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Sums of Three Prime Squares


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Sums of Three Prime Squares.

HIROSHI MIKAWA - TEMENOUJKA PENEVA (*)

Sunto. – Siano \( A, \varepsilon > 0 \) arbitrari. Supponiamo che \( x \) sia un numero positivo sufficientemente grande. Proviamo che il numero di interi \( n \) appartenenti ad \( (x, x + x^0) \), e soddisfacenti alcune condizioni di congruenza naturali, che non si possono scrivere come somma di tre quadrati di primi \( \ll x^0 (\log x)^{-A} \) con \( 7/16 + \varepsilon \leq \theta \leq 1 \).

Summary. – Let \( A, \varepsilon > 0 \) be arbitrary. Suppose that \( x \) is a sufficiently large positive number. We prove that the number of integers \( n \in (x, x + x^0) \), satisfying some natural congruence conditions, which cannot be written as the sum of three squares of primes is \( \ll x^0 (\log x)^{-A} \), provided that \( 7/16 + \varepsilon \leq \theta \leq 1 \).

1. – Introduction.

It is conjectured that every integer \( n \in \mathbb{H} = \{ m : m \equiv 3 \pmod{24}, m \not\equiv 0 \pmod{5} \} \) can be written as the sum of three squares of primes. The number of possible exceptions up to \( x \),

\[ E(x) = \{ n \leq x : n \in \mathbb{H}, n \not\equiv p_1^2 + p_2^2 + p_3^2, \text{ for all prime } p_i, \ i = 1, 2, 3 \}, \]

was first estimated in 1938 by Hua [3], who showed that \( E(x) \ll x (\log x)^{-A} \) for a certain constant \( A > 0 \). In 1961 Schwarz [11] demonstrated that any \( A > 0 \) is acceptable. In 1993 Leung and Liu [5] proved that \( E(x) \ll x^{1 - \delta} \) for some absolute constant \( \delta > 0 \). In 2000 Bauer, Liu and Zhan [1] gave a specific value for \( \delta \), namely \( \delta = 9/160 - \varepsilon \), which was later improved to \( \delta = 3/50 - \varepsilon \) by Liu and Zhan [6].

Liu and Zhan [7] were also the first to investigate the local properties of \( E(x) \). In 1996 they proved that for any \( A > 0 \) one has

\[ (1) \quad E(x + x^0) - E(x) \ll x^0 (\log x)^{-A}, \]

provided that \( 3/4 + \varepsilon \leq \theta \leq 1 \). An important tool in their proof is a new mean value estimate for nonlinear exponential sums over primes [7, Theorem 3]. Shortly

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thereafter, the first named author [8] replaced the constant 3/4 with 1/2 by using a tricky argument based on certain properties of the sequence of squares.

In this paper we refine Liu and Zhan’s approach [7], and establish the following result.

**Theorem 1.** – Let $A$, $\varepsilon > 0$ be arbitrary. Then the inequality (1) holds, provided that $7/16 + \varepsilon \leq \theta \leq 1$.

Theorem 1 can be derived by a standard argument (see [7, §1]) from the average estimate, which we formulate as Theorem 2 below. Let $\Lambda(n)$ and $\varphi(n)$ denote the von Mangoldt function and the Euler function, respectively, and write $e(a) = e^{2\pi ia}$ for real $a$. Define

$$R(n) = R(n, x, y) = \sum_{\substack{k+\mu x n^2 \leq x \\
\tau \leq \delta + n^2 \leq y}} \Lambda(k) \Lambda(l) \Lambda(m),$$

$$s(q, a) = \sum_{\substack{h=1 \\
(h, q) = 1}}^{q} e\left(\frac{ah^2}{q}\right), \quad \sigma(n, P) = \sum_{q \leq P} \sum_{\substack{a=1 \\
(a, q) = 1}}^{q} \left(\frac{s(q, a)}{\varphi(q)}\right)^3 e\left(-\frac{an}{q}\right).$$

Our main result is the following.

**Theorem 2.** – Let $A, \varepsilon > 0$ be given, $x^{7/12+\varepsilon} \leq y \leq x$ and $y^{3/4+\varepsilon} \leq H \leq y/4$. Then there exists a constant $B_0 = B_0(A) > 0$ such that for $B > B_0$ we have

$$\sum_{x < n \leq x + H} \left| R(n) - \frac{\pi}{8} \sqrt{y} \sigma(n, (\log x)^B) \right|^2 \ll_{A, \varepsilon} H y (\log x)^{-A}.$$

To prove Theorem 2 we apply the Hardy–Littlewood circle method, the main difficulties arising in the treatment of the major arcs error term, see Lemma 3. The minor arcs are handled by using a version of Liu and Zhan’s estimate [7, Theorem 3].

2. – Lemmas.

Before we are able to establish Theorem 2, we require some auxiliary results. Our first lemma is a minor modification of [7, Theorem 3].

**Lemma 1.** – Let $X$ be a large number, $Y = X^{\theta}$ and $1/2 \leq \theta < 1$. Suppose that $|a - a/q| \leq 1/q^2$ with $(a, q) = 1$. Then

$$Y \int_{|\lambda| \leq 1/Y} \left| \sum_{X < n^2 \leq 4X} \Lambda(n) e((a + \lambda)n^2) \right|^2 d\lambda \ll X^{3/4}(\log X)^6 + (Y q^{-1/4} + Y X^{-1/8} + Y^{3/4} q^{1/4})(\log X)^{13}.$$
Lemma 2. Let $D, E > 0$ and $0 < \varepsilon < 1/6$ be given. Let $X$ be a large number, $Y = X^\theta$ and $7/12 + \varepsilon \leq \theta \leq 1 - \varepsilon$. Then, uniformly for $1 \leq a \leq q \leq (\log X)^D$ with $(a, q) = 1$, we have

\begin{equation}
\int_{|\lambda| \leq 1/Y} \left| \sum_{n^2 \leq X} A(n) e \left( \left( \frac{\lambda}{\phi(q)} \right)^2 - \frac{s(q, a)}{\phi(q)} \sum_{n^2 \leq X} e(\lambda n^2) \right) \right|^2 d\lambda \ll (\log X)^{-E},
\end{equation}

where the implied constant depends on $D, E, \varepsilon,$ and $\theta$.

Proof. By the orthogonality of the characters mod $q$, we see that the left-hand side of (2) is

\[ \ll \sum_{\chi \mod q} \int_{|\lambda| \leq 1/Y} \left| \sum_{n^2 \leq X} \chi(n) A(n) e(\lambda n^2) \right|^2 d\lambda + Y^{1-1}(\log X)^4, \]

where $\sum^\#$ indicates that, for $\chi = \chi_0$, $\chi(n) A(n)$ is to be replaced by $A(n) - 1$. By Gallagher’s lemma [2, Lemma 1] and the Siegel-Walfisz theorem (see, e.g., [10, Chapter IV, Satz 8.3]), the above sum becomes

\[ \ll Y^{-2} \sum_{\chi \mod q} \int_Y^X \left| \sum_{t < n^2 \leq t+Y/2} \chi(n) A(n) \right|^2 dt + qYX^{-1} + q(\log X)^{-E-D}. \]

Since for a sufficiently small $\varepsilon > 0$,

\[(t + Y/2)^{1/2} - t^{1/2} \gg Y^{1/2} \gg (t^{1/2})^{1/6+\varepsilon},\]


The next statement can be derived easily from Lemma 2.

Lemma 2’. Suppose that the hypothesis of Lemma 2 is satisfied. Then

\[ \int_{|\lambda| \leq 1/Y} \left| \sum_{n^2 \leq X} A(n) e \left( \left( \frac{\lambda}{\phi(q)} \right)^2 - \frac{s(q, a)}{\phi(q)} \sum_{n \leq X} e(\lambda n^2) \right) \right|^2 d\lambda \ll (\log X)^{-E}, \]

where the implied constant depends on $D, E, \varepsilon,$ and $\theta$.

Lemma 3. Let $D, E > 0$ and $7/12 < \theta < 1$ be given. Let $X$ be a large number, $Y = X^\theta$. Then, uniformly for $1 \leq a \leq q \leq (\log X)^D$ with $(a, q) = 1$, we have

\begin{equation}
\sum_{X-Y < m^2 + n^2 \leq X} A(m)A(n) \frac{\alpha}{\phi(q)} \left( \frac{\alpha}{\phi(q)} \right) = \frac{\pi}{4} \left( \frac{s(q, a)}{\phi(q)} \right) Y + O(Y(\log X)^{-E}),
\end{equation}

where the implied constant depends on $D, E,$ and $\theta$.\[\]
**Proof.** – We begin by noting that the double sum in the left-hand side of (3) can be written as

\[
W := \int_{-1/2}^{1/2} S \left( \frac{a}{q} + \beta \right)^2 K(-\beta) \, d\beta,
\]

where

\[
S(a) = \sum_{n^2 \leq X} A(n) e(an^2), \quad K(\beta) = \sum_{X-Y<n\leq X} e(\beta n) \ll \min(Y, ||\beta||^{-1}).
\]

In order to approximate \( S(a/q + \beta) \), we define

\[
T \left( \frac{a}{q} + \beta \right) = \frac{s(q,a)}{\varphi(q)} t(\beta) = \sum_{n^2 \leq X} e(\beta n^2).
\]

For brevity, we write \( S_\beta = S(a/q + \beta) \) and \( T_\beta = T(a/q + \beta) \). Since \( A^2 = B^2 + (A-B)(A+B) \), we have that

\[
W = \left( \frac{s(q,a)}{\varphi(q)} \right)^2 \int_{-1/2}^{1/2} t(\beta)^2 K(-\beta) \, d\beta
\]

\[
+ O \left( \int_{-1/2}^{1/2} |S_\beta - T_\beta| (|S_\beta| + |T_\beta|) K(\beta) \, d\beta \right).
\]

An elementary argument, arising in the Gauss circle problem, shows that the integral in the main term is equal to

\[
\sum_{X-Y<n^2+n^2 \leq X} 1 = \frac{\pi}{4} Y + O(X^{1/2}).
\]

Next we bound the error term in (4). Since

\[
\int_{-1/2}^{1/2} \min(Y, |\beta|^{-1}) e(n\beta) \, d\beta \ll \min \left( \log Y, \frac{Y}{|n|} \right),
\]

we have that

\[
\int_{-1/2}^{1/2} |S_\beta|^2 |K(\beta)| \, d\beta \ll \sum_{m^2,n^2 \leq X} A(m) A(n) \min \left( \log Y, \frac{Y}{m^2-n^2} \right)
\]

\[
\ll \sum_{n^2 \leq X} A(n)^2 \log Y + \sum_{h \leq X} \tau(h) \frac{Y}{h} (\log X)^2
\]

\[
\ll X^{1/2}(\log X)^2 + Y(\log X)^4
\]

(6)

\[
\ll Y(\log X)^4.
\]
Similarly,

\[(7) \quad \int_{-1/2}^{1/2} \left| T_\beta \right|^2 |K(\beta)| \, d\beta \ll Y(\log X)^2.\]

Put \( \tilde{Y} = Y(\log X)^{-8-2E} \), so that \( \tilde{Y} \geq X^{\theta'} \) with some \( \theta' > 7/12 \), provided that \( X \) is large. Then Lemma 2 yields

\[\int_{|\beta| \leq 1/\tilde{Y}} \left| S_\beta - T_\beta \right|^2 |K(\beta)| \, d\beta \ll Y \int_{|\beta| \leq 1/\tilde{Y}} \left| S_\beta - T_\beta \right|^2 \, d\beta \ll Y(\log X)^{-4-2E}.\]

In the remaining range \( 1/\tilde{Y} \leq |\beta| \leq 1/2 \), we see that

\[|K(\beta)| \ll |\beta|^{-1} \leq \min (\tilde{Y}, |\beta|^{-1}).\]

As in (6) and (7), we obtain that

\[\int_{1/\tilde{Y} \leq |\beta| \leq 1/2} \left| S_\beta - T_\beta \right|^2 |K(\beta)| \, d\beta \ll \int_{|\beta| \leq 1/2} (|S_\beta|^2 + |T_\beta|^2) \min (\tilde{Y}, |\beta|^{-1}) \, d\beta \]

\[\ll X^{1/2}(\log X)^2 + \tilde{Y}(\log X)^4 \ll Y(\log X)^{-4-2E}.\]

Hence,

\[\int_{-1/2}^{1/2} \left| S_\beta - T_\beta \right|^2 |K(\beta)| \, d\beta \ll Y(\log X)^{-4-2E}.\]

Combining this estimate with (6) and (7) by means of Schwartz’s inequality, we derive that the error term in (4) is

\[\ll \left( Y(\log X)^{-4-2E} \right)^{1/2} \left( Y(\log X)^4 \right)^{1/2} = Y(\log X)^{-E}.\]

Now the conclusion of Lemma 3 follows from (4), (5) and (8). \( \square \)

3. – Proof of Theorem 2.

We have

\[R(n) = \int_0^1 S_1(a)S_2(a)e(-an) \, da,\]
\[ S_1(a) = \sum_{x-y<k^2+l^2 \leq x} A(k)A(l)e(a(k^2 + l^2)) \]

and

\[ S_2(a) = \sum_{y/4<m^2 \leq y} A(m)e(am^2). \]

Put

\[ P = L^B, \quad Q = HL^{-B}, \]

where \( L = \log x \) and \( B = 4A + 84 \). Define the set of major arcs \( \mathcal{M} \) as the union of all intervals \( \{ a \in \mathbb{R} : |qa - a| \leq 1/Q \} \) with \( 1 \leq a \leq q \leq P \) and \( (a, q) = 1 \). Denote the corresponding set of minor arcs by \( \mathfrak{m} = [1/Q, 1 + 1/Q] \setminus \mathcal{M} \). Hence,

\[ R(n) = \left( \int_{\mathcal{M}} + \int_{\mathfrak{m}} \right) S_1(a)S_2(a)e(-an)da = R_{\mathcal{M}}(n) + R_{\mathfrak{m}}(n), \]

say.

Let us first consider \( R_{\mathfrak{m}}(n) \). Recall that by Dirichlet’s approximation theorem, for any \( a \in \mathfrak{m} \) there exists a rational number \( a/q \) such that \( |a - a/q| \leq 1/q^2 \), \( (a, q) = 1 \) and \( P \leq q \leq Q \). Further, one has

\[ \int_0^1 |S_1(a)|^2 da \ll L^4 \sum_{x-y<k^2+l^2 \leq x} \left( \sum_{k^2+l^2=m} 1 \right)^2 \ll yL^7. \]

Arguing as in [9, §2], we obtain

\[ \sum_{x<n \leq x+H} |R_{\mathfrak{m}}(n)|^2 \ll HL \sup_{\beta \in \mathfrak{m}} \int_{|\beta| \leq 1/H} |S_2(a + \beta)|^2 d\beta \int_{\mathfrak{m}} |S_1(a')|^2 da' \]

\[ \ll (y^{3/4} + HL^{-B/4})L^{14}yL^7 \]

\[ \ll HyL^{-A}, \]

by Lemma 1 and (10).

We proceed to \( R_{\mathcal{M}}(n) \). In order to approximate \( S_2(a) \), we define, for \( a = a/q + \beta \in \mathcal{M} \),

\[ T_2(a) = \frac{s(q, a)}{\varphi(q)} t_2(\beta), \quad t_2(\beta) = \sum_{y/4 < n \leq y} \frac{e(\beta n)}{2\sqrt{n}}. \]
Note that $|s(q, a)| \ll q^{1/2+\varepsilon}$, and $|t_2(\beta)| \ll \min \left( y, y^{-1}|\beta|^{-2} \right)^{1/2}$ for $|\beta| \leq 1/2$. Then

$$
\int_{\mathbb{R}} (|S_2(a)|^2 + |T_2(a)|^2) \, da \\
\ll \int_{0}^{1} |S_2(a)|^2 \, da + \sum_{q \leq P} \sum_{(a, q)=1} \frac{|s(q, a)|^2}{\varphi(q)} \int_{|\beta| \leq 1/qQ} |t_2(\beta)|^2 \, d\beta \\
\ll y^{1/2}L^2 + PL
$$

(12)

$$
\ll y^{1/2}L^2.
$$

We next replace $S_2(a)$ by $T_2(a)$. The resulting cost is

$$
\sum_{x \ll n \leq x+H} \left| \int_{\mathbb{R}} S_1(a)(S_2(a) - T_2(a))e(-an) \, da \right|^2 \\
\ll \left( H \max_{1 \leq a \leq P} \int_{|\beta| \leq 1/qQ} \left| S_2 \left( \frac{a}{q} + \beta \right) - T_2 \left( \frac{a}{q} + \beta \right) \right|^2 \, d\beta \\
+ P^2 \int_{\mathbb{R}} |S_2(a) - T_2(a)|^2 \, da \right) \int_{\mathbb{R}} |S_1(a')|^2 \, da' \\
\ll \left( HL^{-A-7} + P^2y^{1/2}L^2 \right) yL^7
$$

(13)

$$
\ll HyL^{-A},
$$

by Lemma 2', (12) and (10).

Furthermore, we approximate $S_1(a)$, and to this end we need to make the major arcs small. Put

$$
\tilde{y} = yL^{-A-8},
$$

and let $\mathcal{M}^\dagger$ be the union of all intervals $\{ a \in \mathbb{R} : |qa - a| \leq 1/\tilde{y} \}$ with $1 \leq a \leq q \leq P$ and $(a, q) = 1$. Clearly $\mathcal{M}^\dagger \subset \mathcal{M}$. Observe that for $a = a/q + \beta \in \mathcal{M} \setminus \mathcal{M}^\dagger$ one has $1/q\tilde{y} \leq |\beta| \leq 1/qQ$, and

$$
|T_2(a)|^2 \ll Lq^{-1}y^{-1}|\beta|^{-2}.
$$

(14)
As before, we obtain

\[
\sum_{x < n \leq x + H} \left| \int \frac{S_1(a)T_2(a)e(-an)\,da}{y_n^{\frac{1}{q}}} \right|^2 \ll \left( H \max_{1 \leq a \leq P} \int \left| T_2 \left( \frac{a}{q} + \beta \right) \right|^2 \,d\beta + P^2 \int \left| T_2(a) \right|^2 \,da \right)
\times \int \frac{|S_1(a')|^2 \,da'}{y_n^{\frac{1}{q}}} \ll (HL\tilde{y}y^{-1} + P^2PL) \cdot yL^7.
\]

(15) \quad \ll HyL^{-A},

by (14), (12) and (10).

For \( a = \frac{a}{q} + \beta \in \mathfrak{M}^{\frac{1}{q}} \), we can approximate \( S_1(a) \) by

\[
T_1(a) = \frac{\pi}{4} \left( \frac{s(q, a)}{\varphi(q)} \right)^2 t_1(\beta), \quad t_1(\beta) = \sum_{x - y < n \leq x} e(\beta n).
\]

In fact, by partial summation and Lemma 3, we have that

\[
|S_1(a) - T_1(a)| \ll (1 + |\beta|y) \cdot yL^{-2A-13} \ll (1 + \tilde{y}y^{-1}) \cdot yL^{-2A-13}
\]

(16) \quad \ll yL^{-A-5},

uniformly for \( a \in \mathfrak{M}^{\frac{1}{q}} \). We also see that

\[
\int \frac{|T_1(a)|^2 \,da}{y_n^{\frac{1}{q}}} \ll \sum_{q \leq P} \sum_{a=1}^{q} \left| \frac{s(q, a)}{\varphi(q)} \right|^4 \int \left| t_1(\beta) \right|^2 \,d\beta \ll yL.
\]

(17) \quad \ll yL^{-A-5},

Hence, we infer that

\[
\sum_{x < n \leq x + H} \left| \int \frac{(S_1(a) - T_1(a))T_2(a)e(-an)\,da}{y_n^{\frac{1}{q}}} \right|^2 \ll H \sum_{q \leq P} \sum_{a=1}^{q} \int \left| T_2 \left( \frac{a}{q} + \beta' \right) \right|^2 \,d\beta'
\]

\[
\ll H \sum_{a=1}^{q} \int \left| T_2 \left( \frac{a}{q} + \frac{\beta'}{q} \right) \right|^2 \,d\beta'
\]

\[
\ll H \sum_{a=1}^{q} \int \left| T_2 \left( \frac{a}{q} + \frac{\beta'}{q} \right) \right|^2 \,d\beta'^{\frac{1}{2}}.
\]
\[ \times \int_{|\beta| \leq 1/\sqrt{y}} \left| S_1 \left( \frac{a}{q} + \beta \right) - T_1 \left( \frac{a}{q} + \beta \right) \right|^2 d\beta \]

\[ + P^2 \int_{\mathbb{R}} |T_2(a')|^2 da' \int_{\mathbb{R}} |S_1(a) - T_1(a)|^2 da \]

\[ \leq H \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{s(q,a)}{\varphi(q)} \right)^2 \frac{L(qy)^{-1} (yL^{-A-5})^2}{(y+1)^2} \]

\[ + P^2 P \int_{\mathbb{R}} \left( |S_1(a)|^2 + |T_1(a)|^2 \right) da \]

\[ \leq H L^2 y^{A+8} yL^{-2A-10} + P^3 L^3 y \]

(18)

by (16), (12), (10) and (17).

It remains to consider

\[ \int_{\mathbb{R}} T_1(a)T_2(a)e(-an) \, da \]

\[ = \frac{\pi}{4} \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{s(q,a)}{\varphi(q)} \right)^3 \frac{1}{q} e \left( -\frac{an}{q} \right) \int_{|\beta| \leq 1/\sqrt{y}} t_1(\beta)t_2(\beta)e(-\beta n) \, d\beta. \]

We extend the interval of integration in the right-hand side to $|\beta| \leq 1/2$. The resulting expression is equal to

\[ \frac{\pi}{4} \sigma(n, P) \sum_{l+m=n}^{l+\sqrt{m}} \frac{1}{2\sqrt{m}} = \frac{\pi}{4} \sigma(n, P) \left( \frac{\sqrt{y}}{2} + O(1) \right). \]

By Schwartz’s inequality and the dual form of the additive large sieve inequality we then conclude that

\[ \sum_{x < n \leq x + H} \left( \int_{\mathbb{R}} T_1(a)T_2(a)e(-an) \, da - \frac{\pi}{8} \sqrt{y} \sigma(n, P) \right)^2 \leq \int_{1/\sqrt{y} \leq |\beta| \leq 1/2} |t_2(\beta)|^2 \sum_{x < n \leq x + H} \left( \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \frac{s(q,a)}{\varphi(q)} \right)^3 \frac{1}{q} e \left( -\frac{an}{q} \right) \right)^2 d\beta \]
\[
\times \int \frac{|t_1(\beta')|^2}{1+y \leq |\beta'| \leq 1/2} d\beta' + HPL^2
\]
\[
\ll y^{-1} \left( \int_{1/y}^{1/2} \beta'^{-2} d\beta' \right)^2 (H + P^2) \sum_{q \leq P} \sum_{\substack{\sigma = 1 \\ (\alpha, q) = 1}} |s(q, \alpha)|^6 + HPL^2
\]
\[
\ll y^{-1} y^{-2} H + HPL^2
\]
\[
\ll HyL^{-A}.
\]

Combining this with (9), (11), (13), (15) and (18), we complete the proof of Theorem 2. \hfill \Box

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