0^1(G)$ of* (7)*, for all $\lambda \in \mathbb{C} : \Re\lambda \geq 0$.*

The singularities of problem (6) take the form

$$S^{(m)} = \eta r^{\lambda_m} t_m(\theta),$$

where λ_m is a nonnegative real number and λ_m^2, t_m are respectively the eigenvalues and eigenfunctions of the following Sturm-Liouville problem

$$\begin{cases} -t_m''(\theta) = \lambda_m^2 t_m(\theta) & \text{for } \theta \in [-\omega_1, \omega_2], \theta \neq 0, \\ t_m(0^-) = t_m(0^+), \\ a_2 t_m'(0^-) = a_1 t_m'(0^+), \\ t_m(-\omega_1) = t_m(\omega_2) = 0. \end{cases}$$

The singular behavior of the solution of (7) is given by the following theorem (see Theorem 2.27 of [21]).

THEOREM 3.2. – *If $\lambda_m \neq \dfrac{2}{p'}$ for all $m \in \mathbb{N}^*$, then for each $f \in L^p(G)$, there exists a unique variational solution $u \in H_0^1(G)$ of (6) which admits the decomposition*

$$u = u_0 + \sum_{\substack{\lambda_m \neq \frac{2}{p'} \\ \lambda_m \in]0, \frac{2}{p'}[}} c_m S^{(m)}, \tag{9}$$

where $u_0 \in PW^{2,p}(G)$ is the regular part of u and the constants c_m of the singularities $S^{(m)}$ are equal to

$$c_m = \int_G a(f - \lambda u)\overline{K^{(m)}} dx, \tag{10}$$

with $K^{(m)} \in L^{p'}(G)$ defined by

$$K^{(m)} = \frac{1}{2\lambda_m}\left(S^{(-m)} - v^{(m)}\right), \tag{11}$$

where $S^{(-m)} = \eta r^{-\lambda_m} t_m(\theta)$ is the so-called dual singular function and $v^{(m)} \in H_0^1(G)$ is the solution of

$$a_0(v^{(m)}, w) = -\sum_{i=1,2} a_i \int_{G_i} \Delta S_i^{(-m)} w_i \, dx, \forall w \in H_0^1(G). \tag{12}$$

To establish the estimate (8), we distinguish the case $p > 2$ to the case $p < 2$.

FIRST CASE $p > 2$: Here we follow the techniques of [1] who used the approach of Sobolevskii [23] based on the dual mapping $u \mapsto |u|^{p-2}\overline{u}$.
Let $E(\Delta, L^p(G))$ be the space

$$E(\Delta, L^p(G)) = \left\{u \in H^1(G); \Delta u \in L^p(G)\right\}, \quad p \geq 2.$$

This is a Banach space for the norm

$$u \mapsto \|u\|_{1,2} + \|\Delta u\|_{0,p}.$$

We recall the next Green formula (cf. [12, Theorem 1.5.3.11]).

THEOREM 3.3. – *Let $i = 1$ or 2. For $u \in E(\Delta, L^p(G_i))$ and $v \in W^{1,r}(G_i)$, with $p \geq 2$ and $r > 2$ such that $v(0) = 0$, and*

$$v = 0 \text{ on } \partial G_i \setminus \Sigma,$$

we have

$$\int_{G_i} [v_i \Delta u_i + \nabla u_i \cdot \nabla v_i] dx = \left\langle \gamma_\Sigma \frac{\partial u_i}{\partial \nu_i}, \gamma_\Sigma v_i \right\rangle, \tag{13}$$

where γ_Σ denotes the operator trace on Σ.

Let A_p be the operator defined in section 1. The following theorem gives a sufficient condition on A_p, which guarantees that the estimate (8) holds:

THEOREM 3.4. – *Suppose that there exists s with $2 < s < p$ such that $D_{A_p} \subset W^{1,s}(G)$. Then any $u \in D_{A_p}$ satisfies the next Green identity*

$$\begin{aligned} - \int_G a(A_p u)|u|^{p-2}\overline{u} dx = & \frac{p}{2} \int_G a|\nabla u|^2 |u|^{p-2} dx \\ & + \frac{p-2}{2} \int_G a|u|^{p-4}\overline{u}^2 (\nabla u)^2 dx, \end{aligned} \tag{14}$$

where we understand that

$$|\nabla u|^2 = \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i} \right|^2, \quad (\nabla u)^2 = \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2.$$

PROOF. – Let us define

$$v = \begin{cases} |u|^{p-2}\overline{u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases} \tag{15}$$

From [1], we have

$$\nabla \left(|u|^{p-2}\overline{u} \right) = \frac{p}{2} |u|^{p-2} \nabla \overline{u} + \frac{p-2}{2} |u|^{p-4} \overline{u}^2 \nabla u.$$

As $u \in D_{A_p}$, it is clear that $u_i \in E(\Delta, L^p(G_i))$, for $i = 1$ and 2. The inclusion $D_{A_p} \subset W^{1,s}(G)$ implies that $v \in W^{1,s}(G)$, since the functions of $W^{1,s}(G)$ are bounded. Then we can apply Green's formula (13) to get

$$\int_{G_i} [v_i \Delta u_i + \nabla u_i \cdot \nabla v_i] dx = \left\langle \gamma_\Sigma \frac{\partial u_i}{\partial \nu_i}, \gamma_\Sigma v_i \right\rangle, \quad i = 1, 2.$$

Multiplying this identity by a_i and summing up on i, we get

$$\begin{aligned} \sum_{i=1}^{2} a_i \int_{G_i} [v_i \Delta u_i + \nabla u_i \cdot \nabla v_i] dx &= \sum_{i=1}^{2} a_i \left\langle \frac{\partial u_i}{\partial \nu_i}, |u_i|^{p-2} \overline{u_i} \right\rangle \\ &= \left\langle \sum_{i=1}^{2} a_i \frac{\partial u_i}{\partial \nu_i}, |u_1|^{p-2} \overline{u_1} \right\rangle \\ &= 0, \end{aligned}$$

thanks to the transmission conditions. □

We are now in position to establish the following result

THEOREM 3.5. – *Assume that* (14) *holds. Then*

$$\|u\|_{0,p} \leq \frac{1}{\Re\lambda} \|f\|_{0,p}, \quad \Re\lambda > 0, \tag{16}$$

$$\|u\|_{0,p} \leq \frac{p}{2|\Im\lambda|} \|f\|_{0,p}, \quad \Im\lambda \neq 0. \tag{17}$$

Consequently there exists a constant $c(p) > 0$ *such that the estimate* (8) *holds.*

PROOF. – The proof of (16) and (17) is similar to Lemma 1.8 and Lemma 1.9 of [1]. We set

$$I = -\int_G a(A_p u)|u|^{p-2}\overline{u} dx, \;\; R = \int_G a|\nabla u|^2 |u|^{p-2} dx \;\; \text{and} \;\; S = \int_G a|u|^{p-4}\overline{u}^2 (\nabla u)^2 dx.$$

By (14) we clearly have

$$|S| \leq R, \quad \text{and} \quad I = R + \frac{p-2}{2}(R + S). \tag{18}$$

If u is solution of the transmission problem (6), then by multiplying the differential equation in (6) by $v = |u|^{p-2}\overline{u}$ and integration by parts, we have

$$I + \lambda \int_G a|u|^p dx = \int_G af|u|^{p-2}\overline{u} dx. \tag{19}$$

Taking the real part of the left-hand side we obtain

$$\Re I + \Re\lambda \int_G a|u|^p dx \leq \int_G af|u|^{p-1}dx \leq \|f\|_{0,p}\|u\|_{0,p}^{p-1}, \tag{20}$$

by Hölder's inequality. Since (18) guarantees that $\Re I$ is nonnegative for $p > 2$, the last estimate yields (16).

Now, taking the imaginary part in (19), we obtain

$$\Im\lambda\|u\|_{0,p}^{p} \leq \|f\|_{0,p}\|u\|_{0,p}^{p-1} - \Im I. \tag{21}$$

From (18), we have

$$|\Im I| = \frac{p-2}{2}|\Im S| \leq \frac{p-2}{2}R \leq \frac{p-2}{2}\Re I.$$

Owing to (20), we get

$$\Re I \leq \|f\|_{0,p}\|u\|_{0,p}^{p-1}.$$

The two last inequalities in (21) implies (17). (8) is a direct consequence of (16) and (17). □

In view of the previous results, we then need to check the inclusion $D_{A_p} \subset W^{1,s}(G)$, for some $2 < s < p$.

THEOREM 3.6. – *There exists $s > 2$ such that $D_{A_p} \subset W^{1,s}(G)$ and consequently the estimates* (16), (17) *and* (8) *hold, for all $p > 2$.*

PROOF. – From Theorem 3.2, u belongs to the span of $PW^{2,p}(G)$ and a finite number of singular solutions which behaves like $r^{\lambda_m}t_m(\theta)$, $0 < \lambda_m < \frac{2}{p'}$. Since $W^{2,p}(G_i) \hookrightarrow W^{1,s}(G_i)$, for all $s > 2$, it suffices to find $s > 2$ such that $r^{\lambda_m}t_m(\theta) \in W^{1,s}(G)$, i.e., $\frac{2}{s} > 1 - \lambda_m$ thanks to Theorem 1.4.5.3 of [12]. This is satisfied for all $s \in \mathbb{R}$ if $1 - \lambda_m \leq 0$, otherwise it holds for each $s < \frac{2}{1-\lambda_m}$. In that last case, s may be chosen > 2 since $\frac{2}{1-\lambda_m} > 2$ (recall that $\lambda_m > 0$). □

SECOND CASE $1 < p < 2$: We proceed by duality.

Let $1 < p < 2$ and let u be the variational solution of (6). We have

$$\|u\|_{0,p} = \sup_{\varphi\in L_{p'}(G)} \frac{\langle u, \varphi\rangle}{\|\varphi\|_{0,p'}} \tag{22}$$

$$= \sup_{\varphi\in L_{p'}(G)} \frac{\int_G au\overline{\varphi}dx}{\|\varphi\|_{0,p'}}.$$

Thanks to Lemma 3.1, there exists a unique solution $u_\varphi \in H_0^1(G)$ of

$$a_{\bar{\lambda}}(u_\varphi, v) = \int_G a\varphi\bar{v}\,dx \quad \forall v \in V.$$

Moreover as $p' > 2$, Theorem 3.6 yields the following estimate

$$\|u_\varphi\|_{0,p'} \leq \frac{c}{|\lambda|}\|\varphi\|_{0,p'}, \quad \Re\lambda \geq 0,\ \lambda \neq 0. \tag{23}$$

Owing to (22) we get

$$\begin{aligned}
\|u\|_{0,p} &= \sup_{\varphi \in L_{p'}(G)} \frac{\overline{a_{\bar{\lambda}}(u_\varphi, u)}}{\|\varphi\|_{0,p'}} \\
&= \sup_{\varphi \in L_{p'}(G)} \frac{a_\lambda(u, u_\varphi)}{\|\varphi\|_{0,p'}} \\
&= \sup_{\varphi \in L_{p'}(G)} \frac{\int_G af\overline{u}_\varphi}{\|\varphi\|_{0,p'}} \\
&\leq \sup_{\varphi \in L_{p'}(G)} \frac{\|f\|_{0,p}\|u_\varphi\|_{0,p'}}{\|\varphi\|_{0,p'}}.
\end{aligned}$$

Using (23), we arrive at (8).

Similarly using the estimate (16) (resp. (17)) satisfied by u_φ, we deduce that u satisfies (16) (resp. (17)) as well. □

Summing up we have proved the

THEOREM 3.7. – *Let $u \in H_0^1(G)$ be the solution of* (7), *then for all* $1 < p < \infty$, u *satisfies the estimates* (16), (17) *and* (8).

REMARK 3.8. – We easily check (8) when $p = 2$, by taking as usual $v = u$ in (7). □

COROLLARY 3.9. – *For all $1 < p < \infty$, A_p generates a strongly continuous semigroup of contraction in $L^p(G)$ which preserves positivity.*

PROOF. – The estimate (16) implies that $]0, \infty[\ \subset \rho(A_p)$ and

$$\|(A_p - \lambda)^{-1}\| \leq \frac{1}{\lambda} \quad (\lambda > 0),$$

and therefore $-A_p$ is dissipative (see Theorem I.4.2 of [22]).

Consequently $\overline{D(A_p)} = L^p(G)$, because $L^p(G)$ is reflexive, see for example [19, Corollary 1.1.4] or [22, Theorem I.4.6]. Then applying Hille-Yosida Theorem, we deduce that $-A_p$ is the infinitesimal generator of a strongly continuous semi-group of contraction in $L^p(G)$.

To complete the proof we need the following lemma:

LEMMA 3.10. – *Let $f \in L^p(G)$ be real valued, $\lambda \in [0, +\infty[$ and u the variational solution of* (7). *Then*

$$f \geq 0 \Rightarrow u \geq 0.$$

PROOF. – Clearly if f is real valued, then u is real valued. Now we write $u = u^+ - u^-$ where $u^+ = \sup(0, u)$ and $u^- = \sup(0, -u)$. Then u^+ and u^- belong to $H_0^1(G)$.

Applying the variational identity with $v = u^-$ we obtain

$$\int_G a\nabla(u^+ - u^-) \cdot \nabla u^- dx + \lambda \int_G a(u^+ - u^-)u^- dx = \int_G afu^- dx.$$

This gives

$$\int_G a|\nabla u^-|^2 dx + \lambda \int_G a|u^-|^2 dx = -\int_G afu^- dx \leq 0,$$

since $f \geq 0$. This shows that $u^- = 0$ and then $u = u^+ \geq 0$. $\square$

Let us denote by e^{-tA_p} for $t \geq 0$, the semigroup generated by $-A_p$.

We now assume that $f \geq 0$, by Yosida approximation we get

$$\begin{aligned} e^{-tA_p}f &= \lim_{\lambda \to \infty} e^{t\lambda A_p(\lambda - A_p)^{-1}} f \\ &= \lim_{\lambda \to \infty} e^{t\left[\lambda^2(\lambda - A_p)^{-1} - \lambda\right]} f \\ &= \lim_{\lambda \to \infty} \left[e^{-\lambda t} e^{t\lambda^2(\lambda - A_p)^{-1}} f \right] \\ &= \lim_{\lambda \to \infty} \left[e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda^2 t)^k}{k!} (\lambda - A_p)^{-k} f \right]. \end{aligned}$$

Lemma 3.10 gives $(\lambda - A_p)^{-1} f \geq 0$ and then $e^{-tA_p} f \geq 0$. So the corresponding contraction semigroup preserves positivity. $\square$

COROLLARY 3.11. – *For each $1 < p < \infty$, A_p generates an analytic semigroup in $L^p(G)$.*

PROOF. – Direct consequence of (8) (see Theorem I.5.2 of [22]). $\square$

4. – Resolvent of the transmission problem in a corner domain.

The aim of this section is to extend Theorem 4.7 of [17] to the L^p-norms, $1 < p < \infty$. We actually want to obtain uniform estimate with respect to λ for the regular part and the singular one of the variational solution of (6) appearing in the expansion (9). For that purpose, we first transform the expression (10) to obtain an exact formula with respect to f.

LEMMA 4.1. – *For all $m \in \mathbb{N}^*$, there exists $T^{(m)} \in V$ solution of*

$$a_{\overline{\lambda}}(T^{(m)}, w) = \overline{\lambda} \int_G a K^{(m)} \overline{w} dx, \quad \forall w \in V. \tag{24}$$

Consequently under the assumption of Theorem 3.2, we have

$$c_m = \int_G af\overline{w^{(m)}}dx, \tag{25}$$

where

$$w^{(m)} = K^{(m)} - T^{(m)}. \tag{26}$$

PROOF. – The existence of $T^{(m)} \in V$ solution of (24) is a direct consequence of Lax-Milgram Lemma since $K^{(m)} \in L^{p'}(G)$.

From (10) and (24), we may write

$$c_m = \int_G af\overline{K^{(m)}}dx - a_\lambda(u, T^{(m)}).$$

As u is solution of the variational problem (7) and $T^{(m)} \in V$ we may write

$$a_\lambda(u, T^{(m)}) = \int_G af\overline{T^{(m)}}dx.$$

The two previous identities yield (25). □

In order to estimate the $L^{p'}$-norm of $w^{(m)}$ with respect to λ, we write it as follows (thanks to (11) and (26))

$$w^{(m)} = \psi_m - \phi_m \tag{27}$$

where

$$\psi_m = \frac{1}{2\lambda_m} \begin{cases} e^{-r\sqrt{\lambda}} S^{(-m)} & \text{if } \lambda_m < \dfrac{2}{p'} - 1, \\ e^{-r\sqrt{\lambda}}(1 + r\sqrt{\lambda}) S^{(-m)} & \text{if } \dfrac{2}{p'} - 1 \le \lambda_m < \dfrac{2}{p'}, \end{cases} \tag{28}$$

and ϕ_m is the remainder. ψ_m is defined such that ψ_m and $(-\Delta+\lambda)\psi_m$ belong to $L^{p'}(G)$. Indeed by direct calculations one has

$$(-\Delta+\lambda)\psi_m = \begin{cases} (1-2\lambda_m)\sqrt{\lambda}\,\dfrac{\psi_m}{r} + r_m & \text{if } \lambda_m < \dfrac{2}{p'} - 1, \\ \lambda \dfrac{3-2\lambda_m}{1+r\sqrt{\lambda}}\psi_m + r_m & \text{if } \dfrac{2}{p'} - 1 \le \lambda_m < \dfrac{2}{p'}, \end{cases} \tag{29}$$

where we have set

$$r_m = \begin{cases} \dfrac{-\Delta\eta}{2\lambda_m} e^{-r\sqrt{\lambda}} r^{-\lambda_m} t_m - \dfrac{1}{\lambda_m} \vec{\nabla}\eta \vec{\nabla}\left(e^{-r\sqrt{\lambda}} r^{-\lambda_m} t_m\right) & \text{if } \lambda_m < \dfrac{2}{p'} - 1, \\ \dfrac{-\Delta\eta}{2\lambda_m}(1+\sqrt{\lambda}r) e^{-r\sqrt{\lambda}} r^{-\lambda_m} t_m - \dfrac{r\sqrt{\lambda}}{\lambda_m} \vec{\nabla}\eta \vec{\nabla}\left(e^{-r\sqrt{\lambda}} r^{-\lambda_m} t_m\right) & \\ +(e^{-r\sqrt{\lambda}} r^{-\lambda_m} t_m)\vec{\nabla}\eta & \text{if } \dfrac{2}{p'} - 1 \le \lambda_m < \dfrac{2}{p'}. \end{cases}$$

LEMMA 4.2. – *There exists a constant $C > 0$ (independent of λ and m) such that*

$$\|(-\Delta+\lambda)\psi_m\|_{L^{p'}(G)} \le C|\sqrt{\lambda}|^{\lambda_m + \frac{2}{p}}. \tag{30}$$

PROOF. – The terms of r_m have exponential decay in $\sqrt{\lambda}$ in any norm since $\Delta\eta$ and $\vec{\nabla}\eta$ vanishes near the corner. Then for some $\varepsilon > 0$, we can show that

$$\|r_m\|_{L^{p'}(G)} \le C e^{-\varepsilon \Re\sqrt{\lambda}}. \tag{31}$$

On the other hand, from the explicit form (28) of ψ_m, one can write

$$\left\|\frac{\sqrt{\lambda}\psi_m}{r}\right\|_{0,p'}^{p'} \le C \int_0^\infty |\sqrt{\lambda}|^{p'} e^{-p' r \Re\sqrt{\lambda}} r^{-(\lambda_m+1)p'+1} dr$$

if $\lambda_m < \dfrac{2}{p'} - 1$, and

$$\left\|\lambda \frac{3-2\lambda_m}{1+r\sqrt{\lambda}}\psi_m\right\|_{0,p'}^{p'} \le C \int_0^\infty |\lambda|^{p'} e^{-p' r \Re\sqrt{\lambda}} r^{-\lambda_m p'+1} dr$$

if $\dfrac{2}{p'} - 1 \le \lambda_m < \dfrac{2}{p'}$.

By the change of variable $s = r\Re\sqrt{\lambda}$, we arrive at

$$\begin{cases} \left\|\frac{\sqrt{\lambda}\psi_m}{r}\right\|_{0,p'} \le C|\sqrt{\lambda}|^{\lambda_m + 2 - \frac{2}{p'}} & \text{if } \lambda_m < \dfrac{2}{p'} - 1, \\ \left\|\lambda \dfrac{3-2\lambda_m}{1+\sqrt{\lambda}r}\psi_m\right\|_{0,p'} \le C|\sqrt{\lambda}|^{\lambda_m + 2 - \frac{2}{p'}} & \text{if } \dfrac{2}{p'} - 1 \le \lambda_m < \dfrac{2}{p'}. \end{cases} \tag{32}$$

The estimates (31) and (32) drive to (30) thanks to (29). □

LEMMA 4.3. – *There exists a constant* $C > 0$ *such that*

$$\|w^{(m)}\|_{0,p'} \leq C|\sqrt{\lambda}|^{\lambda_m - \frac{2}{p'}}. \tag{33}$$

PROOF. – From the identity (27) it suffices to show that ψ_m and ϕ_m satisfies (33).

Indeed the explicit form of ψ_m given by (28) implies easily that

$$\|\psi_m\|_{0,p'} \leq C|\sqrt{\lambda}|^{\lambda_m - \frac{2}{p'}},$$

while ϕ_m is looked as the variational solution of (due to (12) and (24))

$$a_\lambda(\phi_m, w) = \int_G a(-\Delta + \lambda)\psi_m \overline{w} dx \quad \forall w \in V.$$

We apply to ϕ_m the estimate (8) derived in the previous section

$$\|\phi_m\|_{0,p'} \leq \frac{C}{|\lambda|}\|(-\Delta + \lambda)\phi_m\|_{0,p'}.$$

It is clear that $(-\Delta + \lambda)\phi_m = (-\Delta + \lambda)\psi_m$. Therefore (30) implies that ϕ_m also satisfies (33). □

THEOREM 4.4. – *Under the assumption of Theorem* 3.2, u *admits the decomposition*

$$u = u_R + \sum_{\frac{2}{p'}-1<\lambda_m<\frac{2}{p'}} c_m e^{-r\sqrt{\lambda}} S^{(m)} + \sum_{\lambda_m \leq \frac{2}{p'}-1} c_m e^{-r\sqrt{\lambda}}(1 + r\sqrt{\lambda})S^{(m)} \tag{34}$$

where $u_R \in PW^{2,p}(G)$ *satisfies*

$$\|u_R\|_{PW^{2,p}(G)} + |\sqrt{\lambda}|\|u_R\|_{W^{1,p}(G)} + |\lambda|\|u_R\|_{L^p(G)} \leq C\|f\|_{L^p(G)} \tag{35}$$

and c_m *satisfies*

$$|c_m| \leq C|\sqrt{\lambda}|^{\lambda_m - \frac{2}{p'}}\|f\|_{0,p}. \tag{36}$$

PROOF. – The estimation (36) follows from (25) and (33). It then remains to prove (34) and (35).

The decomposition (34) follows from (9) by setting

$$\begin{aligned} u_R = u_0 - &\sum_{\frac{2}{p'}-1<\lambda_m<\frac{2}{p'}} c_m(e^{-r\sqrt{\lambda}} - 1)S^{(m)} \\ &- \sum_{\lambda_m \leq \frac{2}{p'}-1} c_m\Big[e^{-r\sqrt{\lambda}}(1 + r\sqrt{\lambda}) - 1\Big]S^{(m)}, \end{aligned}$$

because we easily check that $(e^{-r\sqrt{\lambda}}-1)S^{(m)}\in PW^{2,p}(G)$ if $\lambda_m>\dfrac{2}{p'}-1$ while $[e^{-r\sqrt{\lambda}}(1+r\sqrt{\lambda})-1]S^{(m)}\in PW^{2,p}(G)$ if $\lambda_m\le\dfrac{2}{p'}-1$.

From Theorem 2.27 of [21] and Peetre's lemma (see for example [21]), there holds

$$
\begin{aligned}
\|u_R\|_{PW^{2,p}(G)} \le & C\big\{\|\varDelta u_R\|_{L^p(G)}+\|u_R\|_{L^p(G)}\big\}\\
\le & C\big\{\|(-\varDelta+\lambda)u_R\|_{L^p(G)}+(1+|\lambda|)\|u_R\|_{L^p(G)}\big\}.
\end{aligned}
\tag{37}
$$

We apply to u_R the inequality (8) of the previous section to get

$$
\|u_R\|_{L^p(G)}\le\frac{C}{|\lambda|}\|(-\varDelta+\lambda)u_R\|_{L^p(G)}. \tag{38}
$$

We then have to estimate $(-\varDelta+\lambda)u_R$ in the L^p-norm. Thanks to (34) we get

$$
\begin{aligned}
(-\varDelta+\lambda)u_R = & f-\sum_{\frac{2}{p'}-1<\lambda_m<\frac{2}{p'}} c_m(-\varDelta+\lambda)\Big(e^{-r\sqrt{\lambda}}S^{(m)}\Big)\\
& -\sum_{\lambda_m\le\frac{2}{p'}-1} c_m(-\varDelta+\lambda)\Big[e^{-r\sqrt{\lambda}}(1+r\sqrt{\lambda})-1\Big]S^{(m)}.
\end{aligned}
$$

In other words, we have to estimate

$$
F_m=\begin{cases}
(-\varDelta+\lambda)\Big(e^{-r\sqrt{\lambda}}S^{(m)}\Big) & \text{if } \dfrac{2}{p'}-1<\lambda_m<\dfrac{2}{p'},\\[2ex]
(-\varDelta+\lambda)\Big[e^{-r\sqrt{\lambda}}(1+r\sqrt{\lambda})S^{(m)}\Big] & \text{if } \lambda_m\le\dfrac{2}{p'}-1.
\end{cases}
$$

Straightforward calculations yield

$$
\|F_m\|_{L^p(G)}\le C|\sqrt{\lambda}|^{\frac{2}{p'}-\lambda_m}.
$$

Therefore taking (36) into account we see that

$$
\|(-\varDelta+\lambda)u_R\|\le\|f\|_{0,p}. \tag{39}
$$

Finally we use the usual convexity inequality to estimate $\|u_R\|_{W^{1,p}(G)}$. Indeed

$$
\|u_R\|_{W^{1,p}(G)}\le\varepsilon\|u_R\|_{PW^{2,p}(G)}+K\varepsilon^{-1}\|u_R\|_{L^p(G)}\quad\forall\varepsilon>0.
$$

For $\varepsilon=|\sqrt{\lambda}|^{-1}$ we get

$$
|\sqrt{\lambda}|\|u_R\|_{W^{1,p}}\le\|u_R\|_{PW^{2,p}(G)}+K|\lambda|\|u_R\|_{L^p(G)}, \tag{40}
$$

The estimates (37), (38), (39) and (40) yield (35). □

COROLLARY 4.5. – *Under the assumption of Theorem* 3.2, *u admits the* *decomposition*

$$u = u_R + \sum_{0<\lambda_m<\frac{2}{p'}} c_m \psi_m(\lambda), \tag{41}$$

where $u_R \in PW^{2,p}(G)$ *and*

$$\psi_m(\lambda) = \begin{cases} e^{-r\sqrt{\lambda}} S^{(m)} & \text{if } \dfrac{2}{p'} - 1 < \lambda_m < \dfrac{2}{p'}, \\ e^{-r\sqrt{\lambda}}(1 + r\sqrt{\lambda}) S^{(m)} & \text{if } \lambda_m \leq \dfrac{2}{p'} - 1. \end{cases} \tag{42}$$

The behavior in λ *of* u_R *and* c_m *is given by*

$$\|u_R\|_{PW^{2,p}(G)} + |\lambda| \|u_R\|_{L^p(G)} + \sum_{0<\lambda_m<\frac{2}{p'}} |\lambda|^{\frac{1}{p'} - \frac{\lambda_m}{2}} |c_m| \leq C \|f\|_{L^p(G)}, \tag{43}$$

for every λ *in the sector* $|\arg(\lambda)| \leq \theta_0$ *where* $\theta_0 \in]0, \pi[$ *is a given angle.*

With the notation already introduced in section 1, we can write

$$u = (-A_p + \lambda)^{-1} f,$$

consequently the decomposition (41) implies a similar decomposition of the resolvent of A_p. Namely we may write

$$(-A_p + \lambda)^{-1} = R(\lambda) + \sum_{0<\lambda_m<\frac{2}{p'}} T_m(\lambda) \otimes \psi_m(\lambda) \tag{44}$$

where $R(\lambda)$ is the continuous linear operator from $L^p(G)$ into $PW^{2,p}(G)$ defined by

$$R(\lambda) f := u_R,$$

and $T_m(\lambda)$ is the continuous linear functional on $L^p(G)$ defined by (so $T_m(\lambda)$ is identified with an element in $L^{p'}(G)$)

$$\langle T_m(\lambda), f \rangle := c_m. \tag{45}$$

The estimate (43) implies

$$\|R(\lambda)\|_{L^p(G) \to PW^{2,p}(G)} + |\lambda| \|R(\lambda)\|_{L^p(G) \to L^p(G)} \leq C \tag{46}$$

and

$$\|T_m(\lambda)\|_{L^{p'}(G)} \leq C |\lambda|^{\frac{\lambda_m}{2} - \frac{1}{p'}}, \tag{47}$$

for all λ such that $|\arg(\lambda)| \leq \theta_0$.

5. – Transmission problem in a cylinder.

Let $\mathcal{B}$, Σ, G_i be defined as in section 1 and let $\mathcal{B}_i = G_i \times \mathbb{R}$. Along this section, the variables in G will be denoted x and y ($x + iy = re^{i\theta}$) and the third variable in $\mathcal{B}$ by z.

For the sake of clarity, we shall denote $\varDelta$ the Laplace operator in 2d while we shall denote it $\varDelta$ in 3d.

Given $g \in L^p(\mathcal{B})$, we look for v, possibly in $PW^{2,p}(\mathcal{B}) \cap W_0^{1,p}(\mathcal{B})$, solution of

$$\begin{cases} \varDelta v_i = g_i & \text{in } \mathcal{B}_i, i = 1, 2, \\ v_1 = v_2 & \text{on } \Sigma \times \mathbb{R}, \\ \displaystyle\sum_{i=1}^{2} a_i \frac{\partial v_i}{\partial v_i} = 0 & \text{on } \Sigma \times \mathbb{R}. \end{cases} \tag{48}$$

We can equivalently write the equation $\varDelta v_i = g_i$ in $\mathcal{B}_i$ as

$$D_z^2 v_i + \varDelta v_i = g_i \qquad in \mathcal{B}_i. \tag{49}$$

5.1 – *Application of the first strategy.*

We shall apply Theorem 2.1 to the equation (49) in the case of the Banach space

$$E = L^p(\mathcal{B}) = L^p(\mathbb{R}, X)$$

where

$$X = L^p(G),$$

and with the operators A and B defined by

$$-Av = \{\varDelta v_i\}_{i=1,2}$$

for $v \in D(A) = L^p\big(\mathbb{R}, D(A_p)\big)$,

$$-Bv = D_z^2 v$$

for $v \in D(B) = W^{2,p}(\mathbb{R}, X)$.

We denote by C the operator defined by

$$Cv = D_z v$$

for $v \in D(C) = W^{1,p}(\mathbb{R}, X)$

It is well known that the spectrum of C $\sigma(C) = i\mathbb{R}$ and that for $\phi \in E$ we have

$$\left[(C+\mu I)^{-1}\phi\right](t,\sigma) = \begin{cases} -\int\limits_t^{\infty} e^{\mu(s-t)}\phi(s,\sigma)ds & \text{if } \Re\mu < 0, \\ \int\limits_{-\infty}^{t} e^{-\mu(t-s)}\phi(s,\sigma)ds & \text{if } \Re\mu > 0. \end{cases} \tag{50}$$

From these expressions, we deduce the estimate

$$\|(C+\mu I)^{-1}\|_{\mathcal{L}(E)} \le \frac{1}{|\Re\mu|} \qquad \forall\mu, \quad \Re\mu \neq 0. \tag{51}$$

5.1.1 — Spectral properties of *B*.

The density of $D(B)$ in E follows from the fact that $D(B)$ contains the space $\mathcal{D}(\mathbb{R}, L^p(G))$. We can write $-B = P(C)$ where P is the polynomial $P(z) = z^2$. Now we can state the following results.

LEMMA 5.1. – *Assume $-B = P(C)$ where P is the polynomial*

$$P(z) = z^2 + az + b$$

with two real numbers a and b. Then we have

(i) $\sigma(-B) = \{-\xi^2 + ia\xi + b, \quad \xi \in \mathbb{R}\}$.

(ii) *There exists $R > 0$ and $\theta_B \in \left]0, \frac{\pi}{2}\right[$ such that the resolvent set $\rho(-B)$ contains the sector*

$$S_b = \{z \in \mathbb{C}; \quad |z| \ge R, \quad |\arg z| < \pi - \theta_B\}.$$

(iii) *For every $\lambda \in S_b$ we have*

$$\|(B+\lambda)^{-1}\|_{\mathcal{L}(E)} = 0\left(\frac{1}{(\Re\sqrt{\lambda})^2}\right) \qquad \forall\lambda \in S_b. \tag{52}$$

(vi) *For $b < 0$, there exists $M > 0$ such that*

$$\|(B+t)^{-1}\| \le \frac{M}{t+1} \qquad \forall t \ge 0. \tag{53}$$

PROOF.

(i) is an application of the spectral mapping theorem.

(ii) $\sigma(-B)$ is the parabola intersecting the x axis at the point b and given by the equation

$$y^2 = -a^2(x-b).$$

The two tangents to this parabola at the points of coordinates $(b-1, a)$, $(b-1, -a)$ are given by the equations

$$y \pm a = \pm \frac{|a|}{2}(x - b + 1).$$

They intersect on the x axis at the point $x = b + 1$ with an angle $\varepsilon_B \in \left]0, \frac{\pi}{2}\right[$ of tangent $\tan \varepsilon_B = \frac{|a|}{2}$. Consequently the resolvent set contains the sector

$$\{z \in C;\ \ |z| > 0, |\arg[z - (b+1)]| < \pi - \varepsilon_B\}.$$

Thus for ε small enough, there exists $R > 0$ ($R = |z_0|$ where z_0 is the intersection point of the half-line $|\arg z| = \pi - (\varepsilon_B + \varepsilon)$ with the tangent to the parabola at the point situated above the x axis) such that $\rho(-B)$ contains the sector

$$\{z \in C;\ \ |z| \geq R, |\arg z| < \pi - (\varepsilon_B + \varepsilon)\}.$$

(iii) For every $\lambda \in S_b$, the equation $P(z) = \lambda$ has two complex roots

$$z_{\pm}(\lambda) = \frac{-a \pm \sqrt{a^2 - 4(b - \lambda)}}{2},$$

where the square root is the analytic determination defined on the plane without its negative real axis. Therefore

$$(B - \lambda)^{-1} = (C - z_+(\lambda))^{-1}(C - z_-(\lambda))^{-1}. \tag{54}$$

It is clear that $z_{\pm}(\lambda) \simeq \pm\sqrt{\lambda}$ for $|\lambda|$ large enough. Consequently (52) follows from (51) and (54).

(vi) $b < 0$ implies that $[0, \infty[\subset \rho(-B)$, then the operator $B + t$ is invertible for every $t \geq 0$, and (53) follows from (52). $\square$

Thanks to Lemma 5.1, we conclude that in our particular case $-B = D_z^2$, the operator B satisfies $\mathbf{H}_1$ and $\sigma(-B) =]-\infty, 0]$.

5.1.2 — Spectral properties of A

The properties of A are those of its realization $-A_p$. Thanks to Corollary 3.11, we know that A_p generates an analytic semi-group, thus A fulfils $\boldsymbol{H}_1$ with some $\theta_A > \frac{\pi}{2}$.

On the other hand, it is easy to check that the operator A_2 associated with the quadratic form

$$(u, v) \mapsto (\nabla u, \nabla v)_a$$

on the space $L^2(G)$ with inner product

$$(u,v) \mapsto (u,v)_a = \int_G auvdx,$$

is a self-adjoint and positive operator with a compact inverse. Thus it has a discrete spectrum. Let $0 < \mu_1 < \mu_2 \leq \mu_3 ...$ be its eigenvalues repeated according to their multiplicity. Since A_p has the same spectrum than A_2 and $\sigma(-B) =]-\infty, 0]$, the assumption $\boldsymbol{H}_2$ is clearly fulfilled.
The commutativity assumption $\boldsymbol{H}_3$ follows from the following lemma.

LEMMA 5.2. – *Assume that B is defined as in Lemma* 5.1, *then the resolvents of A and B commute.*

PROOF. – Thanks to (54), it suffices to check that the resolvents of A and C commute. This follows immediately from (50) and the explicit formula of the resolvent of A given by

$$\left[(A-\lambda I)^{-1}\varphi\right](t,\sigma) = \sum_{j\geq 1} \frac{1}{\mu_j - \lambda}\left[\int_G \varphi(t,\xi)w_j(\xi)d\xi\right]w_j(\sigma),$$

where w_j is the eigenfunction of A_2 associated to the eigenvalue μ_j and where φ belongs to $\mathcal{D}(B)$ a dense subspace of E. □

We have checked all the assumptions of Theorem 2.1. Consequently the operator sum L of A and B is closable ans its closure $\overline{L}$ is invertible. This ensures existence and uniqueness of a strong solution $v \in D(\overline{L})$ of

$$-\overline{L}v = h.$$

Moreover, v is explicitely given by

$$v = \frac{1}{2\pi i}\int_\gamma (A+\lambda)^{-1}(\lambda - B)^{-1}hd\lambda, \tag{55}$$

where γ could be, for instance, the imaginary axis with a small right detour to avoid the origin. For each z, we can write

$$\left[(A+\lambda I)^{-1}h\right](z) = (-A_p + \lambda)^{-1}h(z),$$

where we have considered h as a vector valued function of the only variable z. Consequently (44) allows us to write

$$v = v_R + \sum_{0<\lambda_m<\frac{2}{p'}} v_m, \tag{56}$$

where

$$v_R = \frac{1}{2\pi i} \int_\gamma R(\lambda)[(\lambda - B)^{-1} h] d\lambda, \tag{57}$$

and

$$v_m = \frac{1}{2\pi i} \int_\gamma \langle T_m(\lambda), (\lambda - B)^{-1} h \rangle \psi_m(\lambda) d\lambda. \tag{58}$$

In what follows, similarly to section 6 of [16], we shall prove that $v_R \in PW^{2,p}(\mathcal{B})$ and thus it is the regular part of v, while v_m involved the singular behavior of v.

Let us underline that we differ from [16] in the definition of A, the operator B being the same. This comes from the fact that a depends only on the x, y variables, thus the interface has no effect on the variable z. Therefore we only give the main results on the regularity of v_m and we skip the details of the proof, due to their similarities with [16].

5.1.3 — A direct study of v_m.

We have the following result.

THEOREM 5.3. – *Provided* $\frac{2}{p'} - \lambda_m$ *is not an integer, there exists a function*

$$q_m \in W^{\frac{2}{p'} - \lambda_m, p}(\mathbb{R})$$

such that

$$v_m = (K_m * q_m) S^{(m)}, \tag{59}$$

where

$$K_m(r,t) = \begin{cases} \dfrac{r}{\pi(t^2 + r^2)} & \text{for } \lambda_m > 1 - \dfrac{2}{p}, \\[2ex] \dfrac{2r^3}{\pi(t^2 + r^2)^2} & \text{for } \lambda_m \leq 1 - \dfrac{2}{p}, \end{cases} \tag{60}$$

and the convolution is in t. Morever $\Delta v_m \in L^p(\mathcal{B})$.

PROOF. – First we prove that identity (59) holds with

$$q_m = -\frac{1}{2\pi i} \int_\gamma \langle T_m(\lambda); (\lambda - B)^{-1} h \rangle d\lambda. \tag{61}$$

We assume that $h \in \mathcal{D}(B)$, a dense subspace of $L^p(\mathcal{B})$. Applying partial Fourrier transform in z to (58), we get

$$\hat{v}_m = \frac{1}{2\pi i} \int_\gamma \left\langle T_m(\lambda); \frac{\hat{h}}{\lambda - \tau^2} \right\rangle \psi_m(\lambda) d\lambda,$$

where τ denotes the dual variable of z.

The decay at infinity of $T_m(\lambda)$ and $\psi_m(\lambda)$ due to (47) and (42) allows us to apply Cauchy's formula. We obtain

$$\begin{aligned} \hat{v}_m &= - \langle T_m(\tau^2); \hat{h} \rangle \psi_m(\tau^2) \\ &= \begin{cases} -\langle T_m(\tau^2); \hat{h} \rangle e^{-r|\tau|} S^{(m)} & \text{if } \lambda_m > 1 - \dfrac{2}{p}, \\ -\langle T_m(\tau^2); \hat{h} \rangle (1 + r|\tau|) e^{-r|\tau|} S^{(m)} & \text{if } \lambda_m \le 1 - \dfrac{2}{p}. \end{cases} \end{aligned}$$

This can be seen as the Fourier transform of a convolution in z:

$$v_m = (K_m * q_m) S^{(m)},$$

where

$$\hat{q}_m = -\langle T_m(\tau^2); \hat{h} \rangle,$$

and

$$\hat{K}_m = \begin{cases} e^{-r|\tau|} & \text{if } \lambda_m > 1 - \dfrac{2}{p}, \\ (1 + r|\tau|) e^{-r|\tau|} & \text{if } \lambda_m \le 1 - \dfrac{2}{p}. \end{cases}$$

An inverse application of Cauchy's formula shows that

$$\hat{q}_m = -\frac{1}{2\pi i} \int_\gamma \left\langle T_m(\lambda); \frac{\hat{h}}{\lambda - \tau^2} \right\rangle d\lambda,$$

and therefore

$$q_m = -\frac{1}{2\pi i} \int_\gamma \langle T_m(\lambda); (\lambda - B)^{-1} h \rangle d\lambda.$$

By density, this identity is easily extended to any $h \in L^p(\mathcal{B})$.

The regularity of q_m and as a consequence the regularity of Δv_m are obtained exactly as in [16] (see Propositions 6.3, 6.4 and Theorem 6.5 of Loc. cit).

5.2 – *Application of the second strategy.*

We are now able to prove the regularity of v_R. Going back to (56). We have

$$\varDelta v_i = \varDelta v_{R,i} + \sum_{\lambda_m < \frac{2}{p'}} \varDelta v_{m,i} \quad \text{in } \mathcal{B}_i.$$

And consequently

$$\varDelta v_{R,i} - v_{R,i} = g_{R,i} = f_i - \sum_{\lambda_m < \frac{2}{p'}} \varDelta v_{m,i} - v_{R,i}, i = 1, 2. \tag{62}$$

The function g_R belongs to $L^p(\mathcal{B})$ by Theorem 5.3. On the other hand, (46) implies the following estimates

$$\|R(\lambda)\|_{L^p(G)\to \Pi_{i=1}^2 L^p(G_i)} = O\left(\frac{1}{|\lambda|}\right),$$

$$\|R(\lambda)\|_{L^p(G)\to \Pi_{i=1}^2 W^{2,p}(G_i)} = O(1).$$

By interpolation, we get

$$\|R(\lambda)\|_{L^p(G)\to X_\varepsilon} = O\left(\frac{1}{|\lambda|^\varepsilon}\right) \quad 0 < \varepsilon < 1,$$

where

$$\begin{aligned} X_\varepsilon &= \left[\Pi_{i=1}^2 L^p(G_i), \Pi_{i=1}^2 W^{2,p}(G_i)\right]_{\varepsilon,p} \\ &= \left[L^p(G_1), W^{2,p}(G_1)\right]_{\varepsilon,p} \times \left[L^p(G_2), W^{2,p}(G_2)\right]_{\varepsilon,p} \\ &= \Pi_{i=1}^2 W^{s,p}(G_i), \quad s = 2(1-\varepsilon). \end{aligned}$$

This shows that

$$\|R(\lambda)\|_{L^p(G)\to PW^{s,p}(G)} = O\left(\frac{1}{|\lambda|^{1-\frac{s}{2}}}\right), \ \forall s < 2.$$

With the help of (52), this yields

$$v_R \in L^p(\mathbb{R}, PW^{s,p}(G)), \tag{63}$$

for every $s < 2$ (we recall that $u \in PW^{s,p}(G)$ if and only if $u_i \in W^{s,p}(G_i)$, $i = 1, 2$), since the integral (57) converges in that space as a consequence of the estimate

$$\|R(\lambda)(\lambda - B)^{-1} h\|_{L^p(\mathbb{R}, PW^{s,p}(G)} = O\left(\frac{1}{|\lambda|^{2-\frac{s}{2}}}\right).$$

We shall apply Theorem 2.2 to study the equation (62). For this purpose we write it

$$D_z^2 v_{R,i} + \Delta v_{R,i} - v_{R,i} = g_{R,i}, i = 1, 2.$$

First we must specify E, A and B. We define the space E and the operator A exactly as in section 5.1. We differ in the definition of the operator B. Here we take

$$-Bv = D_z^2 v - v$$

for $v \in D(B) = W^{2,p}(\mathbb{R}, X)$.

We shall check assumptions $\boldsymbol{H}_3$, $\boldsymbol{H}_4$, $\boldsymbol{H}_5$ and $\boldsymbol{H}_6$ of Theorem 2.2: First the space E is U.M.D and consequently $\boldsymbol{H}_4$ holds.

Lemma 5.2 implies, in the particular case $P(z) = z^2 - 1$, that the commutativity assumption $\boldsymbol{H}_3$ is fulfilled.

A direct application of the property (vi) of Lemma 5.1 allows us to conclude that the operator B satisfies the assumption $\boldsymbol{H}_5$. For A, this assumption follows from Theorem 3.7.

It remains to check $\boldsymbol{H}_6$ for A and B. Thanks to Corollary 3.9, A_p is the infinitesimal generator of a semigroup of contraction in X which preserves positivity. By Coifman-Weiss theorem [6] this yields

$$\exists K > 0 : \forall s \in \mathbb{R}, \ \ \|A_p^{is}\| \leq K(1 + |s|)e^{\frac{\pi}{2}|s|}.$$

This estimation just gives the existence of some constants ε and $K(\varepsilon)$ such that

$$\exists K > 0 : \forall s \in \mathbb{R}, \ \ \|A_p^{is}\| \leq K(\varepsilon)e^{(\varepsilon + \frac{\pi}{2})|s|}.$$

Then the operator A verifies $\boldsymbol{H}_6$ with a constant $\tau > \dfrac{\pi}{2}$. This is not sufficient to apply Theorem 2.2 since the condition $\tau_A + \tau_B < \pi$ is not fulfilled. However in the particular case when $p = 2$, A_2 is a non negative self-adjoint operator in $X = L^2(G)$. Accordingly A_2^{is} is a contraction for all $s \in \mathbb{R}$, i.e.,

$$\|A_2^{is}\| = 1.$$

By interpolation, we deduce the existence of $\tau_A < \dfrac{\pi}{2}$ such that

$$\|A_p^{is}\| = 0(e^{|s|\tau_A}).$$

On the other hand, the symbol of the operator B^{is} is $(1 + \tau^2)^{is}$. Consequently we have

$$\|B^{is}\| = 0(e^{|s|\varepsilon})$$

for every $\varepsilon > 0$ by Mikhlin's theorem.

We then apply Theorem 2.2 and show the existence and uniqueness of

$$w_R \in D(L) = D(A) \cap D(B) = L^p(\mathbb{R}, D(A_p)) \cap W^{2,p}(\mathbb{R}, X),$$

solution of

$$Lw_R = g_R.$$

w_R do not coïncide necessarly with v_R. However we can easily verify that the difference

$$y_R = v_R - w_R$$

belongs to $L^p(\mathbb{R}, H_0^1(G))$ and solves the equation

$$D_z^2 y_{Ri} + \Delta y_{Ri} - y_{Ri} = 0, \quad \text{ in } \mathcal{B}_i, i = 1, 2.$$

Thus y_R can be extended over the system of the eigenfunctions w_j of A_2 in L^2, we get

$$y_R = \sum_{j\geq 1} \left(a_j e^{zc_j} + b_j e^{-zc_j}\right) w_j(\sigma), \tag{64}$$

where $c_j^2 = \mu_j + 1$. All the coefficients a_j and b_j must vanish since the exponentials involved in (64) do not belong to $L^p(\mathbb{R})$. Therefore $y_R = 0$ and consequently

$$v_R = w_R \in L^p(\mathbb{R}, D(A_p)) \cap W^{2,p}(\mathbb{R}, X). \tag{65}$$

At first sight this is not the regularity result we expected for v_R since $D(A_p)$ carries the singular solutions. However according to (63) and (65) we get

$$v_R \in L^p(\mathbb{R}, PW^{s,p}(G) \cap D(A_p)) \quad \text{ for every } s < 2.$$

We now take advantage of the following lemma.

LEMMA 5.4. – *For large enough $s < 2$ we have*

$$D(A_p) \cap PW^{s,p}(G) \subset PW^{2,p}(G) \cap W_0^{1,p}(G)$$

PROOF. – Let $u \in D(A_p) \cap PW^{s,p}(G)$. By the definition of $D(A_p)$, u vanishes on ∂G. Then to get $D(A_p) \cap PW^{s,p}(G) \subset W_0^{1,p}(G)$ it is enough to choose $s \geq 1$. On the other hand, by (9) a function $u \in D(A_p)$ can be written

$$u = u_0 + \sum_{0<\lambda_m<2/p'} c_m \eta r^{\lambda_m} t_m(\theta),$$

with $u_0 \in PW^{2,p}(G)$ and $c_m \in \mathbb{C}$. Therefore as $u \in PW^{s,p}(G)$, we get

$$r^{\lambda_m} t_m \in PW^{s,p}(G).$$

As Theorem 1.4.5.3 of [12] shows that this last inclusion holds if and only if

$$\lambda_m > s - \frac{2}{p},$$

we obtain that

$$c_m = 0 \text{ if } \lambda_m \le s - \frac{2}{p}.$$

This means that $u = u_0$ if we choose s close enough to 2 so that no singular exponent λ_m are between $s - \frac{2}{p}$ and $\frac{2}{p'} = 2 - \frac{2}{p}$. □

Accordingly we have

$$v_R \in W^{2,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W_0^{1,p}(G) \cap PW^{2,p}(G))$$

and then $v_{R,i} \in W^{2,p}(\mathbb{R}, L^p(G_i)) \cap L^p(\mathbb{R}, W^{2,p}(G_i))$, $i = 1, 2$. Applying Lemma 4.4 of [16] we deduce that $v_{R,i} \in W^{2,p}(\mathcal{B}_i)$, and consequently $v_R \in PW^{2,p}(\mathcal{B})$.

In summary we have proved the

THEOREM 5.5. – *Let $v \in H_0^1(\mathcal{B})$ be the variational solution of problem* (48). *Then there exist $v_R \in PW^{2,p}(\mathcal{B}) \cap W_0^{1,p}(\mathcal{B})$ and functions $q_m \in W^{\frac{2}{p'}-\lambda_m,p}(\mathbb{R})$ such that*

$$v = v_R + \sum_{\lambda_m < \frac{2}{p'}} (K_m * q_m) S^{(m)}$$

where the kernel K_m is given in Theorem 5.3 *and the singular functions $S^{(m)}$ are given in section* 3.

REMARK 5.6. – For the sake of simplicity, we restrict ourselves to the case of two sectors $G_i, i = 1, 2$ with a common interface Σ. The case of more than two sectors can be treated similarly using the results from [21].

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